Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 70, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ELLIPTIC SYSTEMS AT RESONANCE FOR JUMPING NON-LINEARITIES

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Abstract. In this article, we study the existence of nontrivial solutions for the problem

$$\begin{gathered}
-\Delta u=\alpha_1 u^+-\beta_1 u^-+f(x,u,v)+h_1(x)\quad \text{in } \Omega,\\
-\Delta v=\alpha_2 v^+-\beta_2 v^-+g(x,u,v)+h_2(x)\quad \text{in } \Omega,\\
u=v=0\quad \text{on } \partial\Omega,
\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, and $h_1,h_2\in L^2(\Omega)$. Here $[\alpha_j,\beta_j]\cap\sigma(-\Delta)=\lambda$, where $\sigma(\cdot)$ is the spectrum. We use the Leray-Schauder degree theory.

1. Introduction and statement of results

This article is devoted to the study of nonlinear elliptic systems at resonance. The study of resonant problems started with the seminal work of Landesman and Lazer (1969/1970), who produced sufficient conditions (which in certain circumstances are also necessary) for the existence of solutions for some smooth semilinear Dirichlet problems. The corresponding scalar case considered in [6] has shown the existence of solutions to the problem $Au=\alpha u^+-\beta u^-+f(x,u)+h$, where A is a self-adjoint operator with compact resolvent in $L^2(\Omega)$, $f(\cdot,\cdot)$ maps $\Omega\times\mathbb{R}$ into $\mathbb{R}$, such that $\lim_{s\to\infty}\frac{f(x,s)}{s}=0$ and $[\alpha,\beta]\cap\sigma(A)=\lambda$, ($\lambda$ a simple eigenvalue of A). The study of nonlinear elliptic systems at resonance has been extensively studied during recent years (see [9, 10]). In this work we establish the existence of weak solutions of the problem

$$\begin{gathered}
-\Delta u=\alpha_1 u^+-\beta_1 u^-+f(x,u,v)+h_1(x)\quad \text{in } \Omega,\\
-\Delta v=\alpha_2 v^+-\beta_2 v^-+g(x,u,v)+h_2(x)\quad \text{in } \Omega,\\
u=v=0\quad \text{on } \partial\Omega,
\end{gathered}\tag{1.1}$$

Where Ω is a bounded domain in $\mathbb{R}^N$ ($N\geq 2$) with smooth boundary $\partial\Omega$ and $h=(h_1,h_2)$ is an $(L^2(\Omega))^2$ function. Let $\overline{\lambda}$ and $\underline{\lambda}$ be defined as follows

$$\begin{gathered}
\underline{\lambda}=\sup\{\lambda_k:\lambda_k<\lambda,k\in\mathbb{N}^*\},\\
\overline{\lambda}=\inf\{\lambda_k:\lambda_k>\lambda,k\in\mathbb{N}^*\}.
\end{gathered}$$

2010 *Mathematics Subject Classification.* 35Q30, 65N12, 65N30, 76M25.
Key words and phrases. Topological degree; elliptic systems; homotopy.

Submitted July 26, 2015. Published March 15, 2016.

For the rest of this article, we suppose that $\alpha_j, \beta_j \in]\underline{\lambda}, \overline{\lambda}[= I_\lambda$ satisfy

$$[\alpha_j, \beta_j] \cap \sigma(A) = \lambda, j = 1, 2$$

we denote by $\sigma(A)$ the spectrum of A. For $u \in D(A)$, we define the real function $C(.,.)$ on the square $I_\lambda \times I_\lambda$ satisfying

$$Au = \alpha u^+ - \beta u^- + C(\alpha, \beta)\varphi, \int_\Omega u\varphi = 1,$$

where φ is a normalized eigenfunction corresponding to λ,

$$Au = \lambda u, \quad \|u\|_{L^2(\Omega)} = 1.$$

The function $C(.,.)$ is continuous on $I_\lambda \times I_\lambda$ and strictly decreasing with respect to each variable. Moreover, the curve

$$\Sigma = \{(\alpha, \beta) \in I_\lambda \times I_\lambda, C(\alpha, \beta) = \{0\}\}$$

is continuous, passing through the point (λ, λ) of $I_\lambda \times I_\lambda$. Let

$$C^{+,j} = C(\alpha_j, \beta_j), \quad C^{-,j} = C(\beta_j, \alpha_j), j = 1, 2.$$

The main idea in [10] is to present a priori bounds for the solutions of (1.1) where $C^{+,j} \cdot C^{-,j} \neq 0, j = 1, 2$. Always in the system case, the interested reader may refer to [1, 2, 3] and [4]. In the present paper we study the case where $C^{+,j} \cdot C^{-,j} = 0, (j = 1, 2)$. Let

$$N(\alpha, \beta) = \{u \in D(A), Au = \alpha u^+ - \beta u^-\},$$

then $N(\alpha, \beta) = \{0\}$ if and only if $C(\alpha, \beta) \cdot C(\beta, \alpha) \neq 0$ note that $N(\lambda, \lambda) = N_\lambda = \ker(A - \lambda I)$. The equation of existence of solution for (1.1) when $N(\alpha, \beta) = \{0\}$ has been studied in [10]. The main idea of the paper is to prove the existence of solutions of semilinear elliptic system of the form (1.1) in the case where $N(\alpha, \beta) \neq \{0\}$. There are two cases:

- If $C(\beta, \alpha) = C(\alpha, \beta) = 0$, we have (resonance),
- If $C(\alpha, \beta) = 0 \neq C(\beta, \alpha)$, or $C(\beta, \alpha) = 0 \neq C(\alpha, \beta)$, we have (semi resonance).

