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# NON-EXTINCTION OF SOLUTIONS TO A FAST DIFFUSION SYSTEM WITH NONLOCAL SOURCES

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Abstract. In this short article, we give a positive answer to the problem proposed by Zheng et al [5], and show that the fast diffusion system

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \int_\Omega v^\alpha \mathrm{d}x,$$
$$v_t = \operatorname{div}(|\nabla v|^{q-2}\nabla v) + \int_\Omega u^\beta \mathrm{d}x$$

under homogeneous Dirichlet boundary condition admits at least one non-extinction solution when $\alpha\beta < (p-1)(q-1)$ and the initial data are strictly positive.

## 1. Introduction

This short note concerns the non-extinction properties of solutions to the fast diffusion parabolic system

$$\begin{gathered}
u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \int_\Omega v^\alpha \mathrm{d}x, \quad x \in \Omega,\ t > 0, \\
v_t = \operatorname{div}(|\nabla v|^{q-2}\nabla v) + \int_\Omega u^\beta \mathrm{d}x, \quad x \in \Omega,\ t > 0, \\
u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where $1 < p$, $q < 2$, $\alpha$, $\beta > 0$, $\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$ and the initial data $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, $v_0 \in L^\infty(\Omega) \cap W_0^{1,q}(\Omega)$.

We refer to [5] and the references therein for the motivation of studying problem (1.1). In particular, the authors in [5] investigated the extinction properties of solutions to the above problem. More precisely, by combining the methods of energy estimates with the comparison principle they showed that if $\alpha\beta > (p-1)(q-1)$, then every weak solution of problem (1.1) vanishes in finite time when the initial data are comparable in some sense; if $\alpha\beta = (p-1)(q-1)$ and the diameter of the domain $\Omega$ is sufficiently small, then problem (1.1) admits at least one extinction solution for small initial data. However, for the case $\alpha\beta < (p-1)(q-1)$, they did not

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give any result and conjectured that problem (1.1) should admit at least one non-extinction solution for any nonnegative initial data. Since to give some sufficient conditions for the non-extinction of solutions to systems like (1.1) is much more challenging, one can not expect a full answer to this problem. In this short note, we give a partial answer to the problem proposed by Zheng et al.

It is well known that the equations in (1.1) are singular when $1 < p,\ q < 2$, and hence there is no classical solution in general. Therefore, we have to consider its solutions in some weak sense. We first introduce some notation which will be used throughout this paper. For any $T \in (0, \infty)$ and $0 < t_1 < t_2 < \infty$, we denote $Q_T = \Omega \times (0, T)$, $\Gamma_T = \partial\Omega \times (0, T)$ and

$$
\begin{gathered}
Q = \Omega \times (0, \infty), \quad Q_{(t_1,t_2)} = \Omega \times (t_1, t_2), \\
E_\beta = \big\{w \in L^{2\beta}(Q_T) \cap L^2(Q_T);\ \frac{\partial w}{\partial t} \in L^2(Q_T), \nabla w \in L^p(Q_T)\big\}, \\
E_\alpha = \big\{w \in L^{2\alpha}(Q_T) \cap L^2(Q_T);\ \frac{\partial w}{\partial t} \in L^2(Q_T), \nabla w \in L^q(Q_T)\big\}, \\
E_p = \big\{w \in L^2(Q_T); \nabla w \in L^p(Q_T)\big\}, \quad E_q = \big\{w \in L^2(Q_T); \nabla w \in L^q(Q_T)\big\}, \\
E_{p0} = \big\{w \in E_p;\ w\big|_{\partial\Omega} = 0\big\}, \quad E_{q0} = \big\{w \in E_q;\ w\big|_{\partial\Omega} = 0\big\}.
\end{gathered}
$$

**Definition 1.1.** A nonnegative vector-valued function $(u, v)$ with $u \in E_\beta$ and $v \in E_\alpha$ is called a nonnegative subsolution of (1.1) in $Q_T$ provided that for any $0 \le \varphi_1 \in E_{p0}$ and $0 \le \varphi_2 \in E_{q0}$

$$
\begin{gathered}
\iint_{Q_T} \Big(\frac{\partial u}{\partial t}\varphi_1 + |\nabla u|^{p-2}\nabla u \nabla\varphi_1\Big)\,\mathrm{d}x\,\mathrm{d}\tau \le \iint_{Q_T} \Big(\int_\Omega v^\alpha(y,\tau)\,\mathrm{d}y\Big)\varphi_1(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau, \\
\iint_{Q_T} \Big(\frac{\partial v}{\partial t}\varphi_2 + |\nabla v|^{q-2}\nabla v \nabla\varphi_2\Big)\,\mathrm{d}x\,\mathrm{d}\tau \le \iint_{Q_T} \Big(\int_\Omega u^\beta(y,\tau)\,\mathrm{d}y\Big)\varphi_2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau, \\
u(x,t) \le 0, \quad v(x,t) \le 0, \quad x \in \Gamma_T, \\
u(x,0) \le u_0(x), \quad v(x,0) \le v_0(x), \quad x \in \Omega.
\end{gathered}
$$

