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EXISTENCE OF SOLUTIONS FOR DIRICHLET QUASILINEAR SYSTEMS INCLUDING A NONLINEAR FUNCTION OF THE DERIVATIVE

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Abstract. In this article we establish the existence of at least one non-trivial classical solution for Dirichlet quasilinear systems with a nonlinear dependence on the derivative. We use variational methods for smooth functionals defined on reflexive Banach spaces, and assumptions on the asymptotic behaviour of the nonlinear data.

1. Introduction

In this article we consider the quasilinear system

$$\begin{gathered} -(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x) = \lambda F_{u_i}(x,u_1,\dots,u_n)h_i(x,u_i'), \quad x\in(a,b), \\ u_i(a)=u_i(b)=0 \end{gathered} \tag{1.1}$$

for $1 \le i \le n$ where $p_i > 1$ for $1 \le i \le n$, $\lambda > 0$, $a, b \in \mathbb{R}$ with $a < b$, $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to x, for every $(t_1,\dots,t_n) \in \mathbb{R}^n$, continuously differentiable in $(t_1,\dots,t_n)$, for almost every $x \in [a,b]$, and $F_{t_i}(x,0,\dots,0) = 0$ for all $x \in [a,b]$ and for $1 \le i \le n$, $h_i : [a,b]\times\mathbb{R} \to [0,+\infty[$ is a bounded and continuous function with $m_i := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t) > 0$ for $1 \le i \le n$. Here, F_{t_i} denotes the partial derivative of F with respect to t_i.

Owing to the importance of second-order differential equations with nonlinear derivative dependence in physics, in the last decade or so, many authors applied the variational method to study the existence of solutions of the problems of the form of (1.1) or its variations, see, for example, [2, 3, 5, 8, 9, 10, 11]. We note that the main tools in these cited papers are critical points theorems due to Ricceri and their variants.

Our goal is to establish some new criteria for system (1.1) to have at least one non-trivial classical solution by applying the following critical points theorem due to Ricceri [15, Theorem 2.1].

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Theorem 1.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for every $r > \inf_X \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

We refer the interested reader to the papers [1, 6, 7, 12, 13, 14] in which Theorem 1.1 has been used for the existence of at least one nontrivial solution for boundary value problem.

2. Preliminaries

Let X be the Cartesian product of n Sobolev spaces $W_0^{1,p_1}([a,b]),\dots,$ and $W_0^{1,p_n}([a,b])$, i.e., $X = W_0^{1,p_1}([a,b]) \times \cdots \times W_0^{1,p_n}([a,b])$, equipped with the norm

$$\|(u_1,\dots,u_n)\| = \sum_{i=1}^{n} \|u_i'\|_{p_i}$$

where

$$\|u_i'\|_{p_i} = \Big(\int_a^b |u_i'(x)|^{p_i} dx\Big)^{1/p_i}, \quad i = 1,\dots,n.$$

Since $p_i > 1$ for $i = 1,\dots,n$, X is compactly embedded in $(C([a,b]))^n$.

By a classical solution of (1.1), we mean a function $u = (u_1,\dots,u_n)$ such that, for $i = 1,\dots,n$, $u_i \in C^1[a,b]$, $u_i' \in AC[a,b]$, and $u_i(x)$ satisfies (1.1) a.e. on $[a,b]$. We say that a function $u = (u_1,\dots,u_n) \in X$ is a weak solution of the system (1.1) if

$$\begin{aligned}&\sum_{i=1}^{n}\int_a^b \Big(\int_0^{u_i'(x)} \frac{(p_i-1)|\tau|^{p_i-2}}{h_i(x,\tau)} d\tau\Big) v_i'(x)dx \\ &- \lambda\int_a^b \sum_{i=1}^{n} F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)dx = 0\end{aligned}$$

for every $v = (v_1,\dots,v_n) \in X$. Using standard methods, we see that a weak solution of system (1.1) is indeed a classical solution (see [8, Lemma 2.2]).

