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UNIFORM ESTIMATE AND STRONG CONVERGENCE OF MINIMIZERS OF A p-ENERGY FUNCTIONAL WITH PENALIZATION

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Abstract. This article concerns the asymptotic behavior of minimizers of a p-energy functional with penalization as a parameter ε approaches zero. By establishing $W^{1,p}$ uniform estimates, we obtain $W^{1,p}$ convergence of the minimizer to a p-harmonic map.

1. Introduction

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G, and $B_1 = \{x \in \mathbb{R}^2; x_1^2 + x_2^2 < 1\}$. Denote $S^1 = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $S^2 = \{x \in \mathbb{R}^3; x_1^2+x_2^2+x_3^2 = 1\}$. Sometimes we write the vector value function $u = (u_1, u_2, u_3)$ as (u', u_3). Let $g = (g', 0)$ be a smooth map from ∂G into S^1 satisfying $d = \deg(g', \partial G) \neq 0$. Without loss of generality, we may assume $d > 0$. Consider the energy functional

$$E_\varepsilon(u, G) = \frac{1}{p} \int_G |\nabla u|^p dx + \frac{1}{2\varepsilon^p} \int_G u_3^2 dx, \quad p > 2$$

with a small parameter $\varepsilon > 0$. From the direct method in the calculus of variations it is easy to see that the functional achieves its minimum in the function class $W_g^{1,p}(G, S^2)$. Obviously, the minimizer u_ε on $W_g^{1,p}(G, S^2)$ is a weak solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p + \frac{1}{\varepsilon^p}(uu_3^2 - u_3e_3), \quad \text{on } G,$$

where $e_3 = (0, 0, 1)$. Namely, for any $\psi \in W_0^{1,p}(G, \mathbb{R}^3)$, u_ε satisfies

$$\int_G |\nabla u|^{p-2}\nabla u \nabla\psi dx = \int_G u\psi|\nabla u|^p dx + \frac{1}{\varepsilon^p} \int_G \psi(uu_3^2 - u_3e_3)dx. \tag{1.1}$$

Without loss of generality, we assume $u_3 \geq 0$, otherwise we may consider $|u_3|$ in view of the expression of the functional.

When $p = 2$, the functional $E_\varepsilon(u, G)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planner ferromagnets and antiferromagnets (see [10, 18]). The asymptotic behavior of minimizers of $E_\varepsilon(u, G)$ has been considered by Fengbo Hang and Fanghua Lin in [8]. When

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the term $\frac{u_3^2}{2\varepsilon^2}$ replaced by $\frac{(1-|u|^2)^2}{4\varepsilon^2}$ and S^2 replaced by $\mathbb{R}^2$, the problem becomes the simplified model of the Ginzburg-Landau theory for superconductors and was well studied in many papers such as [3, 4, 17, 19]. These works enunciate that the study of minimizers of the functional with some penalization terms is connected tightly with the study of harmonic maps with S^1-value. When $p > 2$, it also shows an enlightenment, namely, the properties (such as the partial regularity, the properties of singularities) of p-harmonic maps can be seen via studying the asymptotic properties of minimizers of some p-energy functional with penalization (cf. [1, 2, 11, 13, 14, 16, 20]).

In this article, as in [3, 4, 8], we concern with the asymptotic behavior of minimizers of functional $E_\varepsilon(u, G)$ on $W_g^{1,p}(G, S^2)$ where $p > 2$ as $\varepsilon \to 0$.

Theorem 1.1 ([15, Theorem 1.1]). *Assume u_ε is a minimizer of $E_\varepsilon(u, G)$ on $W_g^{1,p}(G, S^2)$. Then all the zeros of $|u'_\varepsilon|$ are included in finite, disintersected discs $B(x_j^\varepsilon, h\varepsilon), j = 1, 2, \dots, N_1$ where N_1 and $h > 0$ do not depend on $\varepsilon \in (0, 1)$.*

As $\varepsilon \to 0$, there exists a subsequence $x_i^{\varepsilon_k}$ of the center x_i^ε and $a_i \in \overline{G}$ such that $x_i^{\varepsilon_k} \to a_i$, $i = 1, 2, \dots, N_1$. Perhaps there may be at least two subsequences converging to the same point, we denote by $a_1, a_2, \dots, a_N$, $N \le N_1$, the collection of distinct points in $\{a_i\}_{i=1}^{N_1}$. Although the relationship between N and d is unknown, the integer N is independent of $\varepsilon \in (0, 1)$. By virtue of Theorem 1.1, we see that all the zeros of $|u'_\varepsilon|$ converge to $a_1, a_2, \dots, a_N$ as ε tends to 0. In addition, (2.3) in [15] shows

$$|u'_\varepsilon| \ge 1/2 \quad \text{on } K, \tag{1.2}$$

where K is an arbitrary compact subset of $G \setminus \cup_{i=1}^N \{a_i\}$.

Theorem 1.2 ([15, Theorem 1.2]). *Assume u_ε is a minimizer of $E_\varepsilon(u, G)$ on $W_g^{1,p}(G, S^2)$. K is an arbitrary compact subset of $\overline{G} \setminus \cup_{j=1}^N \{a_j\}$. Then there exists a subsequence u_{ε_k} of u_ε such that as $k \to \infty$,*

$$u_{\varepsilon_k} \to u_p = (u'_p, 0), \quad \textit{weakly in } W^{1,p}(K, \mathbb{R}^3),$$

where u'_p is a map of the least p-energy $\int_K |\nabla u|^p dx$ in $W^{1,p}(K, \partial B_1)$.

We shall give the uniform $L^p_{\rm loc}$ estimate of ∇u_ε in §3. Recalling the case that the parameter p equals to the dimension 2, we know it is available to estimate the upper bound and the lower bound of $\int |\nabla u_\varepsilon|^2 dx$ since we can use the property of conformal transformation of $\int |\nabla u_\varepsilon|^2 dx$ (the idea of which can be seen in [4, 7, 8, 9]). In fact, when scaling $x = y\varepsilon$ in $E_\varepsilon(u, G)$, there is a coefficient ε^λ appearing in the scaled energy functional. when $p = 2$, it can be derived that the exponent λ of ε is zero. Therefore, the estimate of the upper bound

$$E_\varepsilon(u_\varepsilon, G) \le C_1 \ln \frac{1}{\varepsilon} + C$$

and the lower bound

$$\frac{1}{2} \int_{G \setminus \cup_{i=1}^d B(a_i, h\varepsilon)} |\nabla u'_\varepsilon|^2 dx \ge C_2 \ln \frac{1}{\varepsilon} - C$$

can be obtained, where $C_1 = C_2 = \pi d$ (cf. [8, §4]). The uniform estimate is deduced at once. When $p > 2$, the property of conformal transformation of $\int |\nabla u_\varepsilon|^p dx$ is invalid. Therefore, $\lambda \ne 0$. It is impossible to derive such results as the case $p = 2$

if the idea of estimating the upper and the lower bounds of $\int |\nabla u_\varepsilon|^p dx$ is adopted. In fact, the upper bound

$$E_\varepsilon(u_\varepsilon, G) \le C_3\varepsilon^{2-p} + C$$

and the lower bound

$$\frac{1}{p}\int_{G\setminus\cup_{i=1}^N B(a_i,h\varepsilon)} |\nabla u'_\varepsilon|^p dx \ge C_4\varepsilon^{2-p} - C,$$

are also obtained. However, the relationship between C_3 and C_4 is not clear except that C_4 may be smaller. In [15], a comparison method was used to obtain a uniform estimate where the average functions come into plays.