We assume that $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition below:

$$\begin{gathered} |f(x, s, t)| \leq c_1(1 + |s| + |t|), \\ |g(x, s, t)| \leq c_2(1 + |s| + |t|), \end{gathered} \tag{1.2}$$

where c_1, c_2 are real positive constants.

$$\begin{gathered} \lim_{s, |t| \to \infty} f(., t, s) = \gamma_1^+, \quad \lim_{-s, |t| \to \infty} f(., t, s) = \gamma_1^-, \\ \gamma_1^-, \gamma_1^+ \in L^2(\Omega), \quad \gamma_1^- \leq f(x, t, s) \leq \gamma_1^+, \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} \lim_{t, |s| \to \infty} g(., t, s) = \gamma_2^+, \quad \lim_{-t, |s| \to \infty} g(., t, s) = \gamma_2^-, \\ \gamma_2^-, \gamma_2^+ \in L^2(\Omega), \quad \gamma_2^- \leq g(x, t, s) \leq \gamma_2^+. \end{gathered} \tag{1.4}$$

Let $\theta_1 = (\mu_3, \mu_4)$ and $\theta_2 = (\mu_1, \mu_2)$ be defined as follows

$$\begin{gathered}
-\Delta\mu_j = \alpha_j\mu_j^+ - \beta_j\mu_j^-, \quad \int_\Omega \mu_j\varphi = -1 \\
\text{when } C(\beta_j, \alpha_j) = 0,\ (j = 1, 2), \\
-\Delta\mu_{j+2} = \alpha_j\mu_{j+2}^+ - \beta_j\mu_{j+2}^-, \quad \int_\Omega \mu_{j+2}\varphi = 1 \\
\text{when } C(\alpha_j, \beta_j) = 0,\ (j = 1, 2).
\end{gathered} \tag{1.5}$$

Our main theorem read as follows:

Theorem 1.1. *Assume that* (1.2), (1.3), (1.4) *and* (1.5) *are fulfilled. For each* $(h_1, h_2) \in (L^2(\Omega))^2$. *We define*

$$H_i(h_j) = \int_\Omega h_j\mu_i dx + \int_\Omega \gamma_j^+\mu_i^+ dx - \int_\Omega \gamma_j^-\mu_i^- dx, \quad i \in \{1, 2, 3, 4\},\ j = 1, 2.$$

(i) *If* $C^{+,j} = C^{-,j} = 0$, (1.1) *has at least one solution. For every* $h_j \in L^2(\Omega)$ *such that* $H_j(h_j).H_{j+2}(h_j) > 0$, $j = 1, 2$.

(ii) *If* $C^{+,j} = 0 \neq C^{-,j}$ *(resp* $C^{-,j} = 0 \neq C^{+,j}$*)*, (1.1) *has at least one solution. For every* $h_j \in L^2(\Omega)$ *such that* $C^{-,j}H_{j+2}(h_j) < 0$ *(resp* $C^{+,j}.H_j(h_j) < 0$*)*, $j = 1, 2$.

In the case $\alpha_j = \beta_j \neq \lambda$, $j = 1, 2$ see [10] (resp $\alpha_j = \beta_j = \lambda, j = 1, 2$ see [9]), we obtain the result of solutions existence.

2. Preliminaries

Let us consider the space

$$U = H_0^1(\Omega) \times H_0^1(\Omega),$$

which is a Banach space endowed with the norm

$$\|(u, v)\|_U^2 = \|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2,$$

and let us take $V = L^2(\Omega) \times L^2(\Omega)$. In the sequel, $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H_0^1(\Omega)}$ will denote the usual norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively. Recalling that the operator A, given by

$$\begin{gathered}
Au = -\Delta u \\
D(A) = \{u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\},
\end{gathered}$$

defines an inverse compact on $L^2(\Omega)$ and his spectrum is formed by the sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ such that $|\lambda_k| \to +\infty$ and λ_1 the first eigenvalue is positive. Throughout this paper, we denote by λ a simple eigenvalue of A, φ is an eigenfunction associated to λ normalized in $L^2(\Omega)$, Pr designates the orthogonal projection of V on $(\varphi^\perp)^2$ ($\varphi^\perp$ is the orthogonal of φ in $L^2(\Omega)$). We recall the following proposition proved by Gallouet and Kavian (see [5]).

Proposition 2.1. *For all* $\alpha, \beta \in]\underline{\lambda}, \overline{\lambda}[$, *there exist a unique* $C(\alpha, \beta) \in \mathbb{R}$, *and a unique* $u \in D(A)$, *such that*

$$\begin{gathered}
-\Delta u = \alpha u^+ - \beta u^- + C(\alpha, \beta)\varphi, \\
\int_\Omega u\varphi = 1.
\end{gathered}$$

The next result is given in a general framework.

Proposition 2.2. *Let $Q(x,s):\Omega\times\mathbb{R}\to\mathbb{R}$, measurable on $x\in\Omega$ and continuous on $s\in\mathbb{R}$, function verifying*

(i) *There exists $\alpha,\beta\in\mathbb{R}$ such that $\underline{\lambda}<\alpha\le\frac{Q(x,s)-Q(x,t)}{s-t}\le\beta<\overline{\lambda}$ for all $s,t\in\mathbb{R}$, a.e. in Ω,*
(ii) $\lim_{|s|\to+\infty}\frac{Q(x,s)}{s}=l$ *a.e. in Ω,*
(iii) *$Q(x,0)=0$ a.e. in Ω. Then for all $s\in\mathbb{R}$ and all $Q_0\in\varphi^\perp$, there exists a unique $v\in D(A)\cap\varphi^\perp$ such that*
$$Av=\Pr Q(.,v+s\varphi)+Q_0.$$

The proof of the above proposition can be found also in [5]. For $t\in[0,1]$ and $(u,v)\in(L^2(\Omega))^2$ we define
$$H(t,u,v)=\begin{pmatrix}A^{-1}&\\&A^{-1}\end{pmatrix})\begin{pmatrix}\alpha_1u^+-\beta_1u^-+tf(x,u,v)+(1-t)(\beta_1-\alpha_1)u^-\\ \alpha_2v^+-\beta_2v^-+tg(x,u,v)+(1-t)(\beta_2-\alpha_2)v^-\end{pmatrix}$$
The following two problems are equivalent:
$$\begin{gathered}-\Delta u=\alpha_1u^+-\beta_1u^-+tf(x,u,v)+(1-t)(\beta_1-\alpha_1)u^-+h_1(x),\\ -\Delta v=\alpha_2v^+-\beta_2v^-+tg(x,u,v)+(1-t)(\beta_2-\alpha_2)v^-+h_2(x),\\ (u,v)\in(D(A))^2,\end{gathered}$$
and
$$\begin{gathered}(u,v)=H(t,u,v)+(A^{-1}h_1,A^{-1}h_2),\\ (u,v)\in(D(A))^2,h\in(L^2(\Omega))^2,\end{gathered}$$
$H(t,u,v):[0,1]\times V\to V$ is compact.