Let

$$\begin{gathered}\underline{p} := \min\{p_i : 1 \le i \le n\}, \quad \overline{p} := \max\{p_i : 1 \le i \le n\},\\ m_i := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t) > 0, \quad \text{for } 1 \le i \le n,\\ M_i := \sup_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t), \quad textfor 1 \le i \le n,\\ \overline{M} := \max\{M_i : 1 \le i \le n\}, \quad \underline{M} := \min\{m_i : 1 \le i \le n\}.\end{gathered}$$

Then, $\overline{M} \geq M_i \geq m_i \geq \underline{M} > 0$ for each $i = 1, \dots, n$. Put

$$H_i(x,t) = \int_0^t \Big(\int_0^\tau \frac{(p_i - 1)|\delta|^{p_i - 2}}{h_i(x,\delta)} d\delta \Big) d\tau$$

for all $(x,t) \in [a,b] \times \mathbb{R}$, $1 \leq i \leq n$.

3. Main results

We formulate our min result as follows. Put

$$p^* = \begin{cases} \overline{p} & \text{if } b - a \geq 1, \\ \underline{p} & \text{if } 0 < b - a < 1. \end{cases}$$

Theorem 3.1. *Assume that*

$$\sup_{r>0} \frac{r}{\int_a^b \max_{(t_1,\dots,t_n) \in Q(\frac{(b-a)^{p^*-1}\overline{p}\overline{M}r}{2^{\underline{p}}})} F(x,t_1,\dots,t_n)dx} > 1 \tag{3.1}$$

where

$$Q(\frac{(b-a)^{p^*-1}\overline{p}\overline{M}r}{2^{\underline{p}}}) = \Big\{ (t_1,\dots,t_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq \frac{(b-a)^{p^*-1}\overline{p}\overline{M}r}{2^{\underline{p}}} \Big\}$$

and there are a non-empty open set $D \subseteq (a,b)$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)} \frac{\operatorname{ess\,inf}_{x\in B} F(x,\xi_1,\dots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{\underline{p}}} = +\infty,$$

$$\liminf_{(\xi_1,\dots,\xi_n)\to(0^+,\dots,0^+)} \frac{\operatorname{ess\,inf}_{x\in D} F(x,\xi_1,\dots,\xi_n)}{\sum_{i=1}^n |\xi_i|^{\underline{p}}} > -\infty.$$

Then, for each

$$\lambda \in \Lambda = \Big(0, \sup_{r>0} \frac{r}{\int_a^b \max_{(t_1,\dots,t_n) \in Q(\frac{(b-a)^{p^*-1}\overline{p}\overline{M}r}{2^{\underline{p}}})} F(x,t_1,\dots,t_n)dx} \Big),$$

system (1.1) admits at least one non-trivial classical solution $u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda})$ in X. Moreover, we have

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \to \sum_{i=1}^n \int_a^b H_i(x, u'_{i\lambda}(x))dx - \lambda \int_a^b F(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x))dx$$

is negative and strictly decreasing in the open interval Λ.

Proof. Our aim is to apply Theorem 1.1 to system (1.1). Let the functionals Φ, Ψ be defined by

$$\Phi(u) = \sum_{i=1}^n \int_a^b H_i(x, u'_i(x))dx, \tag{3.2}$$

$$\Psi(u) = \int_a^b F(x, u_1(x), \dots, u_n(x))dx \tag{3.3}$$

for $u = (u_1, \dots, u_n) \in X$, and put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for $u \in X$. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 1.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)dx$$

for every $v = (v_1, \dots, v_n) \in X$, and Ψ is sequentially weakly upper semicontinuous. Moreover, according to the definition Φ, we see that Φ is continuous. Since $0 < \underline{M} \leq h_i(x,t) \leq \overline{M}$ for each $(x,t) \in [a,b] \times \mathbb{R}$ and $i = 1, \dots, n$, from (3.2) we see that

$$\frac{1}{\overline{M}} \sum_{i=1}^n \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} \leq \Phi(u) \leq \frac{1}{\underline{M}} \sum_{i=1}^n \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} \quad \text{for all } u \in X. \tag{3.4}$$

