Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 58, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY OF SOLUTIONS TO IMPULSIVE CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Stability of the solutions to a nonlinear impulsive Caputo fractional differential equation is studied using Lyapunov like functions. The derivative of piecewise continuous Lyapunov functions among the nonlinear impulsive Caputo differential equation of fractional order is defined. This definition is a natural generalization of the Caputo fractional Dini derivative of a function. Several sufficient conditions for stability, uniform stability and asymptotic uniform stability of the solution are established. Some examples are given to illustrate the results.

1. Introduction

The study of stability for fractional order systems is quite recent. There are several approaches in the literature to study stability, one of which is the Lyapunov approach. One of the main difficulties on the application of a Lyapunov function to fractional order differential equations is the appropriate definition of its derivative among the fractional differential equations. We give a brief brief overview of the literature and we use the so called Caputo fractional Dini derivative.

The presence of impulses in fractional differential equations lead to complications with the concept of the solution. Mainly there are two different approaches: either keeping the lower limit at the initial time t_0 or change the nature of fractional differential equation by moving the lower limits of the fractional derivative to the points of impulses. In this paper the second approach is used. The Caputo fractional Dini derivative is generalized to piecewise continuous Lyapunov functions among the studied nonlinear fractional equations with impulses. Comparison results using this definition and scalar impulsive fractional differential equations are presented. Several sufficient conditions for stability, uniform stability and asymptotic uniform stability are obtained. Some examples illustrate the obtained results.

2. Notes on fractional calculus

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [11, 19, 24, 26] and there are several definitions of fractional

2010 *Mathematics Subject Classification.* 34A34, 34A08, 34D20.
Key words and phrases. Stability; Caputo derivative; Lyapunov functions; impulses; fractional differential equations.

Submitted December 16, 2015. Published February 25, 2016.

derivatives and fractional integrals. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0,1)$.
(1) The Riemann–Liouville (RL) fractional derivative of order $q \in (0,1)$ of $m(t)$ is given by (see for example [11, Section 1.4.1.1])

$$ {}_{t_0}^{RL}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \quad t \geq t_0, $$

where $\Gamma(\cdot)$ denotes the usual Gamma function.
(2) The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see for example [11, Section 1.4.1.3])

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0. \tag{2.1} $$

The properties of the Caputo derivative are quite similar to those of ordinary derivatives. Also, the initial conditions of fractional differential equations with the Caputo derivative has a clear physical meaning and as a result the Caputo derivative is usually used in real applications.
(3) The Grunwald-Letnikov fractional derivative is given by (see for example [11, Section 1.4.1.2])

$$ {}_{t_0}^{GL}D^q m(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0, $$

and the Grunwald-Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \limsup_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0, \tag{2.2} $$

where $qCr = \frac{q(q-1)(q-1)\dots(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

Proposition 2.1 ([13, Theorem 2.25])**.** *Let $m \in C^1[t_0, b]$. Then*

$$ {}_{t_0}^{GL}D^q m(t) = {}_{t_0}^{RL}D^q m(t) \quad \text{for } t \in (t_0, b]. $$

Also, by [13, Lemma 3.4] we have ${}_{t_0}^{c}D_t^q m(t) = {}_{t_0}^{RL}D_t^q m(t) - m(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}$.

From the relation between the Caputo fractional derivative and the Grunwald-Letnikov fractional derivative using (2.2) we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{GL}D_+^q [m(t) - m(t_0)], \tag{2.3} $$

i.e.

$$ \begin{aligned} &{}_{t_0}^{c}D_+^q m(t) \\ &= \limsup_{h \to 0+} \frac{1}{h^q} \Big[m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} (qCr) \big(m(t-rh) - m(t_0) \big) \Big]. \end{aligned} \tag{2.4} $$

Definition 2.2 ([12])**.** We say $m \in C^q([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{c}D^q m(t)$ exists and satisfies (2.1) for $t \in [t_0, T]$.

Remark 2.3. Definition 2.2 could be extended to any interval $I \subset \mathbb{R}_+$.

If $m \in C^q([t_0, T], \mathbb{R}^n)$ then ${}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{c}D^q m(t)$.

3. Impulses in fractional differential equations

Consider the initial value problem (IVP) for the system of *fractional differential equations* (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}_{\tau_0}^{c}D^q x = f(t,x) \quad \text{for } t \geq \tau_0 \text{ with } x(\tau_0) = x_0, \tag{3.1}$$

where $x \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, and $(\tau_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ is an arbitrary initial data.

We suppose that the function $f(t,x)$ is smooth enough on $\mathbb{R}_+ \times \mathbb{R}^n$, such that for any initial data $(\tau_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the IVP for FrDE (3.1) has a solution $x(t) = x(t; \tau_0, x_0) \in C^q([\tau_0, \infty), \mathbb{R}^n)$. Some sufficient conditions for the existence of global solutions to (3.1) are given in [8, 19].

The IVP for FrDE (3.1) is equivalent to the following integral equation

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{\tau_0}^{t} (t-s)^{q-1} f(s, x(s)) ds \quad \text{for } t \geq \tau_0. $$

In this article we assume the points t_i, $i = 1, 2, \dots$ are fixed such that $t_1 < t_2 < \dots$ and $\lim_{k\to\infty} t_k = \infty$. Let $\tau \in \mathbb{R}_+$ and define the set $\Omega_\tau = \{k : t_k > \tau\}$.

Consider the initial value problem for the system of *impulsive fractional differential equations* (IFrDE) with a Caputo derivative for $0 < q < 1$,

$$ \begin{gathered} {}_{t_0}^{c}D^q x = f(t,x) \quad \text{for } t \geq t_0,\ t \neq t_i, \\ x(t_i + 0) = \Phi_i(x(t_i)) \quad \text{for } i \in \Omega_{t_0}, \\ x(t_0) = x_0, \end{gathered} \tag{3.2}$$

where $x, x_0 \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$, $\Phi_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, 3, \dots$. Without loss of generality we will assume $0 \leq t_0 < t_1$.

Remark 3.1. In the literature the second equation in (3.2), the so called impulsive condition is also given in the equivalent form $\Delta x(t_i) = I_i(x(t_i))$, $i \in \Omega_{t_0}$ where $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$ and the function $I_i(x) = \Phi_i(x) - x$ gives the amount of the jump of the solution at the point t_i.

Let $J \subset \mathbb{R}_+$ be a given interval and $\Delta \subset \mathbb{R}^n$. Let $J_{\text{imp}} = \{t \in J : t \neq t_k,\ k = 1, 2, \dots\}$ and introduce the following classes of functions

$$ \begin{aligned} C^q(J_{\text{imp}}, \Delta) &= \cup_{k=0}^{\infty} C^q((t_k, t_{k+1}), \Delta), \quad C(J_{\text{imp}}, \Delta) = \cup_{k=0}^{\infty} C((t_k, t_{k+1}), \Delta), \\ PC^q(J, \Delta) &= \Big\{ u \in C^q(J_{\text{imp}}, \Delta) : u(t_k) = \lim_{t \uparrow t_k} u(t) < \infty, u(t_k + 0) = \lim_{t \downarrow t_k} u(t) < \infty \\ &\qquad u'(t_k) = \lim_{t \uparrow t_k} u'(t) < \infty, u'(t_k + 0) = \lim_{t \downarrow t_k} u'(t) < \infty \\ &\qquad \text{for all } k : t_k \in J \Big\}, \\ PC(J, \Delta) &= \Big\{ u \in C(J_{\text{imp}}, \Delta) : u(t_k) = \lim_{t \uparrow t_k} u(t) < \infty, u(t_k + 0) = \lim_{t \downarrow t_k} u(t) < \infty \\ &\qquad \text{for all } k : t_k \in J \Big\}. \end{aligned} $$

Impulsive fractional differential equations is an important area of study. There are many qualitative results obtained for equations of type (3.2). We look at the concept of a solutions to fractional differential equations with impulses. There are mainly two viewpoints:

(V1) using the classical Caputo derivative and working in each subinterval, determined by the impulses (see for example [1, 2, 7, 9, 10]). This approach is based on the idea that on each interval between two consecutive impulses (t_k, t_{k+1}) the solution is determined by the differential equation of fractional order. Since the Caputo fractional derivative depends significantly on the initial point (which is different for the ordinary derivative) it leads to a change of the equation on each interval (t_k, t_{k+1}). This approach neglects the lower limit of the Caputo fractional derivative at t_0 and moves it to each impulsive time t_k. Then the IVP for IFrDE (3.2) is equivalent to the integral equation

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds \quad \text{for } t \in [t_0, t_1] \\ x_0 + \frac{1}{\Gamma(q)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s, x(s)) ds \\ + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s)) ds + \sum_{i=1}^{k} I_i(x(t_i - 0)) \\ \quad \text{for } t \in (t_k, t_{k+1}],\ k = 1, 2, 3, \dots \end{cases} \tag{3.3}$$

where $I_k(x) = \Phi_k(x) - x$, $k = 1, 2, \dots$.