3. A priori bounds for solutions of (1.1)

Lemma 3.1. *Under the assumptions of theorem 1.1, and assuming that $H_j(h_j)<0$, and $H_{j+2}(h_j)<0$, with $\alpha_j<\beta_j, j=1,2$. There exist $R>0$ such that for all $t\in[0,1]$ and all $(u,v)\in U$,*
$$(u,v)-H(t,u,v)=0\Longrightarrow\|(u,v)\|_U<R.$$

Proof. To prove this lemma we assume by contradiction, that for all $R>0$ there exists $(t,u,v)\in[0,1]\times U$ such that
$$(u,v)-H(t,u,v)=0\quad\text{and}\quad\|(u,v)\|_U>R,$$
In other words, we can find a sequence $(t_n,u_n,v_n)\in[0,1]\times U$ such that
$$(u_n,v_n)-H(t_n,u_n,v_n)=0\quad\text{and}\quad b_n=\|(u_n,v_n)\|_U>n.\tag{3.1}$$
Taking
$$w_n=(w_{n,1},w_{n,2})=\Big(\frac{u_n}{\|(u_n,v_n)\|_U},\frac{v_n}{\|(u_n,v_n)\|_U}\Big),$$
then it follows with this choice of w_n that
$$w_n=(w_{n,1},w_{n,2})\in(D(A))^2\quad\text{and}\quad\|w_n\|_U=1.$$

Indeed, it is easy to see that $\|w_n\|_U = 1$. Let us show that $w_n \in (D(A))^2$. We have

$$
\begin{aligned}
&-\Delta w_{n,1} \\
&= \frac{1}{b_n}[\alpha_1 u_n^+ - \beta_1 u_n^- + t_n f(x,u_n,v_n) + (1-t_n)(\beta_1-\alpha_1)u_n^- + h_1(x)],
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
&-\Delta w_{n,2} \\
&= \frac{1}{b_n}[\alpha_2 v_n^+ - \beta_2 v_n^- + t_n g(x,u_n,v_n) + (1-t_n)(\beta_2-\alpha_2)v_n^- + h_2(x)].
\end{aligned} \tag{3.3}
$$

From (1.2) and noticing that $(a+b)^2 \le 2(a^2+b^2)$, we obtain the following estimate

$$
\begin{aligned}
\int_\Omega |f(x,u_n,v_n)|^2 dx &\le \int_\Omega c_1^2(1+|u_n|+|v_n|)^2 dx \\
&\le 2c_1^2 \int_\Omega ((1+|u_n|)^2 + |v_n|^2) dx \le c'(1+\|u_n\|^2_{H_0^1} + \|v_n\|^2_{H_0^1}),
\end{aligned}
$$

where c' is a positive constant. Therefore,

$$
\int_\Omega \frac{|f(x,u_n,v_n)|^2}{\|(u_n,v_n)\|_U^2} dx \le c'\Big(\frac{1}{\|(u_n,v_n)\|_U^2} + \frac{\|u_n\|^2}{\|(u_n,v_n)\|_U^2} + \frac{\|v_n\|^2}{\|(u_n,v_n)\|_U^2}\Big).
$$

Then

$$
\int_\Omega \frac{|f(x,u_n,v_n)|^2}{\|(u_n,v_n)\|_U^2} dx \le c'\Big(\frac{1}{n^2}+1\Big) \le 2c';
$$

that is, $\frac{f(x,u_n,v_n)}{\|(u_n,v_n)\|_U}$ is bounded in $L^2(\Omega)$. Similarly, the function $\frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_U}$ is bounded in $L^2(\Omega)$. Moreover, by (3.1) we have

$$
\begin{gathered}
\frac{\|h_1\|_{L^2(\Omega)}}{\|(u_n,v_n)\|_U} \le \frac{\|h_1\|_{L^2(\Omega)}}{n} \le \|h_1\|_{L^2(\Omega)}, \\
\frac{\|h_2\|_{L^2(\Omega)}}{\|(u_n,v_n)\|_U} \le \frac{\|h_2\|_{L^2(\Omega)}}{n} \le \|h_2\|_{L^2(\Omega)},
\end{gathered}
$$

then the right hand side of (3.2) is bounded in $L^2(\Omega)$ for all n, thus

$$
\frac{1}{b_n}[\alpha_1 u_n^+ - \beta_1 u_n^- + t_n f(x,u_n,v_n) + (1-t_n)(\beta_1-\alpha_1)u_n^- + h_1(x)] \in L^2(\Omega).
$$

Similarly we have

$$
\frac{1}{b_n}[\alpha_2 v_n^+ - \beta_2 v_n^- + t_n g(x,u_n,v_n) + (1-t_n)(\beta_2-\alpha_2)v_n^- + h_2(x)] \in L^2(\Omega).
$$