By replacing $\le$ by $\ge$ in the above inequalities we obtain the definition of weak supersolutions of (1.1). Furthermore, if $(u, v)$ is a weak supersolution as well as a weak subsolution, then we call it a weak solution of problem (1.1).

Before stating our main result, we first denote by $\phi_1$ and $\phi_2$ the unique solution of the following quasilinear elliptic problems

$$
-\operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) = 1, \quad x \in \Omega; \quad \phi(x) = 0, \quad x \in \partial\Omega, \tag{1.2}
$$

and

$$
-\operatorname{div}(|\nabla\psi|^{q-2}\nabla\psi) = 1, \quad x \in \Omega; \quad \psi(x) = 0, \quad x \in \partial\Omega; \tag{1.3}
$$

respectively. It is known from the strong maximum principle (see [4]) and the regularity theory in the standard $p$-Laplace elliptic equations (see [1]) that both $\phi_1$ and $\phi_2$ are strictly positive in $\Omega$ and belong to $C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$. Denote $M_i = \max_{x\in\overline{\Omega}} \phi_i(x),\ i = 1, 2,\ \mu_1 = \int_\Omega \phi_1^\beta(x)\,\mathrm{d}x$ and $\mu_2 = \int_\Omega \phi_2^\alpha(x)\,\mathrm{d}x$. Finally we define

$$
S_1 = \{u \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) : u(x) \ge k\phi_1(x) \text{ for some } k > 0,\ x \in \Omega\},
$$

and

$$
S_2 = \{u \in L^\infty(\Omega) \cap W_0^{1,q}(\Omega) : u(x) \ge k\phi_2(x) \text{ for some } k > 0,\ x \in \Omega\}.
$$

Our main result reads as follows.

**Theorem 1.2.** *Assume that $1 < p,\ q < 2$ and $\alpha\beta < (p-1)(q-1)$. Then problem (1.1) admits at least one non-extinction solution for any initial data $(u_0, v_0) \in S_1 \times S_2$.*

*Proof.* Similar to the corresponding results of the one-equation model (see [3]), this theorem will be proved by constructing a pair of ordered super and subsolution and utilizing the monotonic iteration process. The whole process is divided into four steps.

**Step 1.** We first construct a non-extinction subsolution of problem (1.1). Since $\alpha\beta < (p-1)(q-1)$, there exists two positive constants $\theta_1, \theta_2$ such that

$$\frac{\alpha}{p-1} < \frac{\theta_1}{\theta_2} < \frac{q-1}{\beta}. \tag{1.4}$$

Define $\underline{u} = k^{\theta_1}\phi_1(x)$, $\underline{v} = k^{\theta_2}\phi_2(x)$, where $k > 0$ will be fixed later. By direct computation we see that $(\underline{u}, \underline{v})$ satisfies (in the weak sense)

$$\underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) - \int_\Omega \underline{v}^\alpha \,\mathrm{d}x = k^{\theta_1(p-1)} - k^{\theta_2\alpha}\mu_2, \tag{1.5}$$

$$\underline{v}_t - \operatorname{div}(|\nabla \underline{v}|^{q-2}\nabla \underline{v}) - \int_\Omega \underline{u}^\beta \,\mathrm{d}x = k^{\theta_2(q-1)} - k^{\theta_1\beta}\mu_1. \tag{1.6}$$

Combining (1.5), (1.6) with (1.4) we know that there exists a constant $k_1 > 0$ such that for all $k \in (0, k_1]$, the following relations hold

$$\begin{gathered}
\underline{u}_t - \operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) - \int_\Omega \underline{v}^\alpha \,\mathrm{d}x \le 0, \quad x \in \Omega,\ t > 0, \\
\underline{v}_t - \operatorname{div}(|\nabla \underline{v}|^{q-2}\nabla \underline{v}) - \int_\Omega \underline{u}^\beta \,\mathrm{d}x \le 0, \quad x \in \Omega,\ t > 0.
\end{gathered} \tag{1.7}$$