Using approach (V1) the solution $x(t; t_0, x_0)$ of (3.2) is

$$x(t; t_0, x_0) = \begin{cases} X_0(t; t_0, x_0) & \text{for } t \in [t_0, t_1] \\ X_1(t; t_1, \Phi_1(X_0(t_1; t_0, x_0))) & \text{for } t \in (t_1, t_2] \\ X_2(t; t_2, \Phi_2(X_1(t_2; t_1, \Phi_2(X_0(t_1; t_0, x_0))) & \text{for } t \in (t_2, t_3] \\ \dots \end{cases} \tag{3.4}$$

- $X_0(t; t_0, x_0)$ is the solution of IVP for FrDE (3.1) with $\tau_0 = t_0$,
- $X_1(t; t_1, \Phi_1(X_0(t_1; t_0, x_0)))$ is the solution of IVP for FrDE (3.1) with $\tau_0 = t_1$, $x_0 = \Phi_1(X_0(t_1; t_0, x_0))$,
- $X_2(t; t_2, \Phi_1(X_1(t_2; t_1, \Phi_1(X_0(t_1; t_0, x_0)))$ is the solution of IVP for the FrDE (3.1) with $\tau_0 = t_2$, $x_0 = \Phi_2(X_1(t_2; t_1, \Phi_1(X_0(t_1; t_0, x_0))$,

and so on.

Viewpoint (V1) and the corresponding equivalent integral equations are based on the presence of impulses in the differential equation (see for example book [18] and the cited references therein).

(V2) Keeping the lower limit t_0 of the Caputo derivative for all $t \geq t_0$ but considering different initial conditions on each interval (t_k, t_{k+1}) (see for example [15, 16, 29, 30, 31]). This approach is based on the fact that the restriction of the fractional derivative ${}_{t_0}^{c}D^q x(t)$ on any interval (t_k, t_{k+1}), $k = 1, 2, \dots$ does not change. Then the fractional equation is kept on each interval between two consecutive impulses with only the initial condition changed. Then the IVP for the IFrDE (3.2) is equivalent to the following integral equation (see [15, formula(10)])

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds \quad \text{for } t \in [t_0, t_1] \\ x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds + \sum_{i=1}^{k} I_i(x(t_i - 0)) \\ \quad \text{for } t \in (t_k, t_{k+1}],\ k = 1, 2, 3, \dots \end{cases} \tag{3.5}$$

As a result using approach (V2) the solution $x(t;t_0,x_0)$ of (3.2) is

$$x(t;t_0,x_0)=\begin{cases}X_0(t;t_0,x_0) & \text{for } t\in[t_0,t_1]\\ X_0(t;t_0,x_0)+\sum_{j=1}^{k}\Phi_j(x(t_j;t_0,x_0)) & \\ \quad\text{for } t\in(t_k,t_{k+1}],\ k=1,2,\dots & \end{cases}\tag{3.6}$$

where $X_0(t;t_0,x_0)$ is the solution of IVP for FrDE (3.1) with $\tau_0=t_0$.

Remark 3.2. From the above any solution of (3.2) is from the class $PC^q([t_0,b))$, $b\le\infty$.

In the case $f(t,x)\equiv 0$ both formulas (3.3) and (3.5) coincide and both approaches (V1) and (V2) are equivalent.

Example 3.3. Consider the initial value problem for the scalar IFrDE with a Caputo derivative for $0<q<1$,

$$\begin{gathered}{}_{t_0}^{c}D^q x=Ax,\quad\text{for } t\ge t_0,\ t\ne t_i,\\ x(t_i+0)=\Phi_i(x(t_i-0))\quad\text{for } i=1,2,\dots,\\ x(t_0)=x_0,\end{gathered}\tag{3.7}$$

where $x\in\mathbb{R}$, A is a given real constant.

Case 1. Let $\Phi_i(x)=a_i+x$ where $a_i\ne 0$, $i=1,2,\dots$. Applying (V1) and (3.3) we obtain the solution of (3.7), namely

$$\begin{aligned}x(t;t_0,x_0)=\Big(x_0\prod_{i=1}^{k}E_q(A(t_i-t_{i-1})^q)+\sum_{i=1}^{k}a_i\prod_{j=i+1}^{k}E_q(A(t_j-t_{j-1})^q)\Big)\\ \times E_q(A(t-t_k)^q)\quad\text{for } t\in(t_k,t_{k+1}],\ k=0,1,2,3,\dots,\end{aligned}\tag{3.8}$$

where the Mittag-Leffler function (with one parameter) is defined by $E_q(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(qk+1)}$.

Applying (V2) and (3.6), we obtain the solution of (3.7), namely

$$x(t;t_0,x_0)=x_0E_q(A(t-t_0)^q)+\sum_{i=1}^{k}a_k\tag{3.9}$$

for $t\in(t_k,t_{k+1}]$, $k=0,1,2,3,\dots$. In this case it looks like (3.8) is closer to the ordinary case ($q=1$).

Case 2. Let $\Phi_i(x)=a_ix$ where $a_i\ne 1$, $i=1,2,\dots$ are constants. Applying (V1) and (3.3) we obtain the solution of (3.7), namely

$$x(t;t_0,x_0)=x_0\Big(\prod_{i=1}^{k}a_iE_q(A(t_i-t_{i-1})^q)\Big)E_q(A(t-t_k)^q)\tag{3.10}$$

for $t\in(t_k,t_{k+1}]$, $k=0,1,2,\dots$.

Applying (V2) and (3.6), using $\frac{\lambda}{\Gamma(q)}\int_0^t\frac{E_q(\lambda s^q)}{(t-s)^q}ds=E_q(\lambda t^q)-1$ we obtain the solution of (3.7), namely

$$x(t;t_0,x_0)=x_0\Big(E_q(A(t-t_0)^q)+\sum_{i=1}^{k}E_q(A(t_i-t_0)^q)(a_i-1)\prod_{j=i+1}^{k}a_j\Big),\tag{3.11}$$

for $t\in(t_k,t_{k+1}]$, $k=0,1,2,3,\dots$. In this case it looks like (3.10) is closer to the ordinary case ($q=1$).

The concept of the FrDE with impulses is rather problematic. In [15], the authors pointed out that the formula, based on (V1) of solutions for IFrDE in [1], [7] is incorrect and gave a new formula using approach (V2). In [30, 31] the authors established a general framework to find solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of solutions to impulsive fractional differential equations based on (V1). In [28] the authors discussed (V1) and criticized the viewpoint (V2) in [15, 30, 31]. Next, in [16] the authors considered the counterexample in [15] and provided further explanations about (V2). In this article we use approach (V1).

Note if for some natural k, a component of the function $\Phi_k : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_k = (\Phi_{k,1}, \Phi_{k,2}, \dots, \Phi_{k,n})$ satisfies the equality $\Phi_{k,j}(x) = x_j$ where $x \in \mathbb{R}^n$: $x = (x_1, x_2, \dots, x_n)$, then there will be no impulse at the point t_k for the component $x_j(t)$ of the solution of IFrDE (3.2) and (3.3) is not correct in this case. To avoid this confusing situation in the application of approach (V1), mentioned above we will assume:

- (H1) If $x \neq 0$ then $\Phi_{k,j}(x) \neq x_j$ for all $j = 1, 2 \dots, n$ and $k = 1, 2, 3, \dots$ where $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$ and $\Phi_k : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_k = (\Phi_{k,1}, \Phi_{k,2}, \dots, \Phi_{k,n})$.

Note that (H1) is equivalent to $I_{k,j}(x) \neq 0$ if $x \neq 0$ for all $k = 1, 2, 3, \dots$ and $j = 1, 2 \dots, n$ where $I_k = (I_{k,1}, I_{k,2}, \dots, I_{k,n})$.

4. Definitions about stability and Lyapunov functions

The goal of the article is to study the stability of zero solution of system IFrDEs (3.2). We will assume the following condition is satisfied

- (H2) $f(t, 0) \equiv 0$ for $t \in \mathbb{R}_+$ and $\Phi_i(0) = 0$ for $i = 1, 2, 3 \dots$.

In the definition below we let $x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ be any solution of (3.2).

Definition 4.1. The zero solution of (3.2) is said to be

- stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$;
- uniformly stable if for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ such that for $t_0 \in \mathbb{R}_+, x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$ the inequality $\|x(t; t_0, x_0)\| < \epsilon$ holds for $t \geq t_0$;
- uniformly attractive if for $\beta > 0$: for every $\epsilon > 0$ there exist $T = T(\epsilon) > 0$ such that for any $t_0 \in \mathbb{R}_+, x_0 \in \mathbb{R}^n$ with $\|x_0\| < \beta$ the inequality $\|x(t; t_0, x_0)\| < \epsilon$ holds for $t \geq t_0 + T$;
- uniformly asymptotically stable if the zero solution is uniformly stable and uniformly attractive.