Since $(w_{n,1}, w_{n,2}) \in (H_0^1(\Omega))^2$ and the embedding $(H_0^1(\Omega) \hookrightarrow L^2(\Omega))$ is compact, we can extract a subsequence $(t_n, w_{n,1}, w_{n,2})$, still denoted by $(t_n, w_{n,1}, w_{n,2})$, which converges in $[0,1] \times V$. Let (t, w_1, w_2) be the limit of $(t_n, w_{n,1}, w_{n,2})$ in $[0,1] \times V$. From the hypothesis (1.3) and (1.4) it follows that

$$
\begin{aligned}
\frac{f(x,u_n,v_n)}{\|(u_n,v_n)\|_U} &= \frac{u_n}{\|(u_n,v_n)\|_U} \frac{f(x,u_n,v_n)}{u_n} \\
&= w_{n,1} \frac{f(x, w_{n,1}\|(u_n,v_n,)\|_U, v_n)}{w_{n,1}\|(u_n,v_n)\|_U} \underset{n\to\infty}{\to 0} \quad \text{a.e. in } \Omega, \\
\frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_U} &= \frac{v_n}{\|(u_n,v_n)\|_U} \frac{g(x,u_n,v_n)}{v_n} \\
&= w_{n,2} \frac{g(x, u_n, w_{n,2}\|(u_n,v_n)\|_U)}{w_{n,2}\|(u_n,v_n)\|_U} \underset{n\to\infty}{\to 0} \quad \text{a.e. in } \Omega,
\end{aligned}
$$

and since the sequences $w_{n,1}$, $w_{n,2}$ are bounded in $L^2(\Omega)$, we get

$$\begin{gathered}
\frac{f(x,u_n,v_n)}{\|(u_n,v_n)\|_U} \le c_1(1+|w_{n,1}|+|w_{n,2}|) \le c' \quad \text{a.e. in } \Omega,\\
\frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_U} \le c_2(1+|w_{n,1}|+|w_{n,2}|) \le c'' \quad \text{a.e. in } \Omega,
\end{gathered}$$

where c', c'' are real positive constants. Thanks to Lebesgue's convergence theorem, we deduce that

$$\begin{gathered}
\frac{f(x,u_u,v_n)}{\|(u_n,v_n)\|_U} \to 0 \quad \text{in } L^2(\Omega), n \to \infty,\\
\frac{g(x,u_n,v_n)}{\|(u_n,v_n)\|_U} \to 0 \quad \text{in } L^2(\Omega), n \to \infty,
\end{gathered}$$

and consequently

$$\begin{gathered}
-\Delta w_{n,1} \to [\alpha_1 w_1^+ - \beta_1 w_1^- + (1-t)(\beta_1-\alpha_1)w_1^-],\\
-\Delta w_{n,2} \to [\alpha_2 w_2^+ - \beta_2 w_2^- + (1-t)(\beta_2-\alpha_2)w_2^-],\\
\|(w_{1,n}, w_{2,n})\|_U = 1.
\end{gathered}$$

Then

$$\begin{gathered}
-\Delta w_1 = \alpha_1 w_1^+ - \beta_1 w_1^- + (1-t)(\beta_1-\alpha_1)w_1^-,\\
-\Delta w_2 = \alpha_2 w_2^+ - \beta_2 w_2^- + (1-t)(\beta_2-\alpha_2)w_2^-.
\end{gathered}$$

Case I: $\int_\Omega w_1\varphi = \int_\Omega w_2\varphi = 0$. Then projecting on $\varphi^\perp$ we have

$$\begin{gathered}
-\Delta w_1 = \mathrm{Pr}[\alpha_1 w_1^+ - \beta_1 w_1^- + (1-t)(\beta_1-\alpha_1)w_1^-],\\
-\Delta w_2 = \mathrm{Pr}[\alpha_2 w_2^+ - \beta_2 w_2^- + (1-t)(\beta_2-\alpha_2)w_2^-].
\end{gathered}$$

Using proposition 2.2, $(s=0, Q_0=0)$ we see that $w_1 = w_2 = 0$, this is contradiction with $\|w\|_U = 1$. Hence $\int_\Omega w\varphi \neq 0$.

Case II: $\int_\Omega w_1\varphi \neq 0$. If $\int_\Omega w_1\varphi = \theta > 0$, then $\mu = \frac{w_1}{\theta}$ verifies

$$A\mu = \alpha_1\mu^+ - (\beta_1 + (1-t)(\alpha_1-\beta_1))\mu^-, \quad \int_\Omega \mu\varphi = 1,$$

from proposition 2.1, we deduce that

$$C(\alpha_1, \beta_1 + (1-t)(\alpha_1-\beta_1)) = 0.$$

The function $C(\cdot,\cdot)$ is strictly decreasing with respect to each variable, with $\beta_1 > \alpha_1$ and $t < 1$, we have

$$C(\alpha_1, \beta_1 + (1-t)(\alpha_1-\beta_1)) > C(\alpha_1,\beta_1) = 0,$$

which is a contradiction. If $\int_\Omega w_1\varphi = \theta < 0$, $\mu = \frac{w_1}{\theta}$, we obtain a contradiction as a similar argument with the above step.

Case III: $\int_\Omega w_2\varphi \neq 0$**.** A similar argument can be made when $\int_\Omega w_2\varphi \neq 0$. Let us assume $t = 1$ i.e $t_n \to 1$. Now, however, we have no contradiction since $(w_1, w_2) \in N(\alpha_j, \beta_j)$ and

$$\begin{aligned} Aw_1 &= \alpha_1 w_1^+ - \beta_1 w_1^-, \quad (w_1, w_2) \in N(\alpha_1, \beta_1),\\ Aw_2 &= \alpha_2 w_2^+ - \beta_2 w_2^-, \quad (w_1, w_2) \in N(\alpha_2, \beta_2), \end{aligned} \tag{3.4}$$

we can write

$$\begin{aligned} w_j &= a_j\mu_{j+2} \text{ if } a_j = \int_\Omega w_j\varphi dx > 0, \quad j = 1, 2,\\ w_j &= a_j\mu_j \text{ if } -a_j = \int_\Omega w_j\varphi dx < 0, \quad j = 1, 2, \end{aligned}$$

defining

$$a_{n,j} \in \mathbb{R}, \quad z_{n,j} \in D(A), \quad a_{n,j} = -\int_\Omega w_{n,j}\varphi dx, z_{n,j} = w_{n,j} - a_{n,j}\mu_j,$$

in such a way that

$$w_{n,j} = z_{n,j} + a_{n,j}\mu_j, \quad a_{n,j} \to a_j, \quad \|z_{n,j}\|_{D(A)} \to 0, \quad z_{n,j} \in \varphi^\perp,$$

if $a_j \neq 0$ we claim that

$$\exists M > 0 \text{ such that } \forall n \geq 1, \quad b_n\|z_{n,j}\|_{D(A)} \leq M, \quad j = 1, 2. \tag{3.5}$$