Here, we use the iteration technique introduced in [12] to obtain the uniform L^p estimate of ∇u_ε. In fact, the term $\int_K |\nabla u_\varepsilon|^p dx$ of the functional $E_\varepsilon(u_\varepsilon, K)$ can be divided into three terms, $\int_K |\nabla|u'_\varepsilon||^p dx$, $\int_K |\nabla u_3|^p dx$ and $\int_K |u'_\varepsilon|^p|\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^p dx$. We will prove that $\int_K |\nabla|u'_\varepsilon||^p dx + \int_K |\nabla u_3|^p dx + \frac{1}{\varepsilon^p}\int_K u_{\varepsilon 3}^2 dx$ may be bounded by $O(\varepsilon^\lambda)$ with $\lambda > 0$ as $\varepsilon \to 0$. Using this estimate we will prove

$$\int_K |\nabla u_\varepsilon|^p dx \le C + O(\varepsilon^\lambda).$$

Based on the Theorem 1.2, we will prove in §3 that the p-harmonic map u_p is a map of least p-energy $\int_K |\nabla u|^p dx$, and the convergence is also in strong $W^{1,p}_{\rm loc}$ sense.

Theorem 1.3. *Assume u_ε is a minimizer of $E_\varepsilon(u, G)$ on $W_g^{1,p}(G, S^2)$. K is an arbitrary compact subset of $\overline{G} \setminus \cup_{j=1}^N\{a_j\}$. Then there exists a subsequence u_{ε_k} of u_ε such that as $k \to \infty$,*

$$u_{\varepsilon_k} \to u_p = (u'_p, 0), \quad in \quad W^{1,p}(K, \mathbb{R}^3),$$

where u'_p is the map in Theorem 1.2.

2. Uniform estimate

The following inverse Hölder inequality will be applied later.

Proposition 2.1. *Assume that $p > 1$, and u_ε is a minimizer of $E_\varepsilon(u, G)$ on $W_g^{1,p}(G, S^2)$. Then there exist constants $t, R_0 \in (0, 1/2)$ and $C > 0$ which is independent of ε, such that for any $B_R \subset G$ $(2R < R_0)$, we have*

$$\Big(\int_{B_R} |\nabla u_\varepsilon|^q dx\Big)^{1/q} \le C\Big(\int_{B_{2R}} (|\nabla u_\varepsilon|^2 + 1)^{p/2} dx\Big)^{1/p}, \quad \forall q \in [p, p + 2t).$$

The above proposition is a corollary from [6, Theorem 4.1], with a rescaling.

Theorem 2.2. *Let $R > 0$ be a small constant such that $B(x, 2R) \Subset G \setminus \cup_{j=1}^N\{a_j\}$. There exist constant $\varepsilon_0 > 0$ and $C_j > 0$, and $R_j = 2R - \frac{jR}{[p]+1}$ such that for $j = 2, 3, \dots, [p]$,*

$$E_\varepsilon(u_\varepsilon, B_j) \le C_j\varepsilon^{j-p} \tag{2.1}$$

where $\varepsilon \in (0, \varepsilon_0), B_j = B(x, R_j)$, and $[p]$ is the integer part of p.

For $j = 2$, the inequality (2.1) is follows from [15, Proposition 2.1]. Suppose that (2.1) holds for all $j \le m$. Then we have, in particular,

$$E_\varepsilon(u_\varepsilon, B_m) \le C_m\varepsilon^{m-p}. \tag{2.2}$$

If $m = [p]$, then we are done. Suppose $m < [p]$, we want to prove (2.1) for $j = m+1$.

Applying (1.2) we have $\frac{1}{2} \le |u'_\varepsilon(y)| \le 1$, for all $y \in B(x, 2R)$. Using the integral mean value theorem we know that there exists $r \in [R_{m+1/2}, R_m]$ such that

$$E_\varepsilon(u_\varepsilon, B_m \setminus B_{m+1/2}) = C_0(r) \int_{\partial B(x,r)} [\frac{1}{p}|\nabla u_\varepsilon|^p + \frac{1}{4\varepsilon^p} u_{\varepsilon 3}^2] d\xi,$$

and applying (2.2), we see that

$$\int_{\partial B(x,r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p} \int_{\partial B(x,r)} u_{\varepsilon 3}^2 d\xi \le C_0^{-1}(r) C_m \varepsilon^{m-p}. \tag{2.3}$$

We denote $B = B(x, r)$, and introduce two propositions.

Proposition 2.3. *If ρ_1 is a minimizer of the functional*

$$E(\rho, B) = \frac{1}{p} \int_B (|\nabla \rho|^2 + 1)^{p/2} dx + \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2 dx,$$

on $W^{1,p}_{|u'_\varepsilon|}(B, \mathbb{R}^+ \cup \{0\})$. Then $E(\rho_1, B) \le C\varepsilon^{m-p+1}$.

Proof. Obviously, the minimizer ρ_1 exists and satisfies

$$-\operatorname{div}(v^{(p-2)/2} \nabla \rho) = \frac{1}{\varepsilon^p}(1 - \rho) \quad \text{on } B, \tag{2.4}$$

$$\rho|_{\partial B} = |u'_\varepsilon|, \tag{2.5}$$

where $v = |\nabla \rho|^2 + 1$. Since $1/2 \le |u'_\varepsilon| \le 1$, it follows from the maximum principle that on $\overline{B}$,

$$\frac{1}{2} \le \rho_1 \le 1. \tag{2.6}$$

Applying (2.2) and noting $(1 - |u'|)^2 \le u_3^2$, we see easily that

$$E(\rho_1, B) \le E(|u'_\varepsilon|, B) \le C E_\varepsilon(u_\varepsilon, B) \le C\varepsilon^{m-p}. \tag{2.7}$$

Multiplying (2.4) by $\partial_\nu \rho$, where ρ denotes ρ_1, and integrating over B, we have

$$\begin{aligned} &- \int_{\partial B} v^{(p-2)/2} (\partial_\nu \rho)^2 d\xi + \int_B v^{(p-2)/2} \nabla \rho \nabla (\partial_\nu \rho) dx \\ &= \frac{1}{\varepsilon^p} \int_B (1 - \rho)(\partial_\nu \rho) dx, \end{aligned} \tag{2.8}$$

where ν denotes the unit outside norm vector on ∂B. Using (2.7) we obtain

$$\begin{aligned} \big| \int_B v^{(p-2)/2} \nabla \rho \cdot \nabla (\partial_\nu \rho) dx \big| &\le C \int_B v^{(p-2)/2} |\nabla \rho|^2 dx + \frac{1}{p} \big| \int_B \nu \cdot \nabla (v^{p/2}) dx \big| \\ &\le C\varepsilon^{m-p} + \frac{1}{p} \int_{\partial B} v^{p/2} d\xi. \end{aligned} \tag{2.9}$$

Combining (2.3), (2.5) and (2.7) we also have

$$\big| \frac{1}{\varepsilon^p} \int_B (1 - \rho)(\partial_\nu \rho) dx \big| \le \frac{1}{2\varepsilon^p} \big| \int_B (1 - \rho)^2 div\nu dx - \int_{\partial B} (1 - \rho)^2 d\xi \big| \le C\varepsilon^{m-p}.$$