In this article we use the followings two sets:

$$\mathcal{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a \text{ is strictly increasing and } a(0) = 0\},$$
$$S(A) = \{x \in \mathbb{R}^n : \|x\| \leq A\}, \ A > 0.$$

Furthermore we consider the initial value problem for a scalar FrDE

$$\begin{gathered} {}_{\tau}^{c}D^q u = g(t, u) \quad \text{for } t \geq \tau, \\ u(\tau) = u_0, \end{gathered} \tag{4.1}$$

where $u, u_0 \in \mathbb{R}$, $\tau \in \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

Consider also the IVP for scalar impulsive fractional differential equations

$$\begin{gathered}
{}_{t_0}^{c}D^q u = g(t,u) \quad \text{for } t \geq t_0,\ t \neq t_i,\\
u(t_i+0) = \Psi_i(u(t_i-0)) \quad \text{for } i = 1,2,\dots,\\
u(t_0) = u_0,
\end{gathered} \tag{4.2}$$

where $u, u_0 \in \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\Psi_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots$.

For the scalar IFrDE (4.2) we consider approach (V1) and similar to condition (H1) we assume the following conditions

- (H3) If $u \neq 0$ then $\Psi_k(u) \neq u$ for all $k = 1,2,3,\dots$.
- (H4) $g(t,0) \equiv 0$ for $t \in \mathbb{R}_+$ and $\Psi_i(0) = 0$ for $i = 1,2,3,\dots$.

Note the stability of the zero solution of the scalar IFrDE (4.2) is defined in a similar manner to that in Definition 4.1.

Remark 4.2. Note in the case $\Psi_i(u) \equiv u$ for $i = 1,2,\dots$ the impulsive fractional equation (4.2) is reduced to the fractional differential equation (4.1).

Example 4.3. Consider the scalar impulsive Caputo fractional differential equation (3.7) where $A < 0$, $a_i \in [-1,0) \cup (0,1]$, $i = 1,2,3,\dots$ are constants.

According to Example 3.3 the IVP for IFrDE (3.7) has a solution $x(t;t_0,x_0)$ defined by (3.10). Therefore, applying $0 < E_q(A(T-\tau)^q) \leq 1$ for $T \geq \tau$ we obtain $|x(t;t_0,x_0)| \leq |x_0|$ which guarantees that the zero solution is uniformly stable.

Example 4.4. Consider the IVP for the scalar impulsive Caputo fractional differential equation

$$\begin{gathered}
{}_{t_0}^{c}D^q u = 0, \quad \text{for } t \geq t_0,\ t \neq t_i,\\
u(t_i+0) = a_i\ u(t_i-0) \quad \text{for } i \in \Omega_{t_0},\\
u(t_0) = u_0,
\end{gathered} \tag{4.3}$$

where $a_i \neq 0,1,\ i = 1,2,3,\dots$ are constants and there exists a constant $M > 0$ with $\prod_{i=1}^{\infty} |a_i| \leq M$.

The IVP for IFrDE (4.3) has a solution defined by $u(t;t_0,v_0) = u_0 \prod_{i=1}^{k} a_i$ for $t \in (t_k, t_{k+1}]$, $k = 0,1,2,\dots$. Therefore, we obtain $|u(t;t_0,u_0)| \leq |u_0| \prod_{i=1}^{k} |a_i|$ for $t \in (t_k, t_{k+1}]$ which guarantees that the zero solution of (4.3) is uniformly stable.

Note the existence of a constant $M > 0$ with $\prod_{i=1}^{\infty} |a_i| \leq M$ is guaranteed if $a_i \in [-1,0) \cup (0,1),\ i = 1,2,3,\dots$.

In this article we study the connection between the stability properties of the solutions of a nonlinear system IFrDE (3.2) and the stability properties of the zero solution of a corresponding scalar IFrDE (4.2) or corresponding scalar FrDE (4.1).

We now introduce the class Λ of piecewise continuous Lyapunov-like functions which will be used to investigate the stability of the system IFrDE (3.2).

Definition 4.5. Let $J \in \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$, $V(t,0) \equiv 0$ belongs to the class $\Lambda(J,\Delta)$ if

1. (1) The function $V(t,x)$ is continuous on $J/\{t_k \in J\} \times \Delta$ and it is locally Lipschitzian with respect to its second argument;
2. (2) For each $t_k \in J$ and $x \in \Delta$ there exist finite limits

$$V(t_k-0,x) = \lim_{t \uparrow t_k} V(t,x), \quad V(t_k+0,x) = \lim_{t \downarrow t_k} V(t,x)$$

and the equalities $V(t_k-0,x) = V(t_k,x)$ are valid.

Remark 4.6. When the function $V(t,x) \in \Lambda(J,\Delta)$ is additionally continuous on the whole interval J, we will say $V(t,x) \in \Lambda^C(J,\Delta)$.

Lyapunov like functions used to discuss stability for differential equations require an appropriate definition of the derivative of the Lyapunov function along the studied differential equations. For nonlinear Caputo fractional differential equations (3.2) the following types of derivatives of Lyapunov functions along the nonlinear Caputo fractional differential equations are used:

- *Caputo fractional derivative of Lyapunov functions* ${}^c_{\tau_0}D^q{}_tV(t,x(t))$, where $x(t)$ is a solution of the studied fractional differential equation (3.1) [21, 22]. This approach requires the function to be smooth enough (at least continuously differentiable). It works well for quadratic Lyapunov functions but in the general case when the Lyapunov function depends on t it can cause some problems (see Example 4.8).

- *Dini fractional derivative of Lyapunov functions* [19, 20] given by

$$D^q_+V(t,x) = \limsup_{h\to 0+} \frac{1}{h^q}\big(V(t,x) - V(t-h, x-h^q f(t,x))\big) \tag{4.4}$$

where $0 < q < 1$. The Dini fractional derivative seems to be a natural generalization of the ordinary case ($q = 1$). This definition requires only continuity of the Lyapunov function. However it can be quite restrictive (see Example 4.8) and it can present some problems (see Example 4.9).

- *Caputo fractional Dini derivative of Lyapunov functions* [3, 4, 5]:

$$\begin{aligned}
&{}^c_{(3.1)}D^q_+V(t,x;\tau_0,x_0)\\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{V(t,x) - V(\tau_0,x_0)\\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}qCr\Big[V(t-rh, x-h^q f(t,x)) - V(\tau_0,x_0)\Big]\Big\}
\end{aligned} \tag{4.5}$$

for $t \in (\tau_0, T)$, where $V(t,x) \in \Lambda^C([\tau_0,T),\Delta)$, $x, x_0 \in \Delta$, and there exists $h_1 > 0$ such that $t - h \in [\tau_0, T)$, $x - h^q f(t,x) \in \Delta$ for $0 < h \le h_1$. The above formula is based on the formula (2.4) from fractional calculus. This definition requires only continuity of the Lyapunov function.

Note in [12] the authors defined a derivative of a Lyapunov function and called it the Caputo fractional Dini derivative of $V(t,x)$ (see [12, Definition 3.2]):

$${}^cD^q_+V(t,x) = \limsup_{h\to 0^+} \frac{1}{h^q}\big[V(t,x) - \sum_{r=1}^{n} V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)\big], \tag{4.6}$$

(we feel $(-1)^{r+1}qCr$ is missing in the formula).

The formula (4.6) is quite different than the the Caputo fractional Dini derivative of a function (2.4). Also, in [12, Definition 5.1] the authors define the Caputo fractional Dini derivative of Lyapunov function $V(s,y(t,s,x))$ by

$$\begin{aligned}
&{}^cD^q_+V(t,y(t,s,x))\\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\big[V(t,y(t,s,x))\\
&\quad - \sum_{r=1}^{n}(-1)^{r+1}qCrV(s-rh, y(t,s-rh,x-h^qF(t,x)))\big].
\end{aligned} \tag{4.7}$$

Formula (4.7) is also quite different than the Caputo fractional Dini derivative of a function (2.4).

We will use definition (4.5) as the definition of Caputo fractional Dini derivative of a Lyapunov function.

Example 4.7. Consider the quadratic Lyapunov function, i.e. $V(t,x) = x^2$ for $x \in \mathbb{R}$. Recall the scalar ordinary case ($q = 1$), i.e. the ordinary differential equation $x' = f(t,x)$, $x \in \mathbb{R}$, and the Dini derivative of the quadratic Lyapunov function applied to it,

$$D_+V(t,x(t)) = 2xf(t,x(t)). \tag{4.8}$$

Let $x \in C^q([\tau_0,T],\mathbb{R})$ be a solution of FrDE(3.1). Then the Caputo fractional derivative of the quadratic Lyapunov function ${}^{c}_{\tau_0}D^q{}_t(x(t))^2$ exists and the equality

$${}^{c}_{\tau_0}D^q{}_t(x(t))^2 = 2x(t)f(t,x(t)) \tag{4.9}$$

holds (see for example [6]).