When $\int_\Omega w_1\varphi dx < 0$, if (3.5) is established, multiplying (3.2) on both sides by μ_1 gives

$$\begin{aligned} b_n\int_\Omega -\Delta w_{n,1}\mu_1 dx &= b_n\int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 dx + (1 - t_n)(\beta_1 - \alpha_1)w_{n,1}^-\mu_1 dx,\\ &+ \int_\Omega t_n f(x, b_n w_{n,1}, v_n)\mu_1 dx + h_1(x)\mu_1 dx\,. \end{aligned}$$

For n large enough, $\int_\Omega w_{n,1}^-\mu_1 \leq 0$, because $w_{n,1}^- \to a_1\mu_1^-$ in $L^2, a_1 > 0$, hence

$$\begin{aligned} &t_n\int_\Omega f(x, b_n w_{n,1}, v_n)\mu_1 dx + h_1(x)\mu_1 dx\\ &\geq b_n\int_\Omega -\Delta w_{n,1}\mu_1 dx - b_n\int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 dx, \end{aligned} \tag{3.6}$$

noticing that

$$E_{n,1} = \int_\Omega -\Delta w_{n,1}\mu_1 dx - \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 dx,$$

because $(A = A^*)$;

$$E_{n,1} = \int_\Omega w_{n,1}(-\Delta\mu_1)dx - \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 dx,$$

then

$$E_{n,1} = \int_\Omega w_{n,1}(\alpha_1\mu_1^+ - \beta_1\mu_1^-)dx - \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 dx,$$

that is

$$E_{n,1} = \alpha_1\int_\Omega (w_{n,1}^+\mu_1^- - w_{n,1}^-\mu_1^+) - \beta_1\int_\Omega (w_{n,1}^+\mu_1^- - w_{n,1}^-\mu_1^+)dx,$$

hence

$$|E_{n,1}| \le |\beta_1 - \alpha_1| \Big(\int_\Omega w_{n,1}^+ \mu_1^- + \int_\Omega w_{n,1}^- \mu_1^+ \Big). \tag{3.7}$$

If $x \in \Omega$ is such that $\mu_1(x) \ge 0$ and $w_{n,1}(x) = z_{n,1}(x) + a_{n,1}\mu_1(x) \le 0$, then

$$z_{n,1}(x) \le 0 \quad \text{and} \quad 0 \le \mu_1(x) = \frac{w_{n,1}(x) - z_{n,1}(x)}{a_{n,1}} \le \frac{|z_{n,1}(x)|}{a_{n,1}},$$

we obtain

$$w_{n,1}^-(x)\mu_1^+(x) \le \frac{|z_{n,1}(x)|^2}{a_{n,1}} \quad \text{a.e. in } \Omega,$$

using the same arguments, one can see that

$$w_{n,1}^+(x)\mu_1^-(x) \le \frac{|z_{n,1}(x)|^2}{a_{n,1}} \quad \text{a.e. in } \Omega,$$

From these inequalities and (3.7), we deduce

$$|E_{n,1}| \le 2|\beta_1 - \alpha_1| \frac{\|z_{n,1}\|_{L^2(\Omega)}^2}{a_{n,1}};$$

hence, (3.5) implies that

$$b_n|E_{n,1}| \le 2M|\beta_1 - \alpha_1| \frac{\|z_{n,1}\|_{D(A)}}{a_{n,1}},$$

and $\lim_{n\to\infty} b_n|E_{n,1}| = 0$. Now coming back to formula (3.6),

$$\begin{aligned} J_{n,1} &= t_n \int_\Omega f(x, b_n w_{n,1}, v_n)\mu_1 dx + h_1(x)\mu_1 dx \\ &\ge b_n \int_\Omega -\Delta w_{n,1}\mu_1 dx - b_n \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 dx. \end{aligned}$$

From the hypothesis (1.3),

$$\gamma_1^- \le f(x,s,t) \le \gamma_1^+,$$

we have

$$J_{n,1} = t_n \int_\Omega f(x,u_n,v_n)\mu_1 dx + h_1(x)\mu_1 dx \le t_n \int_\Omega \gamma_1^+\mu_1^+ - \gamma_1^-\mu_1^- dx + h_1(x)\mu_1 dx,$$

which gives

$$b_n E_{n,1} \le t_n \int_\Omega \gamma_1^+\mu_1^+ - \gamma_1^-\mu_1^- dx + h_1(x)\mu_1 dx.$$

Passing to the limit we obtain

$$0 \le \int_\Omega \gamma_1^+\mu_1^+ - \gamma_1^-\mu_1^- dx + h_1(x)\mu_1 dx = H_1(h_1),$$

which contradicts $H_1(h_1) < 0$.

When $\int_\Omega w_2\varphi dx < 0$, we multiply (3.3) on both sides by μ_2,

$$\begin{aligned} & b_n \int_\Omega -\Delta w_{n,2}\mu_2 dx \\ &= b_n \int_\Omega (\alpha_2 w_{n,2}^+ - \beta_2 w_{n,2}^-)\mu_2 dx + (1-t_n)(\beta_2 - \alpha_2) w_{n,2}^-\mu_2 dx \\ &\quad + \int_\Omega t_n g(x,u_n,b_n w_{n,2})\mu_2 dx + h_2(x)\mu_2 dx\,. \end{aligned}$$

By the same arguments used in the precedent step with

$$\gamma_2^- \le g(x,s,t) \le \gamma_2^+,$$

we have

$$J_{n,2} = t_n \int_\Omega g(x,u_n,v_n)\mu_2 dx + h_2(x)\mu_2 dx \le t_n \int_\Omega \gamma_2^+\mu_2^+ - \gamma_2^-\mu_2^- dx + h_2(x)\mu_2 dx\,.$$

This gives a contradiction with $H_2(h_2) < 0$.