Substituting this result and (2.9) into (2.8) yields

$$\big| \int_{\partial B} v^{(p-2)/2} (\partial_\nu \rho)^2 d\xi \big| \le C\varepsilon^{m-p} + \frac{1}{p} \int_{\partial B} v^{p/2} d\xi. \tag{2.10}$$

Applying (2.3), (2.5), (2.10) and the Young inequality, we obtain that for any $\delta \in (0,1)$,

$$\begin{aligned}\int_{\partial B} v^{p/2} d\xi &= \int_{\partial B} v^{(p-2)/2}[1+(\partial_\nu \rho)^2+(\partial_\tau \rho)^2] d\xi \\ &\le \int_{\partial B} v^{(p-2)/2} d\xi + \int_{\partial B} v^{(p-2)/2}(\partial_\nu \rho)^2 d\xi \\ &\quad + \Big(\int_{\partial B} v^{p/2} d\xi\Big)^{(p-2)/p} \Big(\int_{\partial B} (\tau \cdot \nabla |u_\varepsilon|)^p d\xi\Big)^{2/p} \\ &\le C(\delta)\varepsilon^{m-p} + (\frac{1}{p}+2\delta)\int_{\partial B} v^{p/2} d\xi,\end{aligned}$$

where τ denotes the unit tangent vector on ∂B. Therefore, it follows by choosing $\delta > 0$ sufficiently small that

$$\int_{\partial B} v^{p/2} d\xi \le C\varepsilon^{m-p}. \tag{2.11}$$

We multiply both sides of (2.4) by $(1-\rho)$ and integrate over B. Then

$$\int_B v^{(p-2)/2}|\nabla \rho|^2 dx + \frac{1}{\varepsilon^p}\int_B (1-\rho)^2 dx = -\int_{\partial B} v^{(p-2)/2}(\nu \cdot \nabla \rho)(1-\rho) d\xi,$$

whose left hand side is proportional to $E(\rho_1, B)$. Thus

$$E(\rho_1, B) \le C\Big|\int_{\partial B} v^{(p-2)/2}(\nu \cdot \nabla \rho)(1-\rho) d\xi\Big|.$$

Applying Holder's inequality and (2.3), (2.5), (2.6) and (2.11), we obtain

$$\begin{aligned}E(\rho_1, B) &\le C\Big|\int_{\partial B} v^{p/2} d\xi\Big|^{(p-1)/p}\Big|\int_{\partial B}(1-\rho^2)^2 d\xi\Big|^{1/p} \\ &\le C\varepsilon^{(m-p)(p-1)/p}\Big|\int_{\partial B} u_{\varepsilon 3}^2 d\xi\Big|^{1/p} \le C\varepsilon^{m-p+1}.\end{aligned} \tag{2.12}$$

The proof is complete. □

Proposition 2.4. *Denote $h = |u'_\varepsilon|$. Then there is $t \in (0,1/2)$ such that for any $\delta \in (0,1/2)$,*

$$\begin{aligned}&\frac{1}{p}\int_B |\nabla h|^p dx + \frac{1}{p}\int_B |\nabla u_3|^p dx + \frac{1}{4\varepsilon^p}\int_B (1-h^2)^2 dx \\ &\le C\varepsilon^{m-p+1} + \delta\int_B |\nabla u_\varepsilon|^p dx + C\Big(\int_{B(x,2r)} |\nabla u_\varepsilon|^p dx + 1\Big) \\ &\quad \times \Big[\int_B (1-h^2)^2 dx\Big]^{t/(p+t)}.\end{aligned} \tag{2.13}$$

Proof. Let $U = (\sqrt{2\rho_1 - \rho_1^2}w, 1-\rho_1)$ on B; $U = u_\varepsilon$ on $G \setminus B$, where $w = w_\varepsilon = \frac{u'_\varepsilon}{|u'_\varepsilon|}$. Then $U \in W_g^{1,p}(G, S^2)$. Since u_ε is a minimizer of $E_\varepsilon(u, G)$, we have

$$E_\varepsilon(u_\varepsilon, G) \le E_\varepsilon(U, G) = E_\varepsilon(U, B) + E_\varepsilon(u_\varepsilon, G \setminus B),$$

which means $E_\varepsilon(u_\varepsilon, B) \le E_\varepsilon(U, B)$. Using (2.12) it is not difficult to see that for any $\delta > 0$,

$$\int_B |\nabla \rho_1|^2 |\nabla w|^{p-2} dx \le (\int_B |\nabla \rho_1|^p dx)^{2/p} (\int_B |\nabla w|^p dx)^{\frac{p-2}{p}}$$

$$\le \delta \int_B |\nabla u_\varepsilon|^p dx + C\varepsilon^{m+1-p}.$$

By using (2.6) and the mean value theorem,

$$\begin{aligned}
&\int_B (\frac{(1-\rho_1)^2}{2\rho_1-\rho_1^2}|\nabla\rho_1|^2 + (2\rho_1-\rho_1^2)|\nabla w|^2)^{p/2}dx - \int_B ((2\rho_1-\rho_1^2)|\nabla w|^2)^{p/2}dx \\
&\le C\int_B (|\nabla\rho_1|^p + |\nabla\rho_1|^2|\nabla w|^{p-2})dx,
\end{aligned}$$

and noting $2\rho-\rho^2-1=-(1-\rho)^2\le 0$, we have

$$\begin{aligned}
E_\varepsilon(u_\varepsilon,B) &\le E_\varepsilon(U,B) \\
&\le \frac{1}{p}\int_B ((2\rho_1-\rho_1^2)|\nabla w|^2)^{p/2}dx + C\int_B (|\nabla\rho_1|^p + |\nabla\rho_1|^2|\nabla w|^{p-2})dx \\
&\quad + \frac{1}{4\varepsilon^p}\int_B (1-\rho_1)^2 dx \\
&\le \frac{1}{p}\int_B |\nabla w|^p dx + \delta\int_B |\nabla u_\varepsilon|^p dx + C\varepsilon^{m+1-p} + CE(\rho_1,B).
\end{aligned}$$

From this result and (2.12), we deduce

$$E_\varepsilon(u_\varepsilon,B) \le \frac{1}{p}\int_B |\nabla w|^p dx + C\varepsilon^{m+1-p} + \delta\int_B |\nabla u_\varepsilon|^p dx. \tag{2.14}$$

By Jensen's inequality and (2.14), we obtain

$$\begin{aligned}
&\frac{1}{p}\int_B |\nabla h|^p dx + \frac{1}{p}\int_B (h^p-1)|\nabla w|^p dx + \frac{1}{p}\int_B |\nabla u_3|^p dx \\
&+ \frac{1}{4\varepsilon^p}\int_B (1-h^2)^2 dx \\
&\le E_\varepsilon(u_\varepsilon,B) - \frac{1}{p}\int_B |\nabla w|^p dx \\
&\le C\varepsilon^{m-p+1} + \delta\int_B |\nabla u_\varepsilon|^p dx.
\end{aligned} \tag{2.15}$$