Apply (4.4) to obtain Dini fractional derivative of the quadratic Lyapunov function, namely

$$\begin{aligned}
&D^q_+V(t,x(t))\\
&= D^q_+(x(t))^2\\
&= \limsup_{h\to 0+}\frac{1}{h^q}\big((x(t))^2 - (x(t-h) - h^q f(t,x(t-h)))^2\big)\\
&= \limsup_{h\to 0+}\frac{1}{h^q}\big(x(t) - x(t-h) + h^q f(t,x(t-h))\big)\big(x(t) + x(t-h)\\
&\quad - h^q f(t,x(t-h)\big)\\
&= \limsup_{h\to 0+}\big(\frac{x(t)-x(t-h)}{h}h^{1-q} + f(t,x(t-h))\big)\big(x(t) + x(t-h)\\
&\quad - h^q f(t,x(t-h)\big)\\
&= 2x(t)f(t,x(t)).
\end{aligned} \tag{4.10}$$

Finally, apply (4.5) to obtain the Caputo fractional derivative of the quadratic Lyapunov function

$$\begin{aligned}
&{}^{c}_{(3.1)}D^q_+V(t,x(t);\tau_0,x_0)\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{(x(t))^2 - x_0^2\\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]}(-1)^{r+1}qCr\Big[\Big(x(t-rh) - h^q f(t,x(t-rh))\Big)^2 - (x_0)^2\Big]\Big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCr\Big[(x(t-rh))^2 - (x_0)^2\Big]\\
&\quad + \sum_{r=1}^{[\frac{t-\tau_0}{h}]}(-1)^r qCr\Big[\Big(x(t-rh) - h^q f(t,x(t-rh))\Big)^2 - (x(t-rh))^2\Big]\Big\}.
\end{aligned} \tag{4.11}$$

Using (2.2), (2.3) and $\limsup_{h\to 0^+}\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCr=0$ we obtain

$$\begin{aligned}
&{}_{(3.1)}^{\ c}D_+^q V(t,x(t);\tau_0,x_0)\\
&={}_{\tau_0}^{GL}D_+^q\Big[(x(t))^2-(x_0)^2\Big]\\
&\quad-\limsup_{h\to 0^+}\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCrf(t,x(t-rh))\Big[2x(t-rh)-h^q f(t,x(t-rh))\Big]\\
&={}_{\tau_0}^{c}D^q(x(t))^2-2\limsup_{h\to 0^+}\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCrf(t,x(t-rh))x(t-rh)\\
&\quad+\limsup_{h\to 0^+}h^q\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCrf(t,x(t-rh))f(t,x(t-rh))\\
&={}_{\tau_0}^{c}D^q(x(t))^2=2x(t)f(t,x(t)).
\end{aligned}\tag{4.12}$$

From (4.9), (4.10) and (4.12) we see that (in the scalar case) the above derivatives coincide with the ordinary case (4.8).

Example 4.8. Let $V:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ be given by $V(t,x)=m^2(t)x^2$ for $x\in\mathbb{R}$ where $m\in C^1(\mathbb{R}_+,\mathbb{R})$. Recall the Dini derivative of the Lyapunov function in the ordinary case ($q=1$) is

$$D_+V(t,x)=2x\ m^2(t)f(t,x)+\frac{d}{dt}\Big[m^2(t)x^2\Big].\tag{4.13}$$

If $x\in C^q([\tau_0,T],\mathbb{R})$ is a solution of FrDE(3.1), then to obtain the Caputo fractional derivative ${}_{\tau_0}^{c}D^q{}_t\Big(m^2(t)(x(t))^2\Big)$ we need a multiplication rule from fractional calculus, so it could lead to some difficulties in calculations of the derivative.

Now, let $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ and apply formula (4.4) to obtain Dini fractional derivative of V, namely

$$\begin{aligned}
&D_+^q V(t,x)\\
&=\limsup_{h\to 0^+}\frac{1}{h^q}\Big[m^2(t)x^2-m^2(t-h)\Big(x-h^q f(t,x)\Big)^2\Big]\\
&=\limsup_{h\to 0^+}\Big[\frac{m(t)-m(t-h)}{h}xh^{1-q}+m(t-h)f(t,x)\Big)\\
&\quad\times\Big((m(t)+m(t-h))x-m(t-h)h^q f(t,x)\Big)\Big]\\
&=2x\ m^2(t)f(t,x).
\end{aligned}\tag{4.14}$$

Now we look at (4.13) and (4.14). Both differ significantly. In the fractional Dini derivative (4.14) one term is missing. Additionally, the Dini fractional derivative (4.14) is independent of the order of the differential equation q. However the behavior of solutions of fractional differential equations depends significantly on the order q.

Let $t, \tau_0 \in \mathbb{R}_+,\ x, x_0 \in \mathbb{R}$. Now use (4.5) to obtain the Caputo fractional Dini derivative of V, namely

$$\begin{aligned}
&{}_{(3.1)}^{\;\;c}D_+^q V(t,x;\tau_0,x_0)\\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big[m^2(t)x^2 - m^2(\tau_0)x_0^2 \sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCr\\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]}(-1)^{r+1}qCr\ m^2(t-rh)\Big(x-h^q f(t,x)\Big)^2\Big]\\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big[m^2(t)h^q f(t,x)(2x-h^q f(t,x)) - m^2(\tau_0)x_0^2 \sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCr\\
&\quad + \Big(x-h^q f(t,x)\Big)^2 \sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^r qCr m^2(t-rh)\Big].
\end{aligned} \tag{4.15}$$

Now using (2.2) from (4.15) we obtain

$${}_{(3.1)}^{\;\;c}D_+^q V(t,x;\tau_0,x_0) = 2x\ m^2(t)f(t,x) + {}_{\tau_0}^{RL}D^q\big(m^2(t)x^2 - x_0^2 m^2(\tau_0)\big). \tag{4.16}$$

Note the Caputo fractional Dini derivative depends not only on the fractional order q but also on the initial data (τ_0, x_0) of (3.1) which is similar to the Caputo fractional derivative of a function.

Formula (4.16) is similar to the ordinary case $q = 1$ and formula (4.13) consists of two terms where the ordinary derivative is replaced by the fractional one.

It seems that formula (4.5) is a natural generalization of the one for ordinary differential equations. Also, if the function $V(t,x) \equiv c$, c is a constant, then for any $t, \tau_0 \in \mathbb{R}_+,\ x, x_0 \in \mathbb{R}$ the equality ${}_{(3.1)}^{\;\;c}D_+^q V(t,x;\tau_0,x_0) = 0$ holds.

In this article we use piecewise continuous Lyapunov functions from the class $\Lambda(J,\Delta)$. We define the derivative of piecewise continuous Lyapunov functions using the idea of the Caputo fractional Dini derivative of a function $m(t)$ given by (2.4) and based on (4.5). We define the *generalized Caputo fractional Dini derivative* of the function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along trajectories of solutions of IVP for the system IFrDE (3.2) as follows:

$$\begin{aligned}
&{}_{(3.2)}^{\;\;c}\mathcal{D}_+^q V(t,x;t_0,x_0)\\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{V(t,x) - V(t_0,x_0)\\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}qCr\Big[V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)\Big]\Big\}
\end{aligned} \tag{4.17}$$

for $t \in (t_0,T) : t \neq t_k$, where $x, x_0 \in \Delta$, and there exists $h_1 > 0$ such that $t - h \in [t_0,T)$, $x - h^q f(t,x) \in \Delta$ for $0 < h \leq h_1$.

Example 4.9. Consider the scalar IFrDE (4.3) with $t_0 = 0$, $t_k = k$, $a_k = \frac{1}{\sqrt{2}},\ k = 1,2,\dots$, and $u_0 = 2\sqrt{a}$, $a > 0$ is a constant. According to Example 4.4 the solution of (4.3) is $x(t;t_0,u_0) = 2\sqrt{\frac{a}{2^k}}$ on $(k,k+1]$, $k = 0,1,2,\dots$.

Consider the IFrDE (4.3) with $t_0 = 0$, $t_k = k$, $a_k = \frac{1}{2}$, $k = 1, 2, \dots$, and $u_0 = a$. Then IFrDE (4.3) has an unique solution $u^+(t; t_0, u_0) = \frac{a}{2^k}$ for $t \in (k, k+1]$, $k = 0, 1, 2, \dots$.

Let the Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t, x) = x^2 \sin^2 t$. It is locally Lipshitz with respect to its second argument x. According to Example 4.8 and formula (4.14) we obtain the Dini fractional derivative of V, namely ${}^cD^q_+V(t, x) = 2x \sin^2(t) f(t, x) \equiv 0$.

All the conditions in [27, Theorem 3.1] are satisfied and therefore, the inequality $V(t, x(t; t_0, x_0)) \le u^+(t; t_0, u_0))$ has to be hold for all $t \ge t_0$. However, the inequality

$$V(t, 2\sqrt{\frac{a}{2^k}}) = 4\frac{a}{2^k} \sin^2 t \le \frac{a}{2^k},$$

i.e. $\sin^2 t \le \frac{1}{4}$ is not satisfied for all $t \ge 0$.

5. Comparison results for scalar impulsive Caputo fractional differential equations

We use the following results for Caputo fractional Dini derivative of a continuous Lyapunov function.