From (3.4), it follows $\lim_{\|u\| \to +\infty} \Phi(u) = +\infty$, namely Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \sum_{i=1}^n \int_a^b \Big(\int_0^{u_i'(x)} \frac{(p_i - 1)|\tau|^{p_i - 2}}{h_i(x,\tau)} d\tau \Big) v_i'(x)dx$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous (see [17, Proposition25.20]). Since for $1 \leq i \leq n$,

$$\max_{x \in [a,b]} |u_i(x)| \leq \frac{(b-a)^{\frac{p_i - 1}{p_i}}}{2} \|u_i'\|_{p_i}$$

for each $u_i \in W_0^{1,p_i}([a,b])$ (see [16]), we have

$$\max_{x \in [a,b]} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{(b-a)^{p^* - 1}}{2^{\underline{p}}} \sum_{i=1}^n \frac{\|u_i'\|_{p_i}^{p_i}}{p_i}$$

for each $u \in X$. This, for each $r > 0$, along with (3.4), ensures that

$$\begin{aligned} &\Phi^{-1}(-\infty, r) \\ &= \{u \in X; \Phi(u) < r\} \\ &\subseteq \Big\{ u \in X : \max \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{(b-a)^{p^* - 1} \overline{p} \overline{M} r}{2^{\underline{p}}} \text{ for each } x \in [a,b] \Big\}. \end{aligned} \tag{3.5}$$

So,

$$\max_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) \leq \int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)^{p^*-1} \overline{p} \overline{M} r}{2^{\underline{p}}})} F(x, t_1, \dots, t_n)dx. \tag{3.6}$$

From the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned} \varphi(r) &= \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)}{r} \\ &\leq \frac{\int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)^{p^*-1} \overline{p} \overline{M} r}{2^{\underline{p}}})} F(x, t_1, \dots, t_n)dx}{r}. \end{aligned} \tag{3.7}$$

Hence, putting

$$\lambda^* = \sup_{r>0} \frac{r}{\int_a^b \max_{(t_1,\dots,t_n)\in Q\left(\frac{(b-a)^{p^*-1}\overline{p}\,\overline{M}r}{2^{\underline{p}}}\right)} F(x,t_1,\dots,t_n)dx},$$

Theorem 1.1 ensures that for every $\lambda \in (0,\lambda^*) \subseteq (0, \frac{1}{\varphi(r)})$, the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$. Now for every fixed $\lambda \in (0,\lambda^*)$ we show that $u_\lambda = (u_{1\lambda},\dots,u_{n\lambda}) \neq 0$ and the map

$$(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda),$$

is negative. To this end, let us verify that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.8}$$

Owing to our assumptions at zero, we can fix a sequence $\{\xi^k = (\zeta^k,\dots,\zeta^k)\} \subset (\mathbb{R}^+)^n$ converging to zero and two constants $\sigma,\ \kappa$ (with $\sigma > 0$) such that

$$\lim_{k\to+\infty} \frac{\operatorname{ess\,inf}_{x\in B} F(x,\xi^k)}{\sum_{i=1}^n |\zeta^k|^{\underline{p}}} = +\infty,$$

$$\operatorname{ess\,inf}_{x\in D} F(x,\xi) \geq \kappa \sum_{i=1}^n |\xi_i|^{\underline{p}}$$

where $\xi = (\xi_1,\dots,\xi_n)$, $\xi_i \in [0,\sigma]$, $1 \leq i \leq n$. Now, fix a set $C \subset B$ of positive measure and a function $v = (v_1,\dots,v_n) \in X$ such that:

- (i) $v_i(x) \in [0,1]$, for $1 \leq i \leq n$ and for every $x \in [a,b]$,
- (ii) $v(x) = (1,\dots,1)$ for every $x \in C$,
- (iii) $v(x) = (0,\dots,0)$ for every $x \in (a,b)\setminus D$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{n\eta\,\mathrm{meas}(C) + \kappa\int_{D\setminus C}\sum_{i=1}^n |v_i(x)|^{\underline{p}}dx}{\frac{1}{\underline{p}\,\underline{M}}\sum_{i=1}^n \|v_i'\|_{\underline{p}}^{\underline{p}}}.$$