Since $h\ge 1/2$ and Proposition 2.1, there exists a $t\in(0,1/2)$ such that

$$\begin{aligned}
&\frac{1}{p}\int_B (1-h^p)|\nabla w_\varepsilon|^p dx \\
&\le \frac{2^p}{p}\int_B (1-h^p)|\nabla u_\varepsilon|^p dx \\
&\le C\Big(\int_B |\nabla u_\varepsilon|^{p+t}dx\Big)^{p/(p+t)}\Big(\int_B (1-h^p)^{(p+t)/t}dx\Big)^{t/(p+t)} \\
&\le C\Big(\int_{B(x,2r)} |\nabla u_\varepsilon|^p dx + 1\Big)\Big(\int_B (1-h^2)^2 dx\Big)^{t/(p+t)}.
\end{aligned} \tag{2.16}$$

Combining this with (2.15) we complete the proof. □

Proof of Theorem 2.2.
Step 1. Since $|u'_\varepsilon| \geq 1/2$, there exists $\phi \in W^{1,p}(B(x,3R),[0,2\pi))$ such that $w = \frac{u'_\varepsilon}{|u'_\varepsilon|} = (\cos\phi, \sin\phi)$. Obviously, $|\nabla w|^2 = |\nabla \phi|^2$. Substituting this into (1.1) with the test function $(\psi, 0)$ yields

$$\begin{aligned}&\int_{B(x,3R)} |\nabla u|^{p-2}(w\nabla h + h\nabla w)\nabla\psi dx\\ &= \int_{B(x,3R)} hw|\nabla u|^p\psi dx + \frac{1}{\varepsilon^p}\int_{B(x,3R)} hw\psi(1-h^2)dx\end{aligned}$$

where $\psi \in W_0^{1,p}(G,\mathbb{R}^2)$. Let $e^{i\phi} = \cos\phi + i\sin\phi$. Then

$$\begin{aligned}&\int_{B_{3R}(x)} he^{i\phi}|\nabla u|^p\psi dx + \frac{1}{\varepsilon^p}\int_{B_{3R}(x)} h\psi e^{i\phi}(1-h^2)dx\\ &= \int_{B_{3R}(x)} |\nabla u|^{p-2}(e^{i\phi}\nabla h + hie^{i\phi}\nabla\phi)\nabla\psi dx.\end{aligned}$$

Taking $\psi = e^{-i\phi}\zeta$, where $\zeta \in W_0^{1,p}(B(x,3R),\mathbb{R}^2)$, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^p}\int_{B(x,3R)} h(1-h^2)\zeta dx\\ &= \int_{B(x,3R)} |\nabla u|^{p-2}(\nabla h\nabla\zeta + h(|\nabla\phi|^2 - |\nabla u|^2)\zeta)dx.\end{aligned} \tag{2.17}$$

$$0 = \int_{B(x,3R)} |\nabla u|^{p-2}(h\nabla\phi\nabla\zeta - \zeta\nabla h\nabla\phi)dx. \tag{2.18}$$

Taking $\zeta = h\xi$ in (2.18), where $\xi \in W_0^{1,p}(B(x,3R),\mathbb{R}^2)$, we have

$$0 = \int_{B(x,3R)} |\nabla u|^{p-2}h^2\nabla\phi\nabla\xi dx. \tag{2.19}$$

Assume ρ is an arbitrary constant in $(0,3R/2)$. Let $\zeta \in W_0^{1,p}(B(x,2\rho),[0,1])$, and $\zeta = 1$ on $B(x,\rho)$. Taking $\xi = \phi\zeta^2$ in (2.19) and using the Young inequality, for any $\eta \in (0,1)$ we obtain

$$\int_{B(x,2\rho)} |\nabla u|^{p-2}h^2|\nabla\phi|^2\zeta^2 dx \leq C\int_{B(x,2\rho)} |\nabla u|^{p-2}h^2(\eta|\nabla\phi|^2\zeta^2 + C(\eta))dx.$$

Choosing η sufficiently small and noticing $\zeta = 1$ on $B(x,\rho)$, we obtain

$$\int_{B(x,\rho)} |\nabla u|^{p-2}h^2|\nabla\phi|^2 dx \leq C\Big(\int_{B(x,2\rho)} |\nabla u|^p dx\Big)^{1-2/p}. \tag{2.20}$$

Applying (2.20) with $\rho = r$ we obtain

$$\begin{aligned}
\int_B |\nabla u|^p &\le \int_B |\nabla u|^{p-2}(h^2|\nabla\phi|^2 + |\nabla h|^2 + |\nabla u_3|^2)dx \\
&\le C\Big(\int_{B(x,2r)} |\nabla u|^p dx\Big)^{1-2/p} \\
&\quad + \Big(\int_B (|\nabla h|^p + |\nabla u_3|^p)dx\Big)^{2/p}\Big(\int_B |\nabla u|^p dx\Big)^{(p-2)/p} \\
&\le C\Big(\int_{B(x,2r)} |\nabla u|^p dx\Big)^{1-2/p} + \delta\int_B |\nabla u|^p dx \\
&\quad + C(\delta)\int_B (|\nabla h|^p + |\nabla u_3|^p)dx.
\end{aligned} \tag{2.21}$$

Substituting (2.13) into (2.21) and choosing $\delta > 0$ sufficiently small we derive

$$\begin{aligned}
\int_B |\nabla u|^p dx &\le C\Big(\int_{B(x,2r)} |\nabla u|^p dx\Big)^{1-2/p} + C\varepsilon^{m-p+1} \\
&\quad + C\Big(\int_{B(x,2r)} |\nabla u_\varepsilon|^p dx + 1\Big)\Big[\int_B (1-h^2)^2 dx\Big]^{t/(p+t)}.
\end{aligned} \tag{2.22}$$

From (2.2) it follows that

$$\int_B |\nabla u|^p dx \le C(\varepsilon^{m-p})^{1-2/p} + C\varepsilon^{m-p+1} + C\varepsilon^{m-p+\frac{mt}{p+t}} = I_1 + I_2 + I_3. \tag{2.23}$$

Step 2. When $m \le p/2$, then $m+1-p \le (m-p)(1-2/p)$. Therefore $I_1 \le I_2$. Let $k_0 \in N$ be the minimum with the property $m+1 \le (1+\frac{t}{p+t})^{k_0} m$.

In the following we shall improve the exponent $m-p+\frac{t}{p+t}m$ of ε in I_3 to $m-p+1$. Assume $\zeta \in C_0^\infty(B(x,2R),[0,1])$ satisfying $\zeta = 1$ on $B_{m+1/2}$ and $|\nabla\zeta| \le C$. Taking the test function as $h\zeta(1-h)$ in (2.17), we have

$$\begin{aligned}
&\frac{1}{\varepsilon^p}\int_B h^2(1-h^2)\zeta(1-h)dx + \int_B |\nabla u|^{p-2}|\nabla h|^2 h\zeta dx + \int_B h^2|\nabla u|^p(1-h)\zeta dx \\
&\le \int_B |\nabla u|^{p-2}\nabla h\nabla\zeta h(1-h)dx + \int_B |\nabla u|^p\zeta(1-h) \le C\int_B |\nabla u|^p dx
\end{aligned}$$

Noting $\zeta = 1$ on $B_{m+1/2}$, applying $h \ge 1/2$ and (2.22), we obtain

$$\frac{1}{\varepsilon^p}\int_{B_{m+1/2}} (1-h^2)^2 dx \le \frac{C}{\varepsilon^p}\int_B h^2(1-h^2)(1-h)\zeta dx \le C(1+\varepsilon^{m-p+\frac{t}{p+t}m}),$$

which implies

$$\int_{B_{m+1/2}} (1-h^2)^2 dx \le C\varepsilon^{m(1+\frac{t}{p+t})}, \quad \varepsilon \in (0,\varepsilon_0). \tag{2.24}$$