Lemma 5.1 (Comparison result [3]). *Assume the following conditions are satisfied:*

(1) *The function $x^*(t) = x(t; \tau_0, x_0) \in C^q([\tau_0, \tilde{T}], \Delta)$ is a solution of the FrDE (3.1) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, τ_0, $\tilde{T} \in \mathbb{R}_+$, $\tau_0 < \tilde{T}$ are given constants, $x_0 \in \Delta$.*
(2) *The function $g \in C([\tau_0, \tilde{T}] \times \mathbb{R}, \mathbb{R})$.*
(3) *The function $V \in \Lambda^C([\tau_0, \tilde{T}], \Delta)$ and*

$${}^c_{(3.1)}D^q_+V(t, x; \tau_0, x_0) \le g(t, V(t, x)) \quad \text{for } (t, x) \in [\tau_0, \tilde{T}] \times \Delta\,.$$

(4) *The function $u^*(t) = u(t; \tau_0, u_0)$, $u^* \in C^q([\tau_0, \tilde{T}], \mathbb{R})$, is the maximal solution of the initial value problem (4.1) with $\tau = \tau_0$.*

Then the inequality $V(\tau_0, x_0) \le u_0$ implies $V(t, x^*(t)) \le u^*(t)$ for $t \in [\tau_0, \tilde{T}]$.

When $g(t, x) \equiv 0$ in Lemma 5.1 we obtain the following result.

Corollary 5.2 ([3]). *Let (1) in Lemma 5.1 be satisfied and $V \in \Lambda^C([\tau_0, \tilde{T}], \Delta)$ be such that for any points $t \in [\tau_0, \tilde{T}]$, $x \in \Delta$ the inequality ${}^c_{(3.1)}D^q_+V(t, x; \tau_0, x_0) \le 0$ holds. Then for $t \in [\tau_0, \tilde{T}]$ the inequality $V(t, x^*(t)) \le V(\tau_0, x_0)$ holds.*

If the derivative of the Lyapunov function is negative, the following result is true.

Lemma 5.3 ([3]). *Let Condition (1) of Lemma 5.1 be satisfied and the function $V \in \Lambda^C([t_0, \tilde{T}], \Delta)$ be such that for any points $t \in [\tau_0, \tilde{T}]$, $x \in \Delta$ the*

$${}^c_{(3.1)}D^q_+V(t, x; \tau_0, x_0) \le -c(\|x\|)\,,$$

where $c \in \mathcal{K}$. Then for $t \in [\tau_0, \tilde{T}]$,

$$V(t, x^*(t)) \le V(\tau_0, x_0) - \frac{1}{\Gamma(q)} \int_{\tau_0}^t (t - s)^{q-1} c(\|x^*(s)\|) ds\,. \tag{5.1}$$

Now we prove some comparison results for the system of IFrDE (3.2) and piecewise continuous Lyapunov functions applying the generalized Caputo fractional Dini derivative (4.17). Recall $\lim_{k\to\infty} t_k = \infty$. In this section we assume without loss of generality that $0 \le t_0 < t_1 < T$.

As a comparison scalar equation we use the impulsive Caputo fractional differential equation (4.2) or the Caputo fractional differential equation (4.1).

Lemma 5.4 (Comparison result by scalar IFrDE)**.** *Assume that the following conditions are satisfied:*

1. *Let conditions* (H1) *and* (H3) *be satisfied for all* $k \in \{i : t_i \in (t_0, T)\}$ *where* $t_0,\ T \in \mathbb{R}_+,\ t_0 < T$ *are given constants.*
2. *The function* $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, T], \Delta)$ *is a solution of the IFrDE* (3.2) *where* $\Delta \subset \mathbb{R}^n,\ 0 \in \Delta,\ x_0 \in \Delta$.
3. *The function* $g \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ *and the IVP for the IFrDE* (4.2) *has a unique maximal solution* $u^*(t) = u(t; t_0, u_0) \in PC^q([t_0, T], \mathbb{R})$.
4. *The functions* $\Psi_k :\ \mathbb{R} \to \mathbb{R},\ k \in \{i :\ t_i \in (t_0, T)\}$*, are nondecreasing.*
5. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and*
   - (i) *for any* $\tau_0 \in [t_0, T)$ *and* $x_0 \in \Delta$*, the inequality* ${}^{c}_{(3.2)}D^q_+ V(t, x; \tau_0, x_0) \le g(t, V(t, x))$ *for* $(t, x) \in [\tau_0, T] \times \Delta,\ t \ne t_k$ *holds;*
   - (ii) *for any points* $t_k \in (t_0, T)$ *and* $x \in \Delta$ *we have*
   $$V(t_k + 0, \Phi_k(x)) \le \Psi_k(V(t_k, x)).$$

Then the inequality $V(t_0, x_0) \le u_0$ *implies* $V(t, x^*(t)) \le u^*(t)$ *for* $t \in [t_0, T]$.

Proof. We use induction. Let $t \in [t_0, t_1]$. By Lemma 5.1 the claim in Lemma 5.4 holds on $[t_0, t_1]$.

Let $t \in (t_1, t_2] \cap [t_0, T]$. Then the function $\overline{u}_1(t) \equiv u^*(t)$ is the maximal solution of IVP for FrDE (4.1) for $\tau = t_1$ and $\overline{u}_1(t_1) = \Psi_1(u^*(t_1 - 0))(= \Psi_1(u^*(t_1))) = u^*(t_1 + 0)$ and the function $\overline{x}_1(t) \equiv x^*(t)$ is a solution of IVP for FrDE (3.1) for $\tau_0 = t_1$ and $x_0 = \Phi_1(x^*(t_1 - 0)) = x^*(t_1 + 0)$. Using conditions (4), (5)(ii) and the above proved inequality $V(t_1, x^*(t_1)) = V(t_1, x^*(t_1 - 0)) \le u^*(t_1 - 0)$ we obtain

$$\begin{aligned}
&V(t_1 + 0, \overline{x}_1(t_1)) = V(t_1 + 0, x^*(t_1 + 0)) \\
&= V(t_1 + 0, \Phi_1(x^*(t_1 - 0))) = V(t_1 + 0, \Phi_1(x^*(t_1))) \\
&\le \Psi_1(V(t_1, x^*(t_1))) \le \Psi_1(u^*(t_1 - 0)) \\
&= u^*(t_1 + 0) = \overline{u}_1(t_1).
\end{aligned} \tag{5.2}$$

By Lemma 5.1 for $\tau_0 = t_1$ and $\tilde{T} = \min\{T, t_2\}$ we obtain $V(t, \overline{x}_1(t)) \le \overline{u}_1(t)$ for $t \in [t_1, t_2] \cap [t_0, T]$. Therefore, $V(t, x^*(t)) \le u^*(t)$ for $t \in (t_1, t_2] \cap [t_0, T]$, i.e. the claim of Lemma 5.4 holds on $[t_0, t_2] \cap [t_0, T]$.

Continuing this process and an induction argument proves that the claim is true on $[t_0, T]$. □

Example 5.5. Consider the scalar IFrDE (4.3) with $t_0 = 0$, $t_k = k$, $a_k = \frac{1}{\sqrt{2}}$, $k = 1, 2, \dots$, and $u_0 = 2\sqrt{a}$, $a > 0$ is a constant. According to Example 4.4 the solution of (4.3) is $x(t; t_0, u_0) = 2\sqrt{\frac{a}{2^k}}$ on $(k, k + 1]$, $k = 0, 1, 2, \dots$.

Consider the IFrDE (4.3) with $t_0 = 0$, $t_k = k$, $a_k = \frac{1}{2}$, $k = 1, 2, \dots$, and $u_0 = a$. Then IFrDE (4.3) has an unique solution $u^+(t; t_0, u_0) = \frac{a}{2^k}$ for $t \in (k, k + 1]$, $k = 0, 1, 2, \dots$.

Let the Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = x^2 \sin^2 t$. By Example 4.8 and formula (4.16) we obtain the Caputo fractional Dini derivative of V, namely ${}^{c}_{(4.3)}D^q_+ V(t,x;0,x_0) = x^2 {}^{RL}_{0}D^q[\sin^2 t]$. Using $\sin^2 t - 0.5 - 0.5\cos(2t)$ and ${}^{RL}_{0}D^q \cos(2t) = 2^q \cos(2t + \frac{q\pi}{2})$ it follows that the inequality ${}^{c}_{(4.3)}D^q_+ V(t,x;0,x_0) \leq 0$ is not satisfied, i.e. condition (5)(i) of Lemma 5.4 is not satisfied for $g(t,x) \equiv 0$ so we cannot claim that the inequality $V(t, x(t;0,x_0)) \leq u^+(t;0,u_0))$ has to be hold for all $t \geq t_0$, i.e. the application of Lemma 5.4 and the Caputo fractional Dini derivative does not lead to a contradiction as in [27] (compare with Example 4.9).

The result in Lemma 5.4 is also true on the half line (recall [3] that Lemma 5.1 extends to the half line).

Corollary 5.6. *Suppose all the conditions of Lemma 5.4 are satisfied with* $[t_0, T]$ *replaced by* $[t_0, \infty)$. *Then the inequality* $V(t_0, x_0) \leq u_0$ *implies* $V(t, x^*(t)) \leq u^*(t)$ *for* $t \geq t_0$.