When $\int_\Omega w_1\varphi dx > 0$ defining

$$a_{n,j} \in \mathbb{R}, \quad z_{n,j} \in D(A), \quad a_{n,j} = \int_\Omega w_{n,j}\varphi dx, \quad z_{n,j} = w_{n,j} - a_{n,j}\mu_{j+2},$$

in such a way that

$$w_{n,j} = z_{n,j} + a_{n,j}\mu_{j+2}, \quad a_{n,j} \to a_j, \quad \|z_{n,j}\|_{D(A)} \to 0, \quad z_{n,j} \in \varphi^\perp,$$

we multiply (3.2) on both sides by μ_3,

$$\begin{aligned} b_n \int_\Omega -\Delta w_{n,1}\mu_3 dx &= b_n \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_3 dx + (1-t_n)(\beta_1-\alpha_1)w_{n,1}^-\mu_3 dx \\ &\quad + \int_\Omega t_n f(x, b_n u_n, w_{n,2})\mu_3 dx + h_2(x)\mu_3 dx\,. \end{aligned}$$

By the same arguments used in the precedent step with $\gamma_1^- \le f(x,s,t) \le \gamma_1^+$, we have

$$J_{n,1} = t_n \int_\Omega f(x,u_n,v_n)\mu_3 dx + h_2(x)\mu_3 dx \le t_n \int_\Omega \gamma_1^+\mu_3^+ - \gamma_1^-\mu_3^- dx + h_2(x)\mu_3 dx,$$

gives a contradiction to $H_3(h_2) < 0$.

When $\int_\Omega w_2\varphi dx > 0$ Multiply (3.3) on both sides by μ_4,

$$\begin{aligned} b_n \int_\Omega -\Delta w_{n,2}\mu_4 dx &= b_n \int_\Omega (\alpha_2 w_{n,2}^+ - \beta_2 w_{n,2}^-)\mu_4 dx + (1-t_n)(\beta_2-\alpha_2)w_{n,2}^-\mu_4 dx \\ &\quad + \int_\Omega t_n g(x, u_n, b_n w_{n,2})\mu_4 dx + h_2(x)\mu_4 dx,. \end{aligned}$$

By the same arguments used in the precedent step, with $\gamma_2^- \le g(x,s,t) \le \gamma_2^+$, we have

$$J_{n,2} = t_n \int_\Omega f(x,u_n,v_n)\mu_4 dx + h_2(x)\mu_4 dx \le t_n \int_\Omega \gamma_2^+\mu_4^+ - \gamma_2^-\mu_4^- dx + h_2(x)\mu_4 dx,$$

give a contradiction with $H_4(h_2) < 0$.

Now, if (3.5) does not hold, there exists a subsequence denoted by $b_n\|z_n\|_{(D(A))^2}$, such that $\lim_{n\to\infty} b_n\|z_n\|_{(D(A))^2} \to \infty$. Let

$$\begin{gathered} c_n = \|z_n\|_{(D(A))^2}, \\ y_n = (y_{n,1}, y_{n,2}) = \Big(\frac{z_{n,1}}{\|z_n\|_{(D(A))^2}}, \frac{z_{n,2}}{\|z_n\|_{(D(A))^2}}\Big) = \frac{z_n}{c_n}, \end{gathered}$$

$y_n \in (D(A))^2$, $\|y_n\|_{(D(A))^2} = 1$. The inclusion $D(A) \hookrightarrow L^2(\Omega)$ being compact there is a subsequence (still denoted by) $y_n = (y_{n,1}, y_{n,2})$ such that

$$\begin{gathered} (y_{n,1}, y_{n,2}) \to (y_1, y_2) \text{ in } V, \quad A(y_n) \to A(y) \text{ in } V \text{ weak } y \in (\varphi^\perp)^2, \\ y_n(x) \to y(x) \quad \text{a.e. in } \Omega. \end{gathered} \tag{3.8}$$

There exists $(k_1, k_2) \in V$, such that

$$|y_{n,1}(x)| \leq k_1(x) \text{ a.e}, \quad |y_{n,2}(x)| \leq k_2(x) \text{ a.e.}$$

On the other hand $w_{n,j} = z_{n,j} + a_{n,j}\mu_j, j = 1, 2$ satisfies

$$\begin{gathered}
-\Delta w_{n,1} = \alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^- + t_n \frac{f(., b_n w_{n,1}, v_n)}{b_n} + (1 - t_n)(\beta_1 - \alpha_1) w_{n,1}^- + \frac{h_1}{b_n}, \\
-\Delta w_{n,2} = \alpha_2 w_{n,2}^+ - \beta_2 w_{n,2}^- + t_n \frac{g(., u_n, b_n w_{n,2})}{b_n} \\
+ (1 - t_n)(\beta_2 - \alpha_2) w_{n,2}^- + \frac{h_2}{b_n}.
\end{gathered}$$

Multiplying the first equation by μ_1/c_n, and the second equation by μ_2/c_n, we have

$$\begin{aligned}
\frac{1}{c_n} \int_\Omega -\Delta w_{n,1}\mu_1 &= \frac{1}{c_n} \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 + t_n \int_\Omega (\frac{f(., b_n w_{n,1}, v_n)}{b_n c_n})\mu_1 \\
&\quad + \frac{1}{c_n} \int_\Omega (1 - t_n)(\beta_1 - \alpha_1) w_{n,1}^- \mu_1 + \int_\Omega \frac{h_1}{b_n c_n}\mu_1, \\
\frac{1}{c_n} \int_\Omega -\Delta w_{n,2}\mu_2 &= \frac{1}{c_n} \int_\Omega (\alpha_2 w_{n,2}^+ - \beta_2 w_{n,2}^-)\mu_2 + t_n \int_\Omega \frac{g(., u_n, b_n w_{n,2})}{b_n c_n}\mu_2 \\
&\quad \frac{1}{c_n} \int_\Omega (1 - t_n)(\beta_2 - \alpha_2) w_{n,2}^- \mu_2 + \int_\Omega \frac{h_2}{b_n c_n}\mu_2 \,.
\end{aligned}$$