On the other hand, since $(u_0, v_0) \in S_1 \times S_2$, there exists a constant $k_2 > 0$ such that for all $k \in (0, k_2]$ we have

$$u_0(x) \ge k^{\theta_1}\phi_1(x), \quad v_0(x) \ge k^{\theta_2}\phi_2(x), \quad x \in \Omega. \tag{1.8}$$

Therefore, from (1.7) and (1.8) we know that $(\underline{u}, \underline{v})$ is a non-extinction weak subsolution of (1.1) for all $0 < k \le \min\{k_1, k_2\}$.

**Step 2.** To construct a supersolution of (1.1), let us consider the auxiliary system

$$\begin{gathered}
u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \int_\Omega (v_+ + 1)^\alpha \,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \\
v_t = \operatorname{div}(|\nabla v|^{q-2}\nabla v) + \int_\Omega (u_+ + 1)^\beta \,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \\
u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega.
\end{gathered} \tag{1.9}$$

Here $s_+ = \max\{s, 0\}$. By applying the standard regularization and a priori estimates methods (see [2] for instance) we know that problem (1.9) admits a weak solution $(\overline{u}, \overline{v})$. By the weak maximum principle it is known that $(\overline{u}, \overline{v})$ is nonnegative. Moreover, $(\overline{u}, \overline{v})$ exists globally and is locally bounded if $\alpha\beta \le 1$. If we can show that $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$, then there exists a solution $(u, v)$ of (1.1) satisfying $(\underline{u}, \underline{v}) \le (u, v) \le (\overline{u}, \overline{v})$.

**Step 3.** We will show that $(\underline{u},\underline{v}) \le (\overline{u},\overline{v})$. For this, fix $T \in (0,\infty)$. From the definition of weak super and subsolutions, we obtain, for any $0 \le \varphi_1 \in E_{p0}$ and $0 \le \varphi_2 \in E_{q0}$,

$$\begin{aligned}
&\iint_{Q_T} \Big(\frac{\partial \underline{u}}{\partial t} - \frac{\partial \overline{u}}{\partial t}\Big)\varphi_1 \,\mathrm{d}x\,\mathrm{d}\tau + \iint_{Q_T} (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla\varphi_1 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \iint_{Q_T}\int_\Omega \Big[\underline{v}^\alpha(y,\tau) - (\overline{v}_+(y,\tau)+1)^\alpha\Big]\,\mathrm{d}y\varphi_1 \,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned} \tag{1.10}$$

$$\begin{aligned}
&\iint_{Q_T} \Big(\frac{\partial \underline{v}}{\partial t} - \frac{\partial \overline{v}}{\partial t}\Big)\varphi_2 \,\mathrm{d}x\,\mathrm{d}\tau + \iint_{Q_T} (|\nabla \underline{v}|^{q-2}\nabla \underline{v} - |\nabla \overline{v}|^{q-2}\nabla \overline{v})\nabla\varphi_2 \ ,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \iint_{Q_T}\int_\Omega \Big[\underline{u}^\beta(y,\tau) - (\overline{u}_+(y,\tau)+1)^\beta\Big]\,\mathrm{d}y\varphi_2 \,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{1.11}$$

By Lagrange mean value theorem we know that if $0 < \alpha < 1$, then there exists a $\xi$ between $\underline{v}$ and $\overline{v}_+ + 1$ such that

$$[\underline{v}^\alpha(y,\tau) - (\overline{v}_+(y,\tau)+1)^\alpha]_+ = \alpha\xi^{\alpha-1}[\underline{v} - (\overline{v}_+ + 1)]_+ \le \alpha(\underline{v} - \overline{v})_+; \tag{1.12}$$

if $\alpha = 1$, then

$$\underline{v} - (\overline{v}_+ + 1) \le (\underline{v} - \overline{v})_+; \tag{1.13}$$

if $\alpha > 1$, then there exists an $\eta$ between $\underline{v}$ and $\overline{v}_+ + 1$ such that

$$\begin{aligned}
[\underline{v}^\alpha(y,\tau) - (\overline{v}_+(y,\tau)+1)^\alpha]_+ &= \alpha\eta^{\alpha-1}[\underline{v} - (\overline{v}_+ + 1)]_+ \\
&\le \alpha k^{\theta_2(\alpha-1)} M_2^{\alpha-1}(\underline{v} - \overline{v})_+.
\end{aligned} \tag{1.14}$$