If $\Psi_k(u) \equiv u$ for all $k = 1, 2, \dots$, we consider the scalar FrDE (4.1) as a comparison equation.

Lemma 5.7 (Comparison result by scalar FrDE)**.** *Assume*

(1) *Condition* (H1) *is fulfilled for all* $k \in \{i : \ t_i \in (t_0, T)\}$ *where* $t_0,\ T \in \mathbb{R}_+,\ t_0 < T$ *are given constants.*
(2) *Condition (2) of Lemma 5.4 is fulfilled.*
(3) *The function* $g \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ *and the IVP for the FrDE* (4.1) *with* $\tau = t_0$ *has a unique maximal solution* $u^*(t) = u(t; t_0, u_0) \in C^q([t_0, T], \mathbb{R})$.
(4) *The function* $V \in \Lambda([t_0, T], \Delta)$, *it satisfies the condition (5)(i) of Lemma 5.4 and*
   (ii) *for any points* $t_k \in (t_0, T)$ *and* $x \in \Delta$ *we have*
$$V(t_k + 0, \Phi_k(x)) \leq V(t_k, x).$$

Then the inequality $V(t_0, x_0) \leq u_0$ *implies* $V(t, x^*(t)) \leq u^*(t)$ *for* $t \in [t_0, T]$.

Proof. The proof is similar to the one of Lemma 5.4 where the inequality (5.2) is replaced by
$$\begin{aligned} &V(t_1 + 0, \overline{x}_1(t_1)) = V(t_1 + 0, x^*(t_1 + 0)) \\ &= V(t_1 + 0, \Phi_1(x^*(t_1 - 0))) = V(t_1 + 0, \Phi_1(x^*(t_1))) \\ &\leq V(t_1 - 0, x^*(t_1 - 0)) \leq u^*(t_1 - 0) \\ &= u^*(t_1 + 0) = \overline{u}_1(t_1). \end{aligned} \tag{5.3}$$
□

The result of Lemma 5.7 is also true on the half line.

Corollary 5.8. *Suppose all the conditions of Lemma 5.7 are satisfied with* $[t_0, T]$ *replaced by* $[t_0, \infty)$. *Then the inequality* $V(t_0, x_0) \leq u_0$ *implies* $V(t, x^*(t)) \leq u^*(t)$ *for* $t \geq t_0$.

Recall $\lim_{k \to \infty} t_k = \infty$. In our next result we assume with loss of generality that $t_p < T \leq t_{p+1}$ for some $p \in \{1, 2, \dots\}$. Next we present a comparison result for negative Caputo fractional Dini derivative.

Lemma 5.9. *Assume the following conditions are satisfied:*

(1) *Conditions (1) and (2) of Lemma 5.7 are fulfilled.*
(2) *The function* $V \in \Lambda([t_0, T], \Delta)$ *and*
  (i) *for any* $\tau_0 \in [t_0, T)$ *and* $x_0 \in \Delta$, *the inequality* ${}_{(3.2)}^{\;\;c}D_+^q V(t, x; \tau_0, x_0) \leq -c(\|x\|)$ *for* $(t, x) \in [\tau_0, T] \times \Delta$, $t \neq t_k$ *holds;*
  (ii) *for any points* $t_k \in (t_0, T)$ *and* $x \in \Delta$ *the inequalities* $V(t_k+0, \Phi_k(x)) \leq V(t_k, x)$ *hold.*

Then for $t \in [t_0, T]$ *the following inequalities hold:*

$$V(t, x^*(t)) \leq V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} c(\|x^*(s)\|) ds \tag{5.4}$$

for $t \in [t_0, t_1]$, *and*

$$\begin{aligned} V(t, x^*(t)) \leq V(t_0, x_0) &- \sum_{i=0}^{k-1} \frac{1}{\Gamma(q)} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{q-1} c(\|x^*(s)\|) ds \\ &- \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} c(\|x^*(s)\|) ds \end{aligned} \tag{5.5}$$

for $t \in (t_k, t_{k+1}^\star]$, $k = 1, 2, \dots, p$; *here* $t_{k+1}^\star = t_{k+1}$ *if* $k = 1, \dots, p-1$ *and* $t_{p+1}^\star = T$.

Proof. We use induction. Let $t \in [t_0, t_1]$. By Lemma 5.3 with $\tau_0 = t_0$ and $\tilde{T} = t_1$ inequality (5.4) holds on $[t_0, t_1]$.

Let $t \in (t_1, t_2] \cap [t_0, T]$. Then the function $\overline{x}_1(t) \equiv x^*(t)$ is a solution of IVP for FrDE (3.1) for $\tau_0 = t_1$ and $\overline{x}_1(t_1) = \Phi_1(x^*(t_1 - 0))(= \Phi_1(x^*(t_1))) = x^*(t_1 + 0)$. Using condition (2)(ii) we obtain

$$\begin{aligned} &V(t_1 + 0, \overline{x}_1(t_1)) = V(t_1 + 0, x^*(t_1 + 0)) \\ &= V(t_1 + 0, \Phi_1(x^*(t_1 - 0))) = V(t_1 + 0, \Phi_1(x^*(t_1))) \\ &\leq V(t_1, x^*(t_1)) = V(t_1, x^*(t_1 - 0)). \end{aligned} \tag{5.6}$$

By Lemma 5.3 with $\tau_0 = t_1$ and $\tilde{T} = \min\{T, t_2\}$, inequality (5.6) and inequality (5.4) with $t = t_1$ we obtain

$$\begin{aligned} V(t, \overline{x}_1(t)) &\leq V(t_1 + 0, \overline{x}_1(t_1)) - \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} c(\|x^*(s)\|) ds \\ &\leq V(t_1, x^*(t_1 - 0)) - \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} c(\|x^*(s)\|) ds \\ &\leq V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1-s)^{q-1} c(\|x^*(s)\|) ds \\ &\quad - \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} c(\|x^*(s)\|) ds. \end{aligned}$$

Therefore, inequality (5.5) holds on $(t_1, t_2] \cap [t_0, T]$. Continuing this process and an induction argument proves the claim is true on $[t_0, T]$. □

The result in Lemma 5.9 is also true on the half line (recall [3] that Lemma 5.3 extends to the half line).

Corollary 5.10. *Suppose all the conditions of Lemma 5.9 are satisfied with* $[t_0, T]$ *replaced by* $[t_0, \infty)$. *Then for any* $t \geq t_0$ *the inequalities* (5.4), (5.5) *(where* $k = 1, 2, \dots, p$ *is replaced by* $k = 1, 2, \dots$*) hold.*

Remark 5.11. In this paper we assumed an infinite number of points t_i, $i = 1, 2, \dots$ with $t_1 < t_2 < \dots$ and $\lim_{k\to\infty} t_k = \infty$. However it is worth noting that the results in Section 5 (and elsewhere) hold if we only consider a finite of points t_i, $i = 1, 2, \dots, p$ for some $p \in \{1, 2, \dots\}$ and $t_1 < t_2 < \dots < t_p$.

6. Main result

In this section we obtain sufficient conditions for stability of the zero solution of nonlinear impulsive Caputo fractional differential equations.

Theorem 6.1. *Let the following conditions be satisfied:*

1. *Conditions* (H1)–(H4) *are satisfied.*
2. *The functions $f \in PC(\mathbb{R}_+, \mathbb{R}^n)$, $\Phi_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots$, are such that for any $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the IVP for the scalar of IFrDE* (3.2) *has a solution $x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$.*
3. *The functions $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $\Psi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \dots$, are such that for any $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the IVP for the scalar IFrDE* (4.2) *has a solution $u(t; t_0, u_0) \in PC^q([t_0, \infty), \mathbb{R})$ and in the case of nonuniqueness the IVP has a unique maximal solution.*
4. *The functions $\Psi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots$, are nondecreasing.*
5. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   (i) *for any points $t_0 \in \mathbb{R}_+$ and $x, x_0 \in \mathbb{R}^n$ we have*
   $$ {}^{c}_{(3.2)}D^q_+ V(t, x; t_0, x_0) \le g(t, V(t, x)) $$
   for $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \dots$;
   (ii) *for any points t_k, $k = 1, 2, \dots$ and $x \in \mathbb{R}^n$ we have*
   $$ V(t_k + 0, \Phi_k(x)) \le \Psi_k(V(t_k, x)); $$
   (iii) *$b(\|x\|) \le V(t, x)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $b \in \mathcal{K}$.*
6. *The zero solution of the scalar IFrDE* (4.2) *is stable.*

Then the zero solution of the system of IFrDE (3.2) *is stable.*

Proof. Let $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ be given. Without loss of generality we assume $t_0 < t_1$. According to condition (6) there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that the inequality $|u_0| < \delta_1$ implies
$$ |u(t; t_0, u_0)| < b(\epsilon), \quad t \ge t_0, \tag{6.1} $$
where $u(t; t_0, u_0)$ is a solution of the scalar IFrDE (4.2). Since $V(t_0, 0) = 0$ there exists $\delta_2 = \delta_2(t_0, \delta_1) > 0$ such that $V(t_0, x) < \delta_1$ for $\|x\| < \delta_2$. Let $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta_2$. Then $V(t_0, x_0) < \delta_1$. Consider any solution $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ of the IFrDE (3.2) which exists according to condition (2). Now let $u_0^* = V(t_0, x_0)$. Then $u_0^* < \delta_1$ and inequality (6.1) holds for the unique maximal solution $\overline{u}(t; t_0, u_0^*)$ of the scalar IFrDE (4.2) (with $\tau = t_0$ and $u_0 = u_0^*$).

