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GLOBAL FAST AND SLOW SOLUTIONS OF A SINGLE-SPECIES BACILLUS SYSTEM WITH FREE BOUNDARY

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Abstract. In this article, we consider a free boundary problem for a reaction diffusion equation which describes the dynamics of single bacillus population in higher space dimensions and heterogeneous environment. For simplicity, we assume that the environment and solution are radially symmetric. First, by using the contraction mapping theorem, we prove that the local solution exists and is unique. Then, some sufficient conditions are given under which the solution will blow up in finite time. Our results indicate that the blowup occurs if the initial data are sufficiently large. Finally, the long time behavior of the global solution is discussed. It is shown that the global fast solution does exist if the initial data are sufficiently small, while the global slow solution is possible if the initial data are suitably large.

1. Introduction

It is well known that mathematical aspects of biological population have been considered widely. Most authors have studied growth and diffusions of biological population in a homogeneous or heterogeneous fixed environment (see [7, 27]). They have also studied the nonlinear differential equations involved such as the Logistic and Fisher equation.

In this article, we consider the single bacillus population model

$$u_t - d\Delta u = Kau^2 - bu, \tag{1.1}$$

which was first proposed by Verhulst in [22]. Parameters a, b, d and K are positive constants. Ecologically, a represents the net birth rate, b is the death rate, d denotes the diffusion coefficient, and K measures the living resource for bacillus. Recently, Jin et al [15] considered this model and established a time-dependent dynamic basis to quantitatively clarify the biological wave behavior of the popular growth and propagation. And Ling and Lin in [18] investigated equation (1.1) with a moving boundary in one-dimensional space.

The main purpose of this paper is to show that the results obtained in [18] continue to hold in higher-dimensional space and heterogeneous environment. For simplicity, we assume that the environment and the solution are radially symmetric. So we are primarily interested in the positive solution $u(r,t)$, $r = |x|$, $x \in \mathbb{R}^N$

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($N \geq 2$), to the problem

$$\begin{gathered} u_t - d\Delta u = Kau^2 - bu, \quad 0 < r < h(t), t > 0, \\ u_r(0,t) = u(h(t),t) = 0, \quad t > 0, \\ h'(t) = -\mu u_r(h(t),t), \quad t > 0, \\ h(0) = h_0, \quad u(r,0) = u_0(r), \quad 0 < r \leq h_0, \end{gathered} \tag{1.2}$$

where $\Delta u = u_{rr} + \frac{N-1}{r} u_r$, $r = h(t)$ is the moving boundary to be determined together with the solution $u(r,t)$, and the initial function $u_0(r)$ satisfies

$$u_0 \in C^2([0,h_0]), \quad u_0'(0) = u_0(h_0) = 0, \quad u_0(r) > 0, \quad r \in (0,h_0). \tag{1.3}$$

When $Ka = b = 0$, the problem is reduced to one phase Stefan problem, which accounts for phase transitions between solid and fluid states such as the melting of ice in contact with water [25]. Stefan problems have been studied by many authors. For example, the weak solution was considered by Oleǐnik in [23], and the existence of a classical solution was given by Kinderlehrer and Nirenberg in [16]. For the two-phase Stefan problem, the local classical solution was obtained in [19, 24] and the global classical solution was given by Borodin in [2].

The free boundary problems have been investigated in many areas, for example, the decrease of oxygen in a muscle in the vicinity of a clotted blood vessel [5], the etching problem [29], the combustion process [30], the American option pricing problem [12, 14], chemical vapor deposition in hot wall reactor [20], image processing [1], wound healing [3], tumor growth [4, 6, 28, 31], and the dynamics of population [13, 21].

Definition 1.1. Let $D_T = \{(r,t) \in \mathbb{R}^2 : 0 < r < h(t), 0 < t \leq T\}$. We say $(u(r,t), h(t))$ is a solution of (1.2) if $u(r,t) \in C^{1,0}(\overline{D}_T) \cap C^{2,1}(D_T)$, $h(t) \in C^1([0,T])$ and $u(r,t)$, $h(t)$ satisfy all the equations in (1.2).