Then, there is $k_0 \in \mathbb{N}$ such that $\zeta^k < \sigma$ and

$$\operatorname{ess\,inf}_{x\in B} F(x,\xi^k) \geq \eta \sum_{i=1}^n |\zeta^k|^{\underline{p}}$$

for every $k > k_0$. Now, for every $k > k_0$, recalling the properties of the function v (that is $0 \leq \zeta^k v_i(x) < \sigma$, $1 \leq i \leq n$ for k sufficiently large), one has

$$\begin{aligned}
\frac{\Psi(\xi^k v)}{\Phi(\xi^k v)} &= \frac{\int_C F(x,\zeta^k,\dots,\zeta^k)dx + \int_{D\setminus C} F(x,\zeta^k v_1(x),\dots,\zeta^k v_n(x))dx}{\Phi(\xi^k v)} \\
&> \frac{n\eta\,\mathrm{meas}(C) + \kappa\int_{D\setminus C}\sum_{i=1}^n |v_i(x)|^{\underline{p}}dx}{\frac{1}{\underline{p}\,\underline{M}}\sum_{i=1}^n \|v_i'\|_{\underline{p}}^{\underline{p}}} > M
\end{aligned}$$

where $\xi^k v = (\zeta^k v_1,\dots,\zeta^k v_n)$. Since M could be consider arbitrarily large, it follows that

$$\lim_{k\to\infty} \frac{\Psi(\xi^k v)}{\Phi(\xi^k v)} = +\infty,$$

from which (3.8) follows. Hence, there exists a sequence $\{w_k = (w_{1k}, \dots, w_{nk})\} \subset X$ strongly converging to zero, $w_k \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(w_k) = \Phi(w_k) - \lambda\Psi(w_k) < 0.$$

Since $u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda})$ is a global minimum of the restriction of I_λ to the set $\Phi^{-1}(-\infty, r)$, we deduce that

$$I_\lambda(u_\lambda) < 0, \tag{3.9}$$

so that u_λ is not trivial. From (3.9) we easily see that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.10}$$

is negative.

Now we prove that $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$. Since Φ is coercive and for $\lambda \in (0, \lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \le L$ for every $\lambda \in (0, \lambda^*)$. Therefore, there exists a positive constant $\mathcal{M}$ such that

$$\Big| \int_a^b \sum_{i=1}^n F_{u_{\lambda_i}}(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x))u_\lambda(x)dx \Big| \le \mathcal{M}\|u_\lambda\| \le \mathcal{M}L \tag{3.11}$$

for every $\lambda \in (0, \lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for any $v \in X$ and every $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$; that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_a^b \sum_{i=1}^n F_{u_{\lambda_i}}(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x))u_\lambda(x)dx \tag{3.12}$$

for every $\lambda \in (0, \lambda^*)$. Then, since

$$0 \le \frac{1}{\overline{M}} \sum_{i=1}^n \|u'_i\|_{p_i}^{p_i} \le \Phi'(u_\lambda)(u_\lambda)$$

by (3.12) it follows that

$$0 \le \frac{1}{\overline{M}} \sum_{i=1}^n \|u'_i\|_{p_i}^{p_i} \le \lambda \int_a^b \sum_{i=1}^n F_{u_{\lambda_i}}(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x))u_\lambda(x)dx$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, by (3.11) we have $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$. Further, taking $X \hookrightarrow (C([a, b]))^n$ into account, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0. \tag{3.13}$$

Finally, we show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. For our aim we see that for any $u \in X$,

$$I_\lambda(u) = \lambda\Big(\frac{\Phi(u)}{\lambda} - \Psi(u)\Big). \tag{3.14}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let $\bar{u}_{\lambda_1} = (u_{1\lambda_1}, \dots, u_{n\lambda_1})$ and $\bar{u}_{\lambda_2} = (u_{1\lambda_2}, \dots, u_{n\lambda_2})$ be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} = \Big(\frac{\Phi(\bar{u}_{\lambda_i})}{\lambda_i} - \Psi(\bar{u}_{\lambda_i})\Big) = \inf_{v\in\Phi^{-1}(-\infty,r)} \Big(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\Big)$$

for $i = 1, 2$. Clearly, (3.10) and (3.14), since $\lambda > 0$, imply that

$$m_{\lambda i} < 0, \quad \text{for } i = 1\ 2. \tag{3.15}$$

Moreover, since $0 < \lambda_1 < \lambda_2$, we have

$$m_{\lambda_2} \le m_{\lambda_1}. \tag{3.16}$$

Hence, from (3.14)-(3.16) and again since $0 < \lambda_1 < \lambda_2$, we obtain

$$I_{\lambda_2}(\bar{u}_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(\bar{u}_{\lambda_1}),$$

which means the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. Since $\lambda < \lambda^*$ is arbitrary, we observe $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. The proof is complete. $\square$