According to Corollary 5.6 the inequality $V(t, x^*(t)) \le \overline{u}(t; t_0, u_0^*)$ holds for $t \ge t_0$. Then for any $t \ge t_0$ from condition (5)(iii) and inequality (6.1) we obtain
$$ b(\|x^*(t)\|) \le V(t, x^*(t)) \le \overline{u}(t; t_0, u_0^*) < b(\epsilon), $$
so the result follows. □

If we consider the scalar FrDE (4.1) as a comparison equation then the following result holds.

Theorem 6.2. *Let the following conditions be satisfied:*

1. *Conditions* (H1)–(H2) *are satisfied.*
2. *Conditions (2) and (5) of Theorem 6.1 are satisfied where the condition (5)(ii) is replaced by*
   - (ii) *for any points t_k, $k = 1, 2, \dots$ and $x \in \mathbb{R}^n$ we have*
   $$V(t_k + 0, \Phi_k(x)) \le V(t_k, x).$$
3. *The function $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $g(t, 0) \equiv 0$ is such that for any $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the IVP for the scalar FrDE* (4.1) *has a solution $u(t; t_0, u_0) \in C^q([t_0, \infty), \mathbb{R})$ and in the case of nonuniqueness the IVP has a unique maximal solution.*
4. *The zero solution of the scalar FrDE* (4.1) *is stable.*

Then the zero solution of the system of IFrDE (3.2) *is stable.*

The proof of above theorem is similar to the one of Theorem 6.1, applying Corollary 5.8 instead of Corollary 5.6.

Now we present some sufficient conditions for stability of the zero solution of the IFrDE in the case when the condition for the Caputo fractional Dini derivative of the Lyapunov function is satisfied only on a ball.

Theorem 6.3. *Let the following conditions be satisfied:*

1. *Conditions (1)–(4) of Theorem 6.1 are fulfilled.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, S(\lambda))$ such that*
   - (i) *for any points $t_0 \in \mathbb{R}_+$ and $x, x_0 \in S(\lambda)$ we have*
   $${}_{(3.2)}^{\;\;c}D_+^q V(t, x; t_0, x_0) \le g(t, V(t, x))$$
   for $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \dots$;
   - (ii) *for any points t_k, $k = 1, 2, \dots$ and $x \in S(\lambda)$ we have*
   $$V(t_k + 0, \Phi_k(x)) \le \Psi_k(V(t_k, x));$$
   - (iii) $b(\|x\|) \le V(t, x) \le a(\|x\|)$ *for $t \in \mathbb{R}_+$, $x \in S(\lambda)$, where $a, b \in \mathcal{K}$.*
3. *The zero solution of the scalar IFrDE* (4.2) *is uniformly stable.*

Then the zero solution of the system of IFrDE (3.2) *is uniformly stable.*

Proof. Let $\epsilon \in (0, \lambda]$ and $t_0 \in \mathbb{R}_+$ be given. From condition (3) of Theorem 6.3 there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that for any $\tau_0 \ge 0$ the inequality $|u_0| < \delta_1$ implies
$$|u(t; \tau_0, u_0)| < b(\epsilon), \quad t \ge \tau_0, \tag{6.2}$$
where $u(t; \tau_0, u_0)$ is a solution of (4.2).

Let $\delta_1 < \min\{\epsilon, b(\epsilon)\}$. From $a \in \mathcal{K}$ there exists $\delta_2 = \delta_2(\epsilon) > 0$ so if $s < \delta_2$ then $a(s) < \delta_1$. Let $\delta = \min(\epsilon, \delta_2)$. Choose the initial value $x_0 \in \mathbb{R}^n$ such that $\|x_0\| < \delta$. Therefore $x_0 \in S(\lambda)$. Also, let $u_0^* = V(t_0, x_0)$. From the choice of the point u_0^* and condition (3)(iii) we obtain $u_0^* \le a(\|x_0\|) < a(\delta_2) < \delta_1$. Let $x^*(t) = x(t; t_0, x_0)$, $t \ge t_0$ be a solution of the IVP for IFrDE (3.2) and $u^*(t; t_0, u_0^*)$ be the maximal solution of the IVP for scalar IFrDE (2). Note $u^*(t; t_0, u_0^*)$ satisfies (6.2). We now prove that
$$\|x^*(t)\| < \epsilon, \quad t \ge t_0. \tag{6.3}$$
Assume inequality (6.3) is not true. Denote $t^* = \inf\{t > t_0 : \|x^*(t) \ge \varepsilon\}$. Then
$$\|x^*(t)\| < \varepsilon \quad \text{for } t \in [t_0, t^*) \quad \text{and} \quad \|x^*(t^*)\| = \varepsilon. \tag{6.4}$$

If $t^* \neq t_k$, $k \in \mathbb{Z}_+$ or $t^* = t_p$ for some natural number p and $\|x^*(t_p - 0)\| = \varepsilon$ then (6.3) is true. If for a natural number p we have $t^* = t_p$ and $\|x^*(t_p - 0)\| < \varepsilon$, then according to Lemma 5.4 for $T = t^*$ and $\Delta = S(\lambda)$ we get $V(t, x^*(t)) \leq u^*(t; t_0, u_0^*)$ on $[t_0, t^*]$. Then applying condition (3)(iii) and inequality (6.2) we obtain $b(\varepsilon) = b(\|x^*(t^*)\|) \leq V(t^*, x^*(t^*)) \leq u^*(t^*; t_0, u_0^*)$. Thus $\|x^*(t^*)\| \leq b^{-1}(u^*(t^*)) < \varepsilon$ and this contradicts the choice of t^*. Therefore, (6.3) holds and then the zero solution of IFrDE (3.2) is uniformly stable. $\square$

Corollary 6.4. *Suppose*

(1) *Conditions* (H1)–(H2) *are satisfied.*
(2) *Condition (2) of Theorem 6.1 is satisfied.*
(3) *Condition (3) of Theorem 6.3 is satisfied with* $g(t, x) = Au$ *and* $\Psi_k(u) = a_k u$ *for* $k = 1, 2, \dots$ *where* $A \leq 0$ *and* $a_k \in (0, 1)$.

Then the zero solution of the IFrDE (3.2) *is uniformly stable.*

The above corollary follows from Example 4.3 (if $A < 0$) and Example 4.4 (if $A = 0$) and Theorem 6.3. If we consider the scalar FrDE (4.1) as a comparison equation then the following result for uniform stability is true:

Theorem 6.5. *Let the following conditions be satisfied:*

(1) *Conditions (1) and (3) of Theorem 6.2 are fulfilled.*
(2) *Condition (2) of Theorem 6.1 is fulfilled.*
(3) *There exists a function* $V \in \Lambda(\mathbb{R}_+, S(\lambda))$ *satisfying condition (2)(i) and 2(iii) of Theorem 6.3 and*
    (ii) *for any points* t_k, $k = 1, 2, \dots$ *and* $x \in S(\lambda)$ *we have*
$$V(t_k + 0, \Phi_k(x)) \leq V(t_k, x);$$
(4) *The zero solution of the scalar FrDE* (4.1) *is uniformly stable.*

Then the zero solution of the system of IFrDE (3.2) *is uniformly stable.*

Now we present some sufficient conditions for uniform asymptotic stability of the zero solution of a system of nonlinear IFrDE.

Theorem 6.6. *Let the following conditions be satisfied:*

(1) *Conditions* (H1) *and* (H2) *are fulfilled.*
(2) *Condition (2) of Theorem 6.1 is fulfilled.*
(3) *There exists a function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*
    (i) *for any points* $t_0 \in \mathbb{R}_+$, *and* $x, x_0 \in \mathbb{R}^n$ *we have*
$${}_{(3.2)}^{\ c}D_+^q V(t, x; t_0, x_0) \leq -c(\|x\|)$$
    for $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \dots$, *where* $c \in \mathcal{K}$;
    (ii) *for any points* t_k, $k = 1, 2, \dots$ *and* $x \in \mathbb{R}^n$ *we have*
$$V(t_k + 0, \Phi_k(x)) \leq V(t_k, x);$$
    (iii) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $a, b \in \mathcal{K}$.