For problem (1.2), we lay great emphasis on the blowup property of the solution and the long time behaviors of global solutions, especially the existence of global slow solution. In this paper, we define $T_{\max}$ as the maximal existence time of the solution of problem (1.2). We say that the solution exists globally if $T_{\max} = +\infty$, whereas if the solution ceases to exist for some finite time, i.e., $T_{\max} < +\infty$ and $\limsup_{t \to T_{\max}} \|u(\cdot,t)\|_{L^\infty([0,h(t)])} = +\infty$, we say that blowup occurs. If $T_{\max} = \infty$ and $\lim_{t\to\infty} h(t) < \infty$, then the solution is called global fast solution since it decays uniformly to 0 at an exponential rate, while if $T_{\max} = \infty$ and $\lim_{t\to\infty} h(t) = \infty$, it is called global slow solution, whose decay rate is at most polynomial (see [10, 11] for more details).

The outline of this paper is as follows. In Section 2, we first apply the contraction mapping theorem to prove the local existence and uniqueness of the solution to (1.2), and then present some fundamental results that will be used later. Section 3 is devoted to the investigation of the blowup result. Our arguments are based on the comparison principle and the construction of an appropriate lower solution to (1.2). In Section 4, we deal with the long time behavior of global solutions, including the existence of global fast solution and slow solution.

2. Local existence and comparison principle

In this section, by using the contraction mapping theorem, we first prove the following local existence and uniqueness result.

Theorem 2.1. *For any given u_0 satisfying (1.3) and any $\alpha \in (0,1)$, there is a $T>0$ such that (1.2) admits a unique solution*

$$(u,h) \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_T) \times C^{1+\alpha/2}([0,T]), \tag{2.1}$$

where $D_T = \{(r,t) \in \mathbb{R}^2 : 0 < r < h(t), 0 < t \le T\}$. Moreover,

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_T)} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C, \tag{2.2}$$

where C and T depend only on α, h_0 and $\|u_0\|_{C^2[0,h_0]}$.

Proof. We first make a change of variable to straighten the free boundary by the transformations

$$\xi = \frac{h_0 r}{h(t)}, \quad u(r,t) = v(\xi,t). \tag{2.3}$$

Direct computations yield

$$u_t = v_t - \frac{h'(t)}{h(t)}\xi v_\xi, \quad u_r = \frac{h_0}{h(t)} v_\xi, \quad u_{rr} = \frac{h_0^2}{h^2(t)} v_{\xi\xi}.$$

Then problem (1.2) can be reduced to

$$\begin{gathered}
v_t - \frac{h'(t)}{h(t)}\xi v_\xi - \frac{dh_0^2}{h^2(t)}(v_{\xi\xi} + \frac{N-1}{\xi}v_\xi) = Kav^2 - bv, \quad 0<\xi<h_0, t>0,\\
v_\xi(0,t) = v(h_0,t) = 0, \quad t>0,\\
h'(t) = -\frac{\mu h_0}{h(t)} v_\xi(h_0,t), \quad t>0,\\
h(0) = h_0, \quad v(\xi,0) = v_0(\xi) := u_0(\xi), \quad 0 \le \xi \le h_0.
\end{gathered} \tag{2.4}$$

Transformations (2.3) change the free boundary $x = h(t)$ to the fixed line $\xi = h_0$ at the expense of making the equation more complicated, since the coefficients of the first equation in (2.4) include the unknown function $h(t)$.

Let $T < 1$ be a positive constant to be determined later, we denote $h^* = -\mu h_0 v_0'(h_0)$ and set

$$\begin{gathered}
H_T = \{h \in C^1[0,T] : h(0) = h_0, h'(0)h_0 = h^*, 0 \le h'(t)h(t) \le h^* + 1\},\\
U_T = \{v \in C([0,h_0] \times [0,T]) : v(\xi,0) = v_0(\xi), \|v - v_0\|_{C([0,h_0]\times[0,T])} \le 1\}.
\end{gathered} \tag{2.5}$$

Then it is easy to see that $\Sigma_T := U_T \times S_T$ is a complete metric space with the metric

$$\mathcal{D}((v_1,h_1),(v_2,h_2)) = \|v_1 - v_2\|_{C([0,h_0]\times[0,T])} + \|h_1'h_1 - h_2'h_2\|_{C([0,T])} \tag{2.6}$$