Remark 3.2. Here employing Ricceri's variational principle we are looking for the existence of critical points of the functional I_λ naturally associated to system (1.1). We emphasize that by direct minimization, we can not argue, in general for finding the critical points of I_λ. Because, in general, I_λ can be unbounded from the following in X. Indeed, for example, in the case when

$$F(x, t_1, \dots, t_n) = (\sum_{i=1}^{n}(|t_i| + |t_i|^{q_i}))$$

for all $(x, t_1, \dots, t_n) \in [a, b] \times \mathbb{R}^n$ with $q_i \in (\overline{p}, +\infty)$ for $1 \le i \le n$, for any fixed $u \in X\backslash\{0\}$ and $\iota \in \mathbb{R}$, we obtain

$$\begin{aligned} I_\lambda(\iota u) &= \Phi(\iota u) - \lambda \int_a^b F(x, \iota u_1(x), \dots, \iota u_n(x))dx \\ &\le \frac{\iota^{\overline{p}}}{\underline{p}\underline{M}} \sum_{i=1}^{n} \|u_i'\|_{p_i}^{p_i} - \lambda \iota \sum_{i=1}^{n} \|u_i\|_{L^1} - \lambda \sum_{i=1}^{n} \iota^{q_i} \|u_i\|_{L^{q_i}}^{q_i} \to -\infty \end{aligned}$$

as $\iota \to +\infty$.

Remark 3.3. We want to point out that the energy functional I_λ associated with system (1.1) is not coercive. Indeed, fix $u \in X\backslash\{0\}$ and $\iota \in \mathbb{R}$, then, for $F(x, t_1, \dots, t_n) = \sum_{i=1}^{n} |t_i|^{s_i}$ for all $(x, t_1, \dots, t_n) \in [a, b] \times \mathbb{R}^n$ with $s_i \in (\overline{p}, +\infty)$. We have

$$\begin{aligned} I_\lambda(\iota u) &= \Phi(\iota u) - \lambda \int_a^b F(x, \iota u_1(x), \dots, \iota u_n(x))dx \\ &\le \frac{\iota^{\overline{p}}}{\underline{p}\underline{M}} \sum_{i=1}^{n} \|u_i\|_{p_i}^{p_i} - \lambda \sum_{i=1}^{n} \iota^{s_i} \|u_i\|_{L^{s_i}}^{s_i} \to -\infty \end{aligned}$$

as $\iota \to +\infty$.

Remark 3.4. If in Theorem 3.1, $F_{t_i}(x, t_1, \dots, t_n) \ge 0$ for a.e. $x \in [a, b]$, $1 \le i \le n$, the condition (3.1) becomes to the more simple and significative form

$$\sup_{r>0} \frac{r}{\int_a^b F(x, \sqrt[\underline{p}]{\frac{(b-a)^{p^*-1}\overline{p}^2\overline{M}r}{2^{\underline{p}}}}, \dots, \sqrt[\underline{p}]{\frac{(b-a)^{p^*-1}\overline{p}^2\overline{M}r}{2^{\underline{p}}}})dx} > 1. \tag{3.17}$$

Moreover, if

$$\limsup_{r\to+\infty} \frac{r}{\int_a^b F(x, \sqrt[\underline{p}]{\frac{(b-a)^{p^*-1}\overline{p}^2\overline{M}r}{2^{\underline{p}}}}, \dots, \sqrt[\underline{p}]{\frac{(b-a)^{p^*-1}\overline{p}^2\overline{M}r}{2^{\underline{p}}}})dx} > 1,$$

then, (3.17) automatically holds.

Remark 3.5. For fixed $\bar{r} > 0$ let

$$\frac{\bar{r}}{\int_a^b \max_{(t_1,\dots,t_n)\in Q(\frac{(b-a)^{p^*-1}\overline{p}\overline{M}\bar{r}}{2^{\underline{p}}})} F(x,t_1,\dots,t_n)dx} > 1.$$

Then the result of Theorem 3.1 holds. In fact, it ensures the existence of least one non-trivial classical solution $u_\lambda = (u_{1\lambda},\dots,u_{n\lambda})$ in X such that

$$\max \sum_{i=1}^n \frac{|u_{i\lambda}(x)|^{p_i}}{p_i} \le \frac{(b-a)^{p^*-1}\overline{p}\overline{M}\bar{r}}{2^{\underline{p}}} \quad \text{for each } x \in [a,b].$$