Then the zero solution of the system of IFrDE (3.2) *is uniformly asymptotically stable.*

Proof. From condition (3)(i) we have ${}_{(3.2)}^{\ c}D_+^q V(t, x; t_0, x_0) \leq 0$. Applying Theorem 6.5 with $g(t, u_0) \equiv 0$ we see that the zero solution of the system of IFrDE (3.2) is uniformly stable. Therefore, for the number λ there exists $\alpha = \alpha(\lambda) \in (0, \lambda)$ such that for any $\tilde{t}_0 \in \mathbb{R}_+$ and $\tilde{x}_0 \in \mathbb{R}^n$ the inequality $\|\tilde{x}_0\| < \alpha$ implies

$$\|x(t; \tilde{t}_0, \tilde{x}_0)\| < \lambda \quad \text{for } t \geq \tilde{t}_0 \tag{6.5}$$

where $x(t; \tilde{t}_0, \tilde{x}_0)$ is any solution of IFrDE (3.2) (with initial data $(\tilde{t}_0, \tilde{x}_0)$).

Now we prove the zero solution of IFrDE (3.2) is uniformly attractive. Consider the constant $\beta \in (0, \alpha]$ such that $b^{-1}(a(\beta)) < \alpha$. Let $\epsilon \in (0, \lambda]$ be an arbitrary number and $x^*(t) = x(t; t_0, x_0)$ be any solution of (3.2) such that $\|x_0\| < \beta$, $t_0 \in \mathbb{R}_+$. Then $b(\|x_0\|) \leq a(\|x_0\|) < a(\beta)$, i.e. $\|x_0\| \leq b^{-1}(a(\beta)) < \alpha$ and therefore the inequality

$$\|x^*(t)\| < \lambda \quad \text{for } t \geq t_0 \tag{6.6}$$

holds. Choose a constant $\gamma = \gamma(\epsilon) \in (0, \epsilon]$ such that $a(\gamma) < b(\epsilon)$. Let $T > \big(\frac{q\Gamma(q)a(\alpha)}{c(\gamma)}\big)^{1/q}$, $T = T(\epsilon) > 0$ and $m \in \{1, 2, \dots\}$ with $t_m < t_0 + T < t_{m+1}$. We now prove that

$$\|x^*(t)\| < \epsilon \quad \text{for } t \geq t_0 + T. \tag{6.7}$$

Assume

$$\|x^*(t)\| \geq \gamma \quad \text{for every } t \in [t_0, t_0 + T]. \tag{6.8}$$

Then from Lemma 5.7 (applied to the interval $[t_m, t_0 + T]$ and $\Delta = \mathbb{R}^n$) and the inequality $a^q + b^q \geq (a + b)^q$ for $a, b > 0$ we obtain

$$\begin{aligned}
&V(t_0 + T, x^*(t_0 + T)) \\
&\leq V(t_0, x_0) - \sum_{i=0}^{m-1} \frac{1}{\Gamma(q)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{q-1} c(\|x^*(s)\|) ds \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_m}^{t_0+T} (t_0 + T - s)^{q-1} c(\|x^*(s)\|) ds \\
&\leq a(\|x_0\|) - \sum_{i=0}^{m-1} \frac{c(\gamma)}{\Gamma(q)} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{q-1} ds - \frac{c(\gamma)}{\Gamma(q)} \int_{t_m}^{t_0+T} (t_0 + T - s)^{q-1} ds \\
&< a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)} \Big(\sum_{i=0}^{m-1} (t_{i+1} - t_i)^q + (T + t_0 - t_m)^q \Big) \\
&\leq a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)} \Big(\sum_{i=0}^{m-1} (t_{i+1} - t_i) + (T + t_0 - t_m) \Big)^q \\
&= a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)} T^q < 0.
\end{aligned}$$

This contradiction proves the existence of $t^* \in [t_0, t_0 + T]$ such that $\|x^*(t^*)\| < \gamma$. Now there are two cases to be considered, namely $t^* \neq t_k$ for $k = 1, 2, \dots$ or $t^* = t_n$ for some $n \in \{1, 2, \dots\}$.

Case 1. Let $t^* \neq t_k$ for $k = 1, 2, \dots$. Without loss of generality assume there exists $j \in \{1, 2, \dots\}$ with $t_j < t^* < t_{j+1}$. From Corollary 5.8 for any $t \geq t^*$ and $\Delta = \mathbb{R}^n$ we have

$$\begin{aligned}
V(t, x^*(t)) &\leq V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)} \int_{t^*}^{t} (t - s)^{q-1} c(\|x^*(s)\|) ds \\
&\leq V(t^*, x^*(t^*)) \quad \text{for } t \in [t^*, t_{j+1}]
\end{aligned}$$

and

$$\begin{aligned}
V(t,x^*(t)) &\leq V(t^*,x^*(t^*)) - \frac{1}{\Gamma(q)}\Big(\int_{t^*}^{t_{j+1}}(t_{j+1}-s)^{q-1}c(\|x^*(s)\|)ds \\
&\quad + \sum_{i=j+1}^{l-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}c(\|x^*(s)\|)ds \\
&\quad + \int_{t_l}^{t}(t-s)^{q-1}c(\|x^*(s)\|)ds\Big) \\
&\leq V(t^*,x^*(t^*)) \quad \text{for } t\in(t_l,t_{l+1}],\ l=j+1,j+2,\dots .
\end{aligned}$$

Then for any $t \geq t^*$ we obtain

$$b(\|x^*(t)\|) \leq V(t,x^*(t)) \leq V(t^*,x^*(t^*)) \leq a(\|x^*(t^*)\|) \leq a(\gamma).$$

Then $\|x^*(t)\| \leq b^{-1}(a(\gamma)) < \varepsilon$ for any $t \geq t^*$.

Case 2. Let $t^* = t_n$ for some $n \in \{1,2,\dots\}$. Applying Corollary 5.8 for any $t > t^* = t_n$, $t \in (t_l,t_{l+1}]$, $l = n, n+1, \dots$, and $\Delta = \mathbb{R}^n$ and obtain

$$\begin{aligned}
V(t,x^*(t)) &\leq V(t_n+0,x^*(t_n+0)) - \frac{1}{\Gamma(q)}\Big(\sum_{i=n}^{l-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}c(\|x^*(s)\|)ds \\
&\quad + \int_{t_l}^{t}(t-s)^{q-1}c(\|x^*(s)\|)ds\Big) \\
&\leq V(t_n+0,x^*(t_n+0)).
\end{aligned}$$

Then for any $t > t^* = t_n$ from conditions (2)(ii) and (2)(iii) we get

$$\begin{aligned}
b(\|x^*(t)\|) &\leq V(t,x^*(t)) \leq V(t_n,x^*(t_n+0)) \\
&= V(t_n,\Phi_n(x^*(t_n-0))) \leq V(t_n,x^*(t_n-0)) \\
&\leq a(\|x^*(t_n-0)\|) \leq a(\gamma).
\end{aligned}$$

Then $\|x^*(t)\| \leq b^{-1}(a(\gamma)) < \varepsilon$ and therefore (6.7) holds for all $t > t^*$ (hence for $t \geq t_0 + T$). $\square$

Remark 6.7. The study of stability of a nonzero solution $x^*(t)$ of the IVP for IFrDE (3.2) could be easily reduced to studing stability of the zero solution of an appropriately chosen system of IFrDE.

7. Applications

Consider the generalized Caputo population model.

Example 7.1. Let the points t_k, $t_k < t_{k+1}$, $lim_{k\to\infty} t_k = \infty$ be fixed. Consider the scalar impulsive Caputo fractional differential equation

$$\begin{gathered}
{}_0^cD^q x = -g(t)x(1+x^2) \quad \text{for } t \geq t_0,\ t \neq t_k,\ k=1,2,\dots, \\
x(t_k+0) = \Phi_k(x(t_k-0)), \quad k=1,2,3,\dots,
\end{gathered} \tag{7.1}$$

where $x \in \mathbb{R}$, the functions $g \in C(\mathbb{R}_+,\mathbb{R}_+) : g(t) \geq \frac{1}{2t^q\Gamma(1-q)}$, $\Phi_k \in C(\mathbb{R},\mathbb{R}) : |\Phi_k(x)| \leq c_k|x|$, $c_k \in (0,1)$, $k = 1,2,\dots$, are given constants.

Consider the function $V(t,x) = x^2$. Then the inequality $(\Phi_k(x))^2 \leq \Psi_k(x^2)$, $k = 1, 2, \dots$ holds with $\Psi_k(x) = c_k^2 x$. The Caputo fractional Dini derivative of the quadratic function for $t > 0$, $t \neq t_k$ is

$$\begin{aligned} {}_{(7.1)}^{\ c}D_+^q V(t,x;0,x_0) &= 2x\Big(-g(t)x(1+x^2)\Big) + (x^2 - x_0^2)\frac{1}{t^q\Gamma(1-q)} \\ &\leq x^2\Big(-2g(t)(1+x^2) + \frac{1}{t^q\Gamma(1-q)}\Big) \\ &\leq -2g(t)x^4 \leq 0. \end{aligned} \tag{7.2}$$

Then by Theorem 6.1, the trivial solution of IFrDE (7.1) is stable.

Acknowledgments. This research was partially supported by the Fund NPD, Plovdiv University, No. MU15-FMIIT-008.

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