On the other hand, similar to the derivation of (2.14), for $B_{m+1/2}$ we still conclude that for any $\delta > 0$,

$$E_\varepsilon(u_\varepsilon, B_{m+1/2}) \le \frac{1}{p}\int_{B_{m+1/2}} |\nabla w|^p dx + C\varepsilon^{m-p+1} + \delta\int_{B_{m+1/2}} |\nabla u_\varepsilon|^p dx.$$

Therefore, (2.15) can be written as

$$\begin{aligned}&\frac{1}{p}\int_{B_{m+1/2}}|\nabla h|^p dx+\frac{1}{p}\int_{B_{m+1/2}}|\nabla u_3|^p dx+\frac{1}{4\varepsilon^p}\int_{B_{m+1/2}}(1-h^2)^2 dx\\&\le C\varepsilon^{m-p+1}+\frac{1}{p}\int_{B_{m+1/2}}(1-h^p)|\nabla w|^p dx+\delta\int_{B_{m+1/2}}|\nabla u_\varepsilon|^p dx.\end{aligned}\tag{2.25}$$

To estimate the second term of the right hand side of (2.25), we apply (2.23) and (2.24) to obtain

$$\frac{1}{p}\int_{B_{m+1/2}}(1-h^p)|\nabla w|^p dx\le C\varepsilon^{(m+\frac{t}{p+t}m)\frac{t}{p+t}+m+\frac{t}{p+t}m-p}=C\varepsilon^{m(1+\frac{t}{p+t})^2-p}$$

by the same way as for (2.16). Substituting this into (2.25) yields

$$\frac{1}{p}\int_{B_{m+1/2}}(|\nabla h|^p+|\nabla u_3|^p)dx\le C(\varepsilon^{m-p+1}+\varepsilon^{m(1+\frac{t}{p+t})^2-p})+\delta\int_{B_{m+1/2}}|\nabla u_\varepsilon|^p dx.$$

Using this instead of (2.13) and by the same argument of Step 1 we can improve (2.23) as

$$\int_{B_{m+1/2}}|\nabla u_\varepsilon|^p dx\le C+C(\varepsilon^{m-p+1}+\varepsilon^{m(1+\frac{t}{p+t})^2-p})\le C\varepsilon^{m(1+\frac{t}{p+t})^2-p}.$$

Now, we use this inequality replacing (2.23) to discuss, thus (2.24) can be written as

$$\int_{B_{m+3/4}}(1-h^2)^2 dx\le C\varepsilon^{m(1+\frac{t}{p+t})^2},\quad \varepsilon\in(0,\varepsilon_0).$$

As a result, it is also follows that, as the derivation of (2.16) and (2.23),

$$\frac{1}{p}\int_{B_{m+3/4}}(1-h^p)|\nabla w|^p dx\le C\varepsilon^{m(1+\frac{t}{p+t})^3-p},$$

$$\int_{B_{m+3/4}}|\nabla u_\varepsilon|^p dx\le C+C(\varepsilon^{m-p+1}+\varepsilon^{m(1+\frac{t}{p+t})^3-p})\le C\varepsilon^{m(1+\frac{t}{p+t})^3-p}.$$

If we do in this way, and noting the definition of k_0, we can derive by k_0 steps that

$$\int_{B_{m+1-1/2^{k_0-1}}}|\nabla u_\varepsilon|^p dx\le C+C(\varepsilon^{m-p+1}+\varepsilon^{m(1+\frac{t}{p+t})^{k_0}-p}).$$

Thus

$$\int_{B_{m+1}}|\nabla u_\varepsilon|^p dx\le\int_{B_{m+1-1/2^{k_0-1}}}|\nabla u_\varepsilon|^p dx\le C(\varepsilon^{m-p+1}+1).$$

This is (2.2) for $j=m+1$.

Step 3. When $m>p/2$, $(m-p)(1-2/p)<m+1-p$. Let $k\ge 1$ be an integer such that $(m-p)(1-2/p)^k\le m+1-p<(m-p)(1-2/p)^{k+1}$. Now, $I_1\ge I_2$ in (2.23). Thus,

$$\int_B|\nabla u|^p dx\le C(\varepsilon^{m-p})^{1-2/p}+C\varepsilon^{m-p+\frac{mt}{(p+t)}}.$$

Similar to Step 2, we may improve the exponent $m-p+\frac{mt}{p+t}$ of ε in I_3 to $(m-p)(1-2/p)$ since we may find $k_0>0$ such that $m(1+\frac{t}{p+t})^{k_0}-p>(m-p)(1-2/p)$.

Namely, there is a constant $r_1 \in (R_{m+1}, r)$ such that

$$\int_{B(x,r_1)} |\nabla u_\varepsilon|^p dx \le C\varepsilon^{(m-p)(1-2/p)}.$$

Therefore, as the derivation of (2.24),

$$\int_{B(x,2r_1/3)} (1-h^2)^2 dx \le C\varepsilon^{(m-p)(1-2/p)+p}.$$

Substituting these into (2.22) we have

$$\begin{aligned}
&\int_{B(x,r_1/2)} |\nabla u_\varepsilon|^p dx \\
&\le C\varepsilon^{m+1-p} + C\Big[\int_{B(x,r)} |\nabla u_\varepsilon|^p dx\Big]^{1-2/p} \\
&\quad + C\Big(\int_{B(x,r)} |\nabla u_\varepsilon|^p dx + 1\Big)\Big[\int_{B(x,r)} (1-h^2)^2 dx\Big]^{\frac{t}{p+t}} \\
&\le C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^2} + C\varepsilon^{(m-p)(1-2/p)+[(m-p)(1-2/p)+p]\frac{t}{p+t}}.
\end{aligned}$$

Noting $(m-p)(1-2/p)^2 < m+1-p$, we can see that

$$\int_{B(x,r_1/2)} |\nabla u_\varepsilon|^p dx \le C\varepsilon^{(m-p)(1-2/p)^2} + C\varepsilon^{(m-p)(1-2/p)+[(m-p)(1-2/p)+p]\frac{t}{p+t}}.$$

Using the idea of Step 2, we can improve the exponent $(m-p)(1-2/p)+[(m-p)(1-2/p)+p]\frac{t}{p+t}$ of ε to $(m-p)(1-2/p)^2$. Namely, there is a constant $r_2 \in (R_{m+1}, r_1/2)$ such that

$$\int_{B(x,r_2)} |\nabla u_\varepsilon|^p dx \le C\varepsilon^{(m-p)(1-2/p)^2}.$$

Suppose that for some $l \le k-1$,

$$\int_{B(x,r_{l-1})} |\nabla u_\varepsilon|^p dx \le C\varepsilon^{(m-p)(1-2/p)^l}$$

holds, where $R_{m+1} < r_{l+1} < r_l/2$ for $l = 2, 3, \cdots, k-1$. Therefore, as the derivation of (2.24),

$$\int_{B(x,r_{l-1})} (1-h^2)^2 dx \le C\varepsilon^{(m-p)(1-2/p)^l+p}.$$

Substituting these inequalities into (2.22) yields

$$\begin{aligned}
&\int_{B(x,r_l)} |\nabla u_\varepsilon|^p dx \\
&\le C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^{l+1}} + C\varepsilon^{(m-p)(1-2/p)^l+[(m-p)(1-2/p)^l+p]\frac{t}{p+t}} \\
&\le C\varepsilon^{(m-p)(1-2/p)^{l+1}} + C\varepsilon^{(m-p)(1-2/p)^l+[(m-p)(1-2/p)^l+p]\frac{t}{p+t}}
\end{aligned}$$