$A = A^*$ gives

$$\begin{aligned}
\frac{1}{c_n} \int_\Omega w_{n,1}(-\Delta\mu_1) &= \frac{1}{c_n} \int_\Omega (\alpha_1 w_{n,1}^+ - \beta_1 w_{n,1}^-)\mu_1 + t_n \int_\Omega (\frac{f(., b_n w_{n,1}, v_n)}{b_n c_n})\mu_1 \\
&\quad + \frac{1}{c_n} \int_\Omega (1 - t_n)(\beta_1 - \alpha_1) w_{n,1}^- \mu_1 + \int_\Omega \frac{h_1}{b_n c_n}\mu_1, \\
\frac{1}{c_n} \int_\Omega w_{n,2}(-\Delta\mu_2) &= \frac{1}{c_n} \int_\Omega (\alpha_2 w_{n,2}^+ - \beta_2 w_{n,2}^-)\mu_2 + t_n \int_\Omega \frac{g(., u_n, b_n w_{n,2})}{b_n c_n}\mu_2 \\
&\quad + \frac{1}{c_n} \int_\Omega (1 - t_n)(\beta_2 - \alpha_2) w_{n,2}^- \mu_2 + \int_\Omega \frac{h_2}{b_n c_n}\mu_2 \,.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{1}{c_n} E_{n,1} - t_n \int_\Omega (\frac{f(., b_n w_{n,1}, v_n)}{b_n c_n})\mu_1 &= \frac{1}{c_n} \int_\Omega (1 - t_n)(\beta_1 - \alpha_1) w_{n,1}^- \mu_1 + \int_\Omega \frac{h_1}{b_n c_n}\mu_1, \\
\frac{1}{c_n} E_{n,2} - t_n \int_\Omega \frac{g(., u_n, b_n w_{n,2})}{b_n c_n}\mu_2 &= \frac{1}{c_n} \int_\Omega (1 - t_n)(\beta_2 - \alpha_2) w_{n,2}^- \mu_2 + \int_\Omega \frac{h_2}{b_n c_n}\mu_2,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\frac{(1 - t_n)(\beta_1 - \alpha_1)}{c_n} \int_\Omega w_{n,1}^- \mu_1 &= \frac{1}{c_n} E_{n,1} - t_n \int_\Omega (\frac{f(., b_n w_{n,1}, v_n)}{b_n c_n})\mu_1 - \int_\Omega \frac{h_1}{b_n c_n}\mu_1, \\
\frac{(1 - t_n)(\beta_2 - \alpha_2)}{c_n} \int_\Omega w_{n,2}^- \mu_2 &= \frac{1}{c_n} E_{n,2} - t_n \int_\Omega \frac{g(., u_n, b_n w_{n,2})}{b_n c_n}\mu_2 - \int_\Omega \frac{h_2}{b_n c_n}\mu_2 \,.
\end{aligned}$$

From (1.3), (1.2) we can write

$$(\beta_j - \alpha_j) \lim_{n\to\infty} \frac{(1 - t_n)}{c_n} \int_\Omega w_{n,j}^- \mu_j = 0, \quad j = 1, 2,$$

such that

$$\lim_{n\to\infty}\int_\Omega w_{n,j}^-\mu_J = -a_j\int_\Omega |\mu_j^-|^2dx \neq 0\,.$$

If μ_j satisfies (1.5) and $\alpha_j \notin \sigma(A)$, then $\mu_j^- \neq 0$. For this index j, $\beta_j - \alpha_j \neq 0$, we find that

$$\lim_{n\to\infty}\frac{(1-t_n)}{c_n} = 0. \tag{3.9}$$

From (1.5) , (3.3), (3.4) and

$$w_{n,j} = z_{n,j} + a_{n,j}\mu_j = y_{n,j}c_n + a_{n,j}\mu_j, j = 1,2$$

we obtain

$$\begin{aligned}
-\Delta(y_{n,1}c_n + a_{n,1}\mu_1) &= \alpha_1(y_{n,1}c_n + a_{n,1}\mu_1)^+ - \beta_1(y_{n,1}c_n + a_{n,1}\mu_1)^- \\
&\quad + t_n\frac{f(.,b_nw_{n,1},v_n)}{b_n} + (1-t_n)(\beta_1-\alpha_1)(w_{1,n})^- + \frac{h_1}{b_n}, \\
-\Delta(y_{n,2}c_n + a_{n,2}\mu_2) &= \alpha_2(y_{n,2}c_n + a_{n,2}\mu_2)^+ - \beta_2(y_{n,2}c_n + a_{n,2}\mu_2)^- \\
&\quad + t_n\frac{g(.,u_n,b_nw_{n,2})}{b_n} + (1-t_n)(\beta_2-\alpha_2)(w_{2,n})^- + \frac{h_2}{b_n}\,.
\end{aligned}$$

From system (1.5) we deduce

$$\begin{aligned}
-\Delta y_{n,1} &= \alpha_1((y_{n,1} + \frac{a_{n,1}}{c_n}\mu_1)^+ - \frac{a_{n,1}}{c_n}\mu_1^+) - \beta_1((y_{n,1} + \frac{a_{n,1}}{c_n}\mu_1)^- - \frac{a_{n,1}}{c_n}\mu_1^-) \\
&\quad + t_n\frac{f(.,b_nw_{n1},v_n)}{c_nb_n} + \frac{(1-t_n)(\beta_1-\alpha_1)}{c_n}(w_{1,n})^- + \frac{h_1}{c_nb_n}, \\
-\Delta y_{n,2} &= \alpha_2((y_{n,2} + \frac{a_{n,2}}{c_n}\mu_2)^+ - \frac{a_{n,2}}{c_n}\mu_2^+) - \beta_2((y_{n,2} + \frac{a_{n,2}}{c_n}\mu_2)^- - \frac{a_{n,2}}{c_n}\mu_2^-) \\
&\quad + t_n\frac{g(.,u_n,b_nw_{n2})}{c_nb_n} + \frac{(1-t_n)(\beta_2-\alpha_2)}{c_n}(w_{2,n})^- + \frac{h_2}{c_nb_n},
\end{aligned} \tag{3.10}$$