Noticing that both $\underline{u}$ and $\overline{u}$ belong to $E_p$ and $\underline{u} \le 0 \le \overline{u}$ on $\partial\Omega \times (0,T)$, it is not hard to check that $\varphi_1 = \chi_{[0,t]}(\underline{u} - \overline{u})_+ \in E_{p0}$ for any $t \in (0,T)$. Taking $\varphi_1 = \chi_{[0,t]}(\underline{u} - \overline{u})_+$ for any $t \in (0,T)$ and noticing (1.12)-(1.14), we see by simple computation that there exists a constant $C_1 > 0$ depending only on $\alpha$, $k$, $\theta_2$ and $M_2$ such that

$$\begin{aligned}
&\int_\Omega (\underline{u} - \overline{u})_+^2 \,\mathrm{d}x + 2\iint_{Q_t} (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla(\underline{u} - \overline{u})_+ \,\mathrm{d}x\,\mathrm{d}\tau \\
&\le C_1 \int_0^t \Big(\int_\Omega (\underline{v} - \overline{v})_+ \,\mathrm{d}x \int_\Omega (\underline{u} - \overline{u})_+ \,\mathrm{d}x\Big)\,\mathrm{d}\tau \\
&\le \frac{C_1|\Omega|}{2}\Big(\iint_{Q_t} (\underline{v} - \overline{v})_+^2 \,\mathrm{d}x\,\mathrm{d}\tau + \iint_{Q_t} (\underline{u} - \overline{u})_+^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big).
\end{aligned}$$

Symmetrically, we also have

$$\begin{aligned}
&\int_\Omega (\underline{v} - \overline{v})_+^2 \,\mathrm{d}x + 2\iint_{Q_t} (|\nabla \underline{v}|^{q-2}\nabla \underline{v} - |\nabla \overline{v}|^{q-2}\nabla \overline{v})\nabla(\underline{v} - \overline{v})_+ \,\mathrm{d}x\,\mathrm{d}\tau \\
&\le C_2 \int_0^t \Big(\int_\Omega (\underline{v} - \overline{v})_+ \,\mathrm{d}x \int_\Omega (\underline{u} - \overline{u})_+ \,\mathrm{d}x\Big)\,\mathrm{d}\tau \\
&\le \frac{C_2|\Omega|}{2}\Big(\iint_{Q_t} (\underline{v} - \overline{v})_+^2 \,\mathrm{d}x\,\mathrm{d}\tau + \iint_{Q_t} (\underline{u} - \overline{u})_+^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big),
\end{aligned}$$

for some $C_2 > 0$ depending only on $\beta$, $k$, $\theta_1$ and $M_1$. Noticing the monotonicity of $p$-Laplace operator we obtain that

$$\int_\Omega [(\underline{u} - \overline{u})_+^2 + (\underline{v} - \overline{v})_+^2]\,\mathrm{d}x \le C \iint_{Q_t} [(\underline{u} - \overline{u})_+^2 + (\underline{v} - \overline{v})_+^2]\,\mathrm{d}x\,\mathrm{d}\tau.$$

Thus, the desired result follow from the above inequality and Gronwall's inequality.

**Step 4.** Define $(u_1, v_1) = (\underline{u}, \underline{v})$ and $\{(u_k, v_k)\}_{k\geq 2}$ iteratively to be a solution of the following problem

$$\begin{gathered} u_{kt} = \operatorname{div}(|\nabla u_k|^{p-2}\nabla u_k) + \int_\Omega v_{k-1}^\alpha \,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \\ v_{kt} = \operatorname{div}(|\nabla v_k|^{p-2}\nabla v_k) + \int_\Omega u_{k-1}^\beta \,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \end{gathered} \tag{1.15}$$

By induction we can prove that $(u_k, v_k) \leq (u_{k+1}, v_{k+1})$ and $(u_k, v_k) \leq (\overline{u}, \overline{v})$ for all $k \geq 1$. Thus the limits $u(x,t) = \lim_{k\to\infty} u_k(x,t)$ and $v(x,t) = \lim_{k\to\infty} v_k(x,t)$ exist for every $x \in \Omega$ and $t > 0$ and it is not hard to show that $(u, v)$ is a weak solution of (1.1) by the regularities of $\{(u_k, v_k)\}_{k\geq 2}$. Therefore, $(u, v)$ is a non-extinction solution of (1.1) since $(u, v) \geq (\underline{u}, \underline{v})$. The proof is complete. $\square$

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