Similar to Step 2, we may improve again the exponent $(m-p)(1-2/p)^l + [(m-p)(1-2/p)^l + p]\frac{t}{p+t}$ of ε to $(m-p)(1-2/p)^{l+1}$. Namely, it can be seen that

$$\int_{B(x,r_l)} |\nabla u_\varepsilon|^p dx \le C\varepsilon^{(m-p)(1-2/p)^{l+1}}.$$

From this result it follows that for $l = k-1$,

$$\int_{B(x,r_{k-1})} |\nabla u_\varepsilon|^p \le C\varepsilon^{(m-p)(1-2/p)^k}.$$

Therefore, as the derivation of (2.24),

$$\int_{B(x,r_{l-1})} (1-h^2)^2 dx \le C\varepsilon^{(m-p)(1-2/p)^k+p}.$$

Combining these with (2.22) we obtain

$$\begin{aligned}
&\int_{B(x,\frac{r_{k-1}}{2})} |\nabla u_\varepsilon|^p dx \\
&\le C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^{k+1}} + C\varepsilon^{(m-p)(1-2/p)^k+[(m-p)(1-2/p)^k+p]\frac{t}{p+t}} \\
&\le C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^k+[(m-p)(1-2/p)^k+p]\frac{t}{p+t}}.
\end{aligned}$$

As in Step 2 and noting the definition of k, we may also improve the exponent of ε to $m+1-p$ finally. Namely, we have

$$\int_{B(x,r_{k-1}/2)} |\nabla u_\varepsilon|^p \le C\varepsilon^{m+1-p}.$$

This is (2.2) for $j = m+1$ and proof of Theorem 2.2 is complete.

Theorem 2.5. *For an arbitrary compact subset K of $G \setminus \{a_1, a_2, \dots, a_N\}$. There exists a constant $C > 0$ which does not depend on $\varepsilon \in (0,1)$ such that $E_\varepsilon(u_\varepsilon, K) \le C$.*

Proof. It is sufficient to prove that $E_\varepsilon(u_\varepsilon, B(x,R)) \le C$, where $B(x,R)$ is the disc in $G \setminus \{a_1, a_2, \dots, a_N\}$. Theorem 2.2 shows that

$$E_\varepsilon(u_\varepsilon, B_{[p]}) \le C\varepsilon^{[p]-p}. \tag{2.26}$$

Using this and the integral mean value theorem, there exists a constant $r \in [R_{[p]+1/2}, R_{[p]}]$ such that

$$\int_{\partial B(x,r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p}\int_{\partial B(x,r)} u_{\varepsilon 3}^2 d\xi \le C(r)\varepsilon^{[p]-p}. \tag{2.27}$$

Consider the functional

$$E(\rho, B) = \frac{1}{p}\int_B (|\nabla\rho|^2+1)^{p/2} dx + \frac{1}{2\varepsilon^p}\int_B (1-\rho)^2 dx,$$

where $B = B(x,r)$. It is easy to prove that the minimizer ρ_2 of $E(\rho, B)$ on $W^{1,p}_{|u_\varepsilon'|}(B, \mathbb{R}^+ \cup \{0\})$ exists. Similar to the proof of proposition 2.3, by (2.26) and (2.27) we can derive

$$E(\rho_2, B) \le C\varepsilon^{[p]-p+1}. \tag{2.28}$$

From this it follows that for any $\delta > 0$,

$$\int_B |\nabla\rho_2|^2 |\nabla w|^{p-2} dx \le \delta\int_B |\nabla u_\varepsilon|^p dx + C\varepsilon^{[p]+1-p}.$$

Since u_ε is a minimizer of $E_\varepsilon(u,G)$, we have

$$\begin{aligned} E_\varepsilon(u_\varepsilon,B) &\le E_\varepsilon((\rho_2 w,\sqrt{1-\rho_2^2}),B)\\ &\le \frac{1}{p}\int_B(\rho_2^2|\nabla w|^2)^{p/2}dx + C\int_B(|\nabla\rho_2|^p+|\nabla\rho_2|^2|\nabla w|^{p-2})dx \\ &\quad+\frac{1}{4\varepsilon^p}\int_B(1-\rho_2^2)^2dx. \end{aligned} \tag{2.29}$$

Therefore,

$$E_\varepsilon(u_\varepsilon,B)\le\frac{1}{p}\int_B|\nabla w|^pdx + C\varepsilon^{[p]+1-p}+\delta\int_B|\nabla u_\varepsilon|^pdx.$$

Combining this with Jensen's inequality yields

$$\begin{aligned} &\frac{1}{p}\int_B|\nabla h|^pdx+\frac{1}{p}\int_B|\nabla u_3|^pdx+\frac{1}{4\varepsilon^p}\int_B(1-h^2)^2\\ &\le E_\varepsilon(u_\varepsilon,B)-\frac{1}{p}\int_B|\nabla w|^pdx+\frac{1}{p}\int_B(1-h^p)|\nabla w|^pdx \\ &\le C\varepsilon^{[p]+1-p}+\delta\int_B|\nabla u_\varepsilon|^pdx+\frac{1}{p}\int_B(1-h^p)|\nabla w|^pdx. \end{aligned} \tag{2.30}$$

To estimate the third term of the right hand side, we proceed in the same way of the proof of Proposition 2.4, and use $\frac{1}{\varepsilon^p}\int_B(1-h^2)^2dx\le C\varepsilon^{[p]-p}$ which is implied by (2.26). As a result, there exists $t\in(0,1/2)$ such that

$$\frac{1}{p}\int_B(1-h^p)|\nabla w|^pdx\le C\varepsilon^{[p]+[p]t/(p+t)-p}.$$

Substituting this into (2.30) yields

$$\begin{aligned} &\frac{1}{p}\int_B(|\nabla h|^p+|\nabla u_3|^p)dx+\frac{1}{4\varepsilon^p}\int_B(1-h^2)^2dx\\ &\le C(\varepsilon^{[p]+1-p}+\varepsilon^{[p]+\frac{[p]t}{p+t}-p})+\delta\int_B|\nabla u_\varepsilon|^pdx. \end{aligned}$$