when $n \to \infty, c_nb_n \to \infty$ and the last three terms of the two equations above converge to zero in $L^2(\Omega)$. The following inequalities hold

$$\begin{aligned}
&|(y_{n,j} + \frac{a_{n,j}}{c_n}\mu_j)^+ - \frac{a_{n,j}}{c_n}\mu_j^+| \le |y_{n,j}| \le k_j \quad \text{a.e.} \\
&|(y_{n,j} + \frac{a_{n,j}}{c_n}\mu_j)^- - \frac{a_{n,j}}{c_n}\mu_j^-| \le |y_{n,j}| \le k_j \quad \text{a.e.}
\end{aligned} \tag{3.11}$$

Extracting a subsequence, we may assume that the last three terms of each equation of (3.10) approach zero a.e in Ω, and there exists $(k_1',k_2') \in L^2(\Omega)\times L^2(\Omega)$ such that

$$\begin{aligned}
&|t_n\frac{f(x,b_nw_{n,1},v_n)}{c_nb_n} + \frac{(1-t_n)(\beta_1-\alpha_1)}{c_n}(w_{1,n})^- + \frac{h_1(x)}{c_nb_n}| \le k_1' \quad \text{a.e. in } \Omega, \\
&|t_n\frac{g(x,u_n,b_nw_{n,2})}{c_nb_n} + \frac{(1-t_n)(\beta_2-\alpha_2)}{c_n}(w_{2,n})^- + \frac{h_2(x)}{c_nb_n}| \le k_2' \quad \text{a.e. in } \Omega.
\end{aligned}$$

From (3.10), (3.11), and the above inequality, we have

$$\begin{aligned}
&|-\Delta y_{n,1}(x)| \le 2\max(|\alpha_1|,|\beta_1|)k_1(x) + k_1'(x), \\
&|-\Delta y_{n,2}(x)| \le 2\max(|\alpha_2|,|\beta_2|)k_2(x) + k_2'(x).
\end{aligned} \tag{3.12}$$

Let $\rho(x)$ be defined a.e in Ω as follows

$$\rho(x)=\begin{cases}\alpha_j & \text{if } \mu_j(x)>0 \text{ or if } \mu_j(x)=0 \text{ and } y_j\ge 0,\\ \beta_j & \text{if } \mu_j(x)<0 \text{ or if } \mu_j(x)=0 \text{ and } y_j<0,\end{cases}$$

from (3.10) and the fact that $c_n\to 0$ one can see that

$$\begin{gathered}-\Delta y_{n,1}(x)\to\rho(x)y_1(x)\quad\text{a.e. in }\Omega,\\ -\Delta y_{n,2}(x)\to\rho(x)y_2(x)\quad\text{a.e. in }\Omega\,.\end{gathered}$$

From (3.12) and Lebesgue's convergence theorem we conclude that

$$-\Delta y_{n,1}\overset{L^2(\Omega)}{\to}\rho y_1,\quad -\Delta y_{n,2}\overset{L^2(\Omega)}{\to}\rho y_2\,.$$

Then

$$-\Delta y_n\overset{(L^2(A))^2}{\to}\rho y,\quad y_n\overset{(L^2(A))^2}{\to}y.$$

The operator A being closed, we have

$$-\Delta y=\rho y,\quad y\in\varphi^{\perp},\quad \|y\|_{(D(A))^2}=1.$$

Since ρ satisfies: $\overline{\lambda}<\alpha_j\le\rho\le\beta_j<\underline{\lambda}$ by [5, Proposition 2.2], we conclude that $y=0$. This contradicts $\|y\|_{(D(\Omega))^2}=1$ and hence (3.5) is established. □

Using a similar argument to that given above, we obtain the following results:

- When $\alpha_j=\beta_j=\lambda$ for $j=1,2$, We assume that (1.3) and (1.4) are fulfilled. Let $(\theta_1,\theta_2)\in N_\lambda\times N_\mu$. Then the problem (1.1) has at least one weak solution if and only if

$$\int_\Omega\gamma_i^+\theta_i^+(x)dx-\int_\Omega\gamma_i^-\theta_i^-(x)(x)dx+\int_\Omega h_i(x)\theta_i(x)dx\ge 0,\quad i\in 1,2$$

.

- A similar argument can be made when $\alpha_j>\beta_j$ and $H_j(h_j),H_{j+2}(h_j)>0$, $j=1,2$.

Now, we give the proof of our main result.

Proof of Theorem (1.1). Let

$$B(0,R)=\{(u,v)\in U,\|(u,v)\|_U<R\}.$$

By invariance of the topological degree, for $t\in[0,1]$, $\deg(H(t,\cdot,\cdot),B(0,R),0)$ is constant. In particular if $t=0$, we have

$$H(0,u,v)=\begin{pmatrix}-\Delta^{-1} & \\ & -\Delta^{-1}\end{pmatrix}\begin{pmatrix}\alpha_1u^+-\alpha_1u^-\\ \alpha_2v^+-\alpha_2v^-\end{pmatrix},$$

on the other hand, for $t=0$ the linear problem

$$\begin{gathered}-\Delta u=\alpha_1u+h_1\text{ in }\Omega,\\ -\Delta v=\alpha_2v+h_2\text{ in }\Omega,\\ u=v=0\text{ on }\partial\Omega.\end{gathered}$$

possesses a unique solution $(u,v)\in U$.
By the homotopy invariance property, we have

$$\begin{aligned}&\deg(I-H(0,\cdot,\cdot),B(0,R),(-\Delta)^{-1}h)\\ &=\deg(I-H(1,\cdot,\cdot),B(0,R),(-\Delta)^{-1}h)=\pm1,\end{aligned}$$

this completes the proof. □

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