Remark 3.6. We observe that Theorem 3.1 is a bifurcation result in the sense that the pair $(0,0)$ belongs to the closure of the set $\{(u_\lambda,\lambda) \in X \times (0,+\infty) : u_\lambda = (u_{1\lambda},\dots,u_{n\lambda})$ is a non-trivial classical solution of (1.1) $\}$ in $X \times \mathbb{R}$. Indeed, by Theorem 3.1 we have that

$$\|u_\lambda\| \to 0 \quad \text{as } \lambda \to 0.$$

Hence, there exist two sequences $\{u_j = (u_{1j}\dots,u_{nj})\}$ in X and $\{(\lambda_1,\dots,\lambda_n)\}$ in $(\mathbb{R}^+)^n$ (here $u_{ij} = u_{\lambda_i}$) such that for $1 \le i \le n$

$$\lambda_i \to 0^+ \quad \text{and} \quad \|u_{ij}\| \to 0,$$

as $j \to +\infty$. Moreover, we want to emphasis that because the map $(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing, for every $\lambda_1,\lambda_2 \in (0,\lambda^\star)$, with $\lambda_1 \neq \lambda_2$, the solutions $\bar{u}_{\lambda_1}$ and $\bar{u}_{\lambda_2}$ ensured by Theorem 3.1 are different.

We present the following example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.7. Consider the system

$$\begin{gathered}
-u_1''(x) = \lambda F_{u_1}(u_1,u_2)h_1(u_1'), \quad x \in (0,1),\\
-u_2''(x) = \lambda F_{u_2}(u_1,u_2)h_2(u_2'), \quad x \in (0,1),\\
u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0
\end{gathered} \tag{3.18}$$

where $F(t_1,t_2) = (t_1^2+t_2^2)\sin^2(t_1) + \cos(t_2)$ for $t_1,t_2 \in \mathbb{R}$,

$$h_1(t_1) = \begin{cases} 2, & \text{if } t_1 \in (-\infty,0),\\ \frac{2}{1+\sqrt{t_1}}, & \text{if } t_1 \in [0,1],\\ 1, & \text{if } t_1 \in (1,+\infty) \end{cases}$$

and $h_2(t_2) = \frac{4}{3+sint_2}$ for $t_2 \in \mathbb{R}$. We observe that F, h_1 and h_2 are continuous, h_1 and h_2 are bounded with $m_1 = \inf h_1 = 1$, $m_2 = \inf h_2 = 1$, $M_1 = \sup h_1 = 2$ and $M_2 = \sup h_2 = 2$. By the definitions of h_1 and h_2 we have

$$H_1(t_1) = \begin{cases} \frac{1}{4}t_1^2, & \text{if } t_1 \in (-\infty,0),\\ \frac{1}{4}t_1^2 + \frac{2}{15}t_1^{\frac{5}{2}}, & \text{if } t_1 \in [0,1],\\ \frac{1}{2}t_1^2 - \frac{1}{4}t_1 + \frac{2}{15}, & \text{if } t_1 \in (1,+\infty) \end{cases}$$

and $H_2(t_2) = \frac{3}{8}t_2^2 - \frac{\sin t_2}{4}$ for $t_2 \in \mathbb{R}$. It is easy to see that all assumptions of Theorem 3.1 are satisfied. By Theorem 3.1 for each $\lambda \in (0,\frac{1}{4}$, system (3.18) admits at least one nontrivial classical solution $u_\lambda = (u_{1\lambda}, u_{2\lambda})$ in X. Moreover, we have $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$ and the real function

$$\begin{aligned}\lambda \mapsto &\int_0^1 \Big(H_1(u'_{1\lambda}(x)) + H_2(u'_{2\lambda}(x))\Big)dx \\ &- \lambda \int_0^1 \big((u_{1\lambda}^2(x) + u_{2\lambda}^2(x))\sin^2(u_{1\lambda}(x)) + \cos(u_{2\lambda}(x))\big)dx\end{aligned}$$

is negative and strictly decreasing in $(0, \frac{1}{4})$.