This and (2.21) imply that

$$\int_B|\nabla u_\varepsilon|^pdx\le C\varepsilon^{[p]-p+1}+C\varepsilon^{[p]-p+\frac{t}{p+t}m}+C\varepsilon^{([p]-p)(1-2/p)}+C, \tag{2.31}$$

as long as we choose $\delta>0$ sufficiently small. Discussing in the same way to Step 2 and Step 3, we may improve the exponent of ε in the second and the third terms of the right hand side of (2.31) step by step such that the improved exponent is not smaller than $[p]-p+1$, thus for some $B_{[p]+1}\subset B$, there exists C independent of $\varepsilon\in(0,\varepsilon_0)$ with ε_0 sufficiently small such that

$$\int_{B_{[p]+1}}|\nabla u_\varepsilon|^pdx\le C+C\varepsilon^{[p]+1-p}\le C.$$

The proof is complete. $\square$

3. Proof of Theorem 1.3

Step 1. Suppose $B(x_0,2\sigma)\subset[G\setminus\cup_{j=1}^N\{a_j\}]$, where the constant σ may be sufficiently small but independent of ε. Since theorem 2.5 implies $E_\varepsilon(u_\varepsilon,B(x_0,2\sigma)\setminus B(x_0,\sigma))\le C$, there is a constant $r\in(\sigma,2\sigma)$ such that

$$\int_{\partial B(x_0,r)}|\nabla u_\varepsilon|^p d\xi+\frac{1}{\varepsilon^p}\int_{\partial B(x_0,r)}u_{\varepsilon 3}^2 d\xi\le C(r).$$

Thus, we can find a subsequence u_{ε_k} of u_ε such that $u_{\varepsilon_k}\to u_p=(u_p',0)$ in $C(\partial B(x_0,r),\mathbb{R}^3)$, where u_p' is the S^1-valued harmonic map, which leads to

$$\frac{u'_{\varepsilon_k}}{|u'_{\varepsilon_k}|}\to u_p',\quad\text{in}\quad C(\partial B(x_0,r)).\tag{3.1}$$

Step 2. Denote $B=B(x_0,r)$. It is easy to see the existence of the solution w_ε of

$$\min\{\int_B|\nabla u|^p dx:u\in W^{1,p}_{\frac{u'_\varepsilon}{|u'_\varepsilon|}}(B,\partial B_1)\}.\tag{3.2}$$

Theorem 2.5 and $|u'_\varepsilon|\ge 1/2$ on B imply $2^{-p}\int_B|\nabla\frac{u'_\varepsilon}{|u'_\varepsilon|}|^p dx\le\int_B|\nabla u_\varepsilon|^p dx\le C$, and hence

$$\int_B|\nabla w_\varepsilon|^p dx\le\int_B|\nabla\frac{u'_\varepsilon}{|u'_\varepsilon|}|^p dx\le C.\tag{3.3}$$

From this and (2.28) it follows that $\int_B|\nabla\rho_2|^2|\nabla w_\varepsilon|^{p-2}dx\le C\varepsilon^{2([p]+1-p)/p}$, where ρ_2 is the minimizer of $E(\rho,B)$ on $W^{1,p}_{|u'_\varepsilon|}(B,\mathbb{R}^+\cup\{0\})$. Substituting this result into (2.29) and using (2.28), we obtain

$$\int_B|\nabla u_\varepsilon|^p dx\le C\varepsilon^{2([p]+1-p)/p}+\int_B|\nabla w_\varepsilon|^p dx.\tag{3.4}$$

Step 3. Let w_ε^τ be a solution of

$$\min\Big\{\int_B(|\nabla w|^2+\tau)^{p/2}dx:w\in W^{1,p}_{\frac{u'_\varepsilon}{|u'_\varepsilon|}}(B,\partial B_1)\Big\},\quad\tau\in(0,1).\tag{3.5}$$

Clearly, w_ε^τ also solves

$$-\operatorname{div}(v_\varepsilon^{\tau\,(p-2)/2}\nabla w)=w|\nabla w|^2 v_\varepsilon^{\tau\,(p-2)/2},\quad v_\varepsilon^\tau=|\nabla w|^2+\tau.\tag{3.6}$$

Noticing $\frac{u'_\varepsilon}{|u'_\varepsilon|}\in W^{1,p}_{\frac{u'_\varepsilon}{|u'_\varepsilon|}}(B,\partial B_1)$, we have

$$\begin{aligned}\int_B|\nabla w_\varepsilon^\tau|^p dx&\le\int_B(|\nabla w_\varepsilon^\tau|^2+\tau)^{p/2}dx\\&\le\int_B(|\nabla\frac{u'_\varepsilon}{|u'_\varepsilon|}|^2+\tau)^{p/2}dx\\&\le\int_B(|\nabla\frac{u'_\varepsilon}{|u'_\varepsilon|}|^2+1)^{p/2}dx\le C\end{aligned}\tag{3.7}$$

by using (3.3), where C is a constant which is independent of ε,τ. Then there exist $w^*\in W^{1,p}_{\frac{u'_\varepsilon}{|u'_\varepsilon|}}(B,\partial B_1)$ and a subsequence of w_ε^τ denoted still by itself such that

$$\lim_{\tau\to 0}w_\varepsilon^\tau=w^*\quad\text{weakly in } W^{1,p}(B,R^2).\tag{3.8}$$

Noting the weak lower semi-continuity of $\int_B |\nabla w|^p$, we have

$$\int_B |\nabla w^*|^p dx \leq \liminf_{\tau\to 0} \int_B |\nabla w_\varepsilon^\tau|^p dx \leq \limsup_{\tau\to 0} \int_B |\nabla w_\varepsilon^\tau|^p dx. \tag{3.9}$$

The fact that w_ε^τ solves (3.5) implies

$$\limsup_{\tau\to 0} \int_B (|\nabla w_\varepsilon^\tau|^2+\tau)^{p/2} dx \leq \lim_{\tau\to 0} \int_B (|\nabla w_\varepsilon|^2+\tau)^{p/2} dx = \int_B |\nabla w_\varepsilon|^p dx,$$

where w_ε is a solution of (3.2). This and (3.9) lead to

$$\int_B |\nabla w^*|^p dx \leq \liminf_{\tau\to 0} \int_B |\nabla w_\varepsilon^\tau|^p dx \leq \limsup_{\tau\to 0} \int_B |\nabla w_\varepsilon^\tau|^p dx \leq \int_B |\nabla w_\varepsilon|^p dx. \tag{3.10}$$

Since $w^* \in W^{1,p}_{\frac{u'_\varepsilon}{|u'_\varepsilon|}}(B, \partial B_1)$, we know w^* also solves (3.2), namely

$$\int_B |\nabla w_\varepsilon|^p dx = \int_B |\nabla w^*|^p dx. \tag{3.11}$$

Combining this with (3.10) yields $\lim_{\tau\to 0}\int_B |\nabla w_\varepsilon^\tau|^p dx = \int_B |\nabla w^*|^p dx$, which and (3.8) imply that as $\tau \to 0$,

$$\nabla w_\varepsilon^\tau \to \nabla w^* \quad \text{in } L^p(B, R^2). \tag{3.12}$$

Step 4. By the same argument as in Step 3, we obtain the following conclusion: Let u^τ be a solution of

$$\min\{\int_B (|\nabla u|^2+\tau)^{p/2} dx : u \in W^{1,p}_{u'_p}(B, \partial B_1)\}, \quad \tau \in (0,1). \tag{3.13}$$

Then u^τ satisfies

$$\int_B |\nabla u^\tau|^p dx \leq C, \tag{3.14}$$

where C is which is independent of τ, and u^τ solves

$$-\operatorname{div}[(v^\tau)^{(p-2)/2}\nabla u] = u|\nabla u|^2 v^{(p-2)/2}, \quad v^\tau = |\nabla u|^2+\tau. \tag{3.15}$$

As $\tau \to 0$, there exists a subsequence of u^τ denoted by itself such that

$$\nabla u^\tau \to \nabla u^* \quad \text{in } L^p(B, R^2), \tag{3.16}$$

where u^* is a minimizer of $\int_B |\nabla u|^p dx$ in $W^{1,p}_{u'_p}(B, \partial B_1)$. It is well-known that u^* is a map of the least p-energy, and also a p-harmonic map.