Let us note that although $\frac{N-1}{\xi}$ is singular at $\xi = 0$, $v_{\xi\xi} + \frac{N-1}{\xi}v_\xi$ actually represents an elliptic operator acting on $v = v(y,t)$ $(= v(|y|,t))$ over the ball $|y| \le h_0$ in view of $v_{\xi\xi} + \frac{N-1}{\xi}v_\xi = \Delta v$ for $|y| \le h_0$. Therefore, we can apply the standard L^p theory (see [17]) and the Sobolev imbedding theorem (see [8, Chaper 5]) to obtain that for any $(v,h) \in \Sigma_T$, the following initial boundary value problem

$$\begin{gathered}
\tilde v_t - \frac{h'(t)}{h(t)}\xi \tilde v_\xi - \frac{dh_0^2}{h^2(t)}(\tilde v_{\xi\xi} + \frac{N-1}{\xi}\tilde v_\xi) = Kav^2 - bv, \quad 0<\xi<h_0, t>0,\\
\tilde v_\xi(0,t) = \tilde v(h_0,t) = 0, \quad t>0,\\
\tilde v(\xi,0) = v_0(\xi) \ge 0, \quad 0 \le \xi \le h_0.
\end{gathered} \tag{2.7}$$

admits a unique solution $\tilde{v} \in C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])$ and

$$\begin{aligned}
&\|\tilde{v}\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])}\\
&\le \|\tilde{v}-v_0\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])} + \|v_0\|_{C^{1+\alpha}([0,h_0])}\\
&\le C_0\|\tilde{v}-v_0\|_{W_p^{2,1}([0,h_0]\times[0,T])} + \|u_0\|_{C^{1+\alpha}([0,h_0])}\\
&\le C_0(\|\tilde{v}\|_{W_p^{2,1}([0,h_0]\times[0,T])} + \|v_0\|_{W_p^{2,1}([0,h_0]\times[0,T])}) + \|u_0\|_{C^{1+\alpha}([0,h_0])}\\
&\le C_0(\|\tilde{v}\|_{W_p^{2,1}([0,h_0]\times[0,T])} + T^{\frac{1}{p}}h_0^{\frac{1}{p}}\|u_0\|_{C^2([0,h_0])}) + \|u_0\|_{C^{1+\alpha}([0,h_0])}\\
&\le C_1,
\end{aligned} \tag{2.8}$$

where $p=3/(1-\alpha)$, C_0 is a Sobolev imbedding constant, C_1 is a constant depending on C_0,α,h_0 and $\|u_0\|_{C^2[0,h_0]}$.

From the third equation in (2.4), we can define a function $\tilde{h}(>0)$ as follows

$$\tilde{h}^2(t) = h_0^2 - 2\mu h_0\int_0^t \tilde{v}_\xi(h_0,\tau)d\tau, \tag{2.9}$$

which implies

$$\tilde{h}'(t)\tilde{h}(t) = -\mu h_0\tilde{v}_\xi(h_0,t), \quad \tilde{h}(0)=h_0, \quad \tilde{h}'(0)h_0 = -\mu h_0\tilde{v}_\xi(h_0,0) = h^*. \tag{2.10}$$

Hence $\tilde{h}'\tilde{h} \in C^{\alpha/2}([0,T])$ with

$$\|\tilde{h}'(t)\tilde{h}(t)\|_{C^{\alpha/2}([0,T])} \le C_2 := \mu h_0 C_1. \tag{2.11}$$

Now, we define a map $\mathcal{F}:\Sigma_T \to C([0,h_0]\times[0,T])\times C^1([0,T])$ by

$$\mathcal{F}(v(\xi,t),h(t)) = (\tilde{v}(\xi,t),\tilde{h}(t)). \tag{2.12}$$

It is easy to see that $(v,h)\in\Sigma_T$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.4). So, we need to prove that $\mathcal{F}$ has a unique fixed point, we use the contraction mapping theorem.

By (2.10) and (2.11), we have

$$\begin{gathered}
\|\tilde{h}'(t)\tilde{h}(t) - \tilde{h}'(0)\tilde{h}(0)\|_{C([0,T])} \le \|\tilde{h}'(t)\tilde{h}(t)\|_{C^{\alpha/2}([0,T])}T^{\alpha/2} \le C_2T^{\alpha/2},\\
\|\tilde{v}(\xi,t)-v_0(\xi)\|_{C([0,h_0]\times[0,T])} \le \|\tilde{v}\|_{C^{0,(1+\alpha)/2}([0,h_0]\times[0,T])}T^{(1+\alpha)/2} \le C_1T^{(1+\alpha)/2}.
\end{gathered}$$

Therefore if we take $T\le\min\{1,C_2^{-2/\alpha},C_1^{-2/(1+\alpha)}\}$, then $\mathcal{F}$ maps Σ_T into itself.