As an application of Theorem 3.1, we consider the problem

$$\begin{gathered}-(p-1)|u'(x)|^{p-2}u''(x) = \lambda f(x,u)h(u'), \quad x \in (a,b), \\ u(a) = u(b) = 0\end{gathered} \tag{3.19}$$

where $p > 1$, $\lambda > 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function such that $f(x,0) = 0$ for a.e. $x \in [a,b]$, and $h : \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m = \inf_{t\in\mathbb{R}} h(t) > 0$.

Let $M = \sup_{t\in\mathbb{R}} h(t)$,

$$\begin{gathered}F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for every } (x,t) \in [a,b] \times \mathbb{R}, \\ H(t) = \int_0^t \Big(\int_0^\tau \frac{(p-1)|\delta|^{p-2}}{h(\delta)}d\delta\Big)d\tau\end{gathered}$$

for all $t \in \mathbb{R}$. For $\gamma > 0$, define

$$W(\gamma) = \{t \in \mathbb{R} : |t|^p \le p\gamma\}.$$

We conclude this section by giving the following consequence of Theorem 3.1.

Theorem 3.8. *Assume that*

$$\sup_{r>0} \frac{r}{\int_a^b \max_{t\in W(\frac{(b-a)^{p-1}pMr}{2^p})} F(x,t)dx} > 1$$

and there are a non-empty open sets $D \subseteq (0,T)$ and $B \subset D$ of positive Lebesgue measure such that

$$\begin{gathered}\limsup_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{x\in B} F(x,\xi)}{\xi^p} = +\infty, \\ \liminf_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{x\in D} F(x,\xi)}{\xi^p} > -\infty.\end{gathered}$$

Then, for each

$$\lambda \in \Lambda = \Big(0, \sup_{r>0} \frac{r}{\int_a^b \max_{t\in W(\frac{(b-a)^{p-1}pMr}{2^p})} F(x,t)dx}\Big),$$

problem (3.19) admits at least one nontrivial classical solution $u_\lambda \in W_0^{1,p}([a,b])$ such that $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$ and the real function

$$\lambda \mapsto \int_a^b H(u'_\lambda(x))dx - \lambda \int_a^b F(x, u_\lambda(x))dx$$

is negative and strictly decreasing in the open interval Λ.

Now, we point out the following result, as a consequence of Theorem 3.8 in which the function f has separated variables.

Theorem 3.9. *Let $\alpha \in L^\infty([a,b])$ with $\operatorname{ess\,inf}_{x\in[a,b]} \alpha(x) > 0$. Further, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. Put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$, and assume that*

$$\frac{1}{\int_a^b \alpha(x)dx} \sup_{r>0} \frac{r}{\max_{\xi\in W(\frac{(b-a)^2 Mr}{2})} F(\xi)} > 1,$$

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for each

$$\lambda \in \Lambda = \Big(0, \frac{1}{\int_a^b \alpha(x)dx} \sup_{r>0} \frac{r}{\max_{\xi\in W(\frac{(b-a)^2 Mr}{2})} F(\xi)}\Big),$$

the problem

$$\begin{gathered} -u'' = \lambda\alpha(x)f(u)h(u'), \quad x \in (a,b), \\ u(a) = u(b) = 0 \end{gathered} \tag{3.20}$$

admits at least one nontrivial classical solution $u_\lambda \in W_0^{1,2}([a,b])$ such that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \to \int_a^b H(u'_\lambda(x))dx - \lambda \int_a^b \alpha(x)F(u_\lambda(x))dx$$

is negative and strictly decreasing in Λ.

Next we given an example to illustrate Theorem 3.9.

Example 3.10. Consider the problem

$$\begin{gathered} -u'' = \lambda(1+e^x)f(u)h(u'), \quad x \in (0,1), \\ u(0) = u(1) = 0 \end{gathered} \tag{3.21}$$

where $f(t) = t^2 \sin^2(t) - \sin(t)$ for $t \in \mathbb{R}$ and

$$h(t) = \begin{cases} 1, & \text{if } t \in (-\infty, 0), \\ \frac{1}{1+t^2}, & \text{if } t \in [0,1], \\ \frac{1}{2}, & \text{if } t \in (1, +\infty). \end{cases}$$