Step 5. From [5, Lemma 1, Page 65], we can write

$$\begin{gathered} w_\varepsilon^\tau = (\cos\phi_\varepsilon^\tau, \sin\phi_\varepsilon^\tau), \quad u^\tau = (\cos\psi^\tau, \sin\psi^\tau), \\ w_\varepsilon = (\cos\phi_\varepsilon^*, \sin\phi_\varepsilon^*), \quad u^* = (\cos\psi^*, \sin\psi^*), \\ \frac{u'_\varepsilon}{|u'_\varepsilon|}|_{\partial B} = (\cos\phi_\varepsilon, \sin\phi_\varepsilon), \quad u'_p|_{\partial B} = (\cos\psi, \sin\psi), \end{gathered}$$

where $\phi_\varepsilon^\tau, \psi^\tau, \phi_\varepsilon^*, \psi^*$ belong to $W^{1,p}(B,R)$, ϕ^*, ψ belong to $W^{1,p}(\partial B, R)$, and they are all single-valued functions since their degrees around ∂B are zero. Therefore,

$$\phi_\varepsilon^\tau|_{\partial B} = \phi_\varepsilon, \quad \psi^\tau|_{\partial B} = \psi, \tag{3.17}$$

and $|\nabla w_\varepsilon^\tau| = |\nabla \phi_\varepsilon^\tau|$, $|\nabla u^\tau| = |\nabla \psi^\tau|$. $|\nabla w_\varepsilon| = |\nabla \phi_\varepsilon^*|$, $|\nabla u^*| = |\nabla \psi^*|$. Moreover, by (3.6) and (3.15), we obtain that both ϕ_ε^τ and ψ^τ satisfy $-\operatorname{div}[(|\nabla\Phi|^2+\tau)^{(p-2)/2}\nabla\Phi]=0$. Thus,

$$-\operatorname{div}[(|\nabla\phi_\varepsilon^\tau|^2+\tau)^{(p-2)/2}\nabla\phi_\varepsilon^\tau-(|\nabla\psi^\tau|^2+\tau)^{(p-2)/2}\nabla\psi^\tau]=0. \tag{3.18}$$

Multiplying both sides of (3.18) by $\phi_\varepsilon^\tau-\psi^\tau$ and integrating over B, we obtain

$$\begin{aligned} &-\int_{\partial B}(v_\varepsilon^{\tau\,(p-2)/2}\phi_\nu - v^{(p-2)/2}\psi_\nu)(\phi-\psi)d\xi \\ &+\int_B (v_\varepsilon^{\tau\,(p-2)/2}\nabla\phi - v^{(p-2)/2}\nabla\psi)\nabla(\phi-\psi)dx=0, \end{aligned} \tag{3.19}$$

where ν denotes the unit outside-norm vector of ∂B.

Let $w=w_\varepsilon^\tau$ be a solution of (3.5). Integrating both sides of (3.6) over B, we have

$$-\int_{\partial B} v_\varepsilon^{\tau\,(p-2)/2} w_\nu d\xi = \int_B w|\nabla w|^2 v_\varepsilon^{\tau\,(p-2)/2}dx,$$

this and (3.7) imply

$$\Big|\int_{\partial B} v_\varepsilon^{\tau\,(p-2)/2}\phi_\nu d\xi\Big| = \Big|\int_{\partial B} v_\varepsilon^{\tau\,(p-2)/2} w_\nu d\xi\Big| \le \int_B v_\varepsilon^{\tau\, p/2}dx \le C. \tag{3.20}$$

An analogous discussion shows that for the solution $u=u^\tau$ of (3.13) which is equipped with (3.14), we may also obtain

$$\Big|\int_{\partial B} v^{(p-2)/2}\psi_\nu d\xi\Big| = \Big|\int_{\partial B} v^{(p-2)/2} u_\nu d\xi\Big| \le \int_B |\nabla u|^p dx \le C. \tag{3.21}$$

Combining (3.17) with (3.19)-(3.21), we derive

$$\int_B (v_\varepsilon^{\tau\,(p-2)/2}\nabla\phi - v^{(p-2)/2}\nabla\psi)\nabla(\phi-\psi)dx \le C\sup_{\partial B}|\phi_\varepsilon^\tau-\psi^\tau| = C\sup_{\partial B}|\phi_\varepsilon-\psi|,$$

where C is independent of ε,τ. Letting $\tau\to 0$ and applying (3.12) and (3.16), we obtain

$$\Big|\int_B (|\nabla\phi_\varepsilon^*|^{(p-2)/2}\nabla\phi_\varepsilon^* - |\nabla\psi^*|^{(p-2)/2}\nabla\psi^*)\nabla(\phi_\varepsilon^*-\psi^*)dx\Big| \le C\sup_{\partial B}|\phi_\varepsilon-\psi|,$$

which implies $\int_B|\nabla\phi_\varepsilon^*-\nabla\psi^*|^p dx \le C\sup_{\partial B}|\phi_\varepsilon-\psi|$. Letting $\varepsilon\to 0$ and using (3.1), we obtain $\int_B|\nabla\phi_\varepsilon^*|^p dx \to \int_B |\nabla\psi^*|^p dx$. That is,

$$\int_B |\nabla w_\varepsilon|^p dx \to \int_B |\nabla u^*|^p dx. \tag{3.22}$$

Step 6. Since $\int_B|\nabla u|^p dx$ is weak lower semi-continuous, from Theorem 1.2 we deduce $\int_B|\nabla u_p|^p dx \le \liminf_{\varepsilon_k\to 0}\int_B|\nabla u_{\varepsilon_k}|^p dx$. Combining this result with (3.4), (3.11) and (3.22), we obtain

$$\begin{aligned} \int_B|\nabla u_p|^p dx &\le \liminf_{\varepsilon_k\to 0}\int_B|\nabla u_{\varepsilon_k}|^p dx \le \limsup_{\varepsilon_k\to 0}\int_B|\nabla u_{\varepsilon_k}|^p dx \\ &\le \lim_{\varepsilon_k\to 0}\int_B|\nabla w_\varepsilon|^p dx = \int_B|\nabla u^*|^p dx. \end{aligned}$$

Recalling the definition of u^* in Step 4, and noticing $u'_p \in W^{1,p}_{u'_p}(B, \partial B_1)$, we know that u'_p is also a minimizer of $\int_B |\nabla u|^p$, and

$$\lim_{\varepsilon_k \to 0} \int_B |\nabla u_{\varepsilon_k}|^p dx = \int_B |\nabla u_p|^p dx = \int_B |\nabla u^*|^p dx.$$

This result and Theorem 1.2 imply $\nabla u_{\varepsilon_k} \to \nabla u_p$ in $L^p(B, R^3)$. when $\varepsilon_k \to 0$. Combining this with the fact $u_{\varepsilon_k} \to u_p$ in $L^p(B, R^3)$, which is implied by Theorem 1.2, we obtain

$$u_{\varepsilon_k} \to u_p, \quad \text{in } W^{1,p}(B, R^3)$$

as $\varepsilon_k \to 0$. Then it is not difficult to complete the proof of this theorem.

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