We observe that f and h are continuous and h is bounded with $m = \inf_{t\in\mathbb{R}} h(t) = \frac{1}{2}$ and $M = \sup_{t\in\mathbb{R}} h(t) = 1$. By the expression of f and h we have

$$F(t) = \frac{1}{6}t^3 - \frac{1}{4}\Big(t^2\sin(2t) + t\cos(2t) - \frac{1}{2}\sin(2t)\Big) + \cos(t) - 1$$

for every $t \in \mathbb{R}$ and

$$H(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } t \in (-\infty, 0), \\ \frac{1}{2}t^2 + \frac{1}{12}t^4, & \text{if } t \in [0,1], \\ t^2 - \frac{1}{2}t + \frac{1}{12}, & \text{if } t \in (1, +\infty). \end{cases}$$

It is easy to see that all assumptions of Theorem 3.9 are satisfied. By Theorem 3.9 for each $\lambda \in (0, \frac{1}{4e})$, problem (3.21) admits at least one nontrivial classical solution u_λ in $W_0^{1,2}([0,1])$. Moreover, $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$ and the real function

$$\lambda \mapsto \int_0^1 H(u'_\lambda(x))dx - \lambda \int_0^1 (1+e^x)F(u_\lambda(x))dx$$

is negative and strictly decreasing in $(0, 1/(4e))$.

References

[1] G. A. Afrouzi, A. Hadjian, G. Molica Bisci; *Some remarks for one-dimensional mean curvature problems through a local minimization principle,* Adv. Nonlinear Anal. 2 (2013) 427-441.

[2] G. A. Afrouzi, S. Heidarkhani; *Three solutions for a quasilinear boundary value problem,* Nonlinear Anal. TMA 69 (2008) 3330-3336.

[3] D. Averna, G. Bonanno; *Three solutions for quasilinear two-point boundary value problem involving the one-dimensional p-Laplacian,* Proc. Edinb. Math. Soc. 47 (2004) 257-270.

[4] H. Brézis; *Analyse Functionelle- Theorie et Applications,* Masson, Paris (1983).

[5] G. D'Aguì, S. Heidarkhani, A. Sciammetta; *Infinitely many solutions for a class of quasilinear two-point boundary value systems,* Electron. J. Qual. Theory Differ. Equ. 2015 (2015), No. 8, 1-15.

[6] M. Ferrara, G. Molica Bisci; *Existence results for elliptic problems with Hardy potential,* Bull. Sci. Math. 138 (2014) 846-859.

[7] M. Galewski, G. Molica Bisci; *Existence results for one-dimensional fractional equations*, Math. Meth. Appl. Sci., to appear.

[8] J. R. Graef, S. Heidarkhani, L. Kong; *A critical points approach for the existence of multiple solutions of a Dirichlet quasilinear system,* J. Math. Anal. Appl. 388 (2012) 1268-1278.

[9] S. Heidarkhani, J. Henderson; *Multiple solutions for a Dirichlet quasilinear system containing a parameter,* Georgian Math. J. 21 (2014) 187-197.

[10] S. Heidarkhani, D. Motreanu; *Multiplicity results for a two-point boundary value problem,* Panamer. Math. J. 19 (2009) 69-78.

[11] R. Livrea; *Existence of three solutions for a quasilinear two point boundary value problem,* Arch. Math. (Basel) 79 (2002) 288-298.

[12] G. Molica Bisci, R. Servadei; *A bifurcation result for non-local fractional equations,* Anal. Appl. 13, No. 4 (2015) 371-394.

[13] G. Molica Bisci, R. Servadei; *Lower semicontinuity of functionals of fractional type and applications to nonlocal equations with critical Sobolev exponent,* Adv. Differential Equations 20 (2015) 635-660.

[14] G. Molica Bisci, V. Rădulescu; *Bifurcation analysis of a singular elliptic problem modelling the equilibrium of anisotropic continuous media,* Topol. Methods Nonlinear Anal. 45 (2015) 493-508.

[15] B. Ricceri; *A general variational principle and some of its applications,* J. Comput. Appl. Math. 113 (2000) 401-410.

[16] G. Talenti; *Some inequalities of Sobolev type on two-dimensional spheres, in: W. Walter (Ed.),* General inequalities, 5 (Oberwolfach, 1986), 401-408, Internat. Schriftenreihe Numer. Math., 80, Birkhauser, Basel (1987).

[17] E. Zeidler; *Nonlinear functional analysis and its applications,* Vol. II, Springer, Berlin-Heidelberg-New York (1985).

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