To prove that $\mathcal{F}$ is a contraction mapping on Σ_T for $T>0$ sufficiently small, we take $(v_i,h_i)\in\Sigma_T, i=1,2$ and denote $(\overline{v}_i,\overline{h}_i)=\mathcal{F}(v_i,h_i)$. Then it follows form (2.8) and (2.11) that

$$\|\overline{v}_i\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])}, \quad \|\overline{h}_i'(t)\overline{h}_i(t)\|_{C^{\alpha/2}([0,T])} \le C_2. \tag{2.13}$$

Set $w=\overline{v}_1-\overline{v}_2$, then $w(\xi,t)$ satisfies

$$\begin{gathered}
w_t - \frac{h_1'}{h_1}\xi w_\xi - \frac{dh_0^2}{h_1^2}(w_{\xi\xi}+\frac{N-1}{\xi}w_\xi) - Ka(v_1^2-v_2^2) + b(v_1-v_2)\\
= (\frac{h_1'}{h_1}-\frac{h_2'}{h_2})\xi\overline{v}_{2\xi} + dh_0^2(\frac{1}{h_1^2}-\frac{1}{h_2^2})(\overline{v}_{2\xi\xi}+\frac{N-1}{\xi}\overline{v}_{2\xi}), \quad 0<\xi<h_0, 0<t<T,\\
w_\xi(0,t)=0, \quad w(h_0,t)=0, \quad 0<t<T,\\
w(\xi,0)=0, \quad 0\le\xi\le h_0.
\end{gathered}$$

Using the L^p estimates for parabolic equations and Sobolev imbedding theorems, we obtain

$$\begin{aligned}&\|\overline{v}_1-\overline{v}_2\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])}\\&\le C_3(\|v_1-v_2\|_{C([0,h_0]\times[0,T])}+\|h_1-h_2\|_{C^1([0,T])}),\end{aligned}\tag{2.14}$$

where C_3 depends on C_1 and C_2. Taking the difference of the equations for $\overline{h}_1'\overline{h}_1$ and $\overline{h}_2'\overline{h}_2$ results in

$$\|\overline{h}_1'(t)\overline{h}_1(t)-\overline{h}_2'(t)\overline{h}_2(t)\|_{C^{\alpha/2}([0,T])}\le\mu h_0\|\overline{v}_{1\xi}(h_0,t)-\overline{v}_{2\xi}(h_0,t)\|_{C^{\alpha/2}([0,T])}.\tag{2.15}$$

Combining inequalities (2.14) and (2.15), we obtain

$$\begin{aligned}&\|\overline{v}_1(\xi,t)-\overline{v}_2(\xi,t)\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])}+\|\overline{h}_1'(t)\overline{h}_1(t)-\overline{h}_2'(t)\overline{h}_2(t)\|_{C^{\alpha/2}([0,T])}\\&\le C_4(\|v_1(\xi,t)-v_2(\xi,t)\|_{C([0,h_0]\times[0,T])}+\|h_1-h_2\|_{C^1([0,T])}),\end{aligned}\tag{2.16}$$

where C_4 depends on C_3. Using a property of $C([0,T])$ norm,

$$\begin{aligned}\|h_1'(t)h_1(t)-h_2'(t)h_2(t)\|_{C([0,T])}&\ge\|h_1'(t)-h_2'(t)\|_{C([0,T])}\|h_1(t)\|_{C([0,T])}\\&\quad-\|h_2'(t)\|_{C([0,T])}\|h_1(t)-h_2(t)\|_{C([0,T])}\end{aligned}\tag{2.17}$$

and the facts $h_1(0)=h_2(0)=h_0$, $h_i(t)\ge h_0$ and $h_i'(t)\le(h^*+1)/h_0$, we derive that if $T\le\frac{h_0^2}{2(h^*+1)}$ then

$$\begin{aligned}\|h_1'-h_2'\|_{C([0,T])}&\le\frac{1}{h_0}\|h_1'h_1-h_2'h_2\|_{C([0,T])}+\frac{h^*+1}{h_0^2}T\|h_1'-h_2'\|_{C([0,T])}\\&\le\frac{2}{h_0}\|h_1'h_1-h_2'h_2\|_{C([0,T])},\end{aligned}\tag{2.18}$$

which implies that

$$\|h_1-h_2\|_{C^1([0,T])}\le(1+T)\|h_1'-h_2'\|_{C([0,T])}\le\frac{2+2T}{h_0}\|h_1'h_1-h_2'h_2\|_{C([0,T])}.\tag{2.19}$$

Hence from (2.16) and (2.19), for

$$T:=\min\big\{1,C_2^{-2/\alpha},C_1^{-2/(1+\alpha)},\frac{h_0^2}{2h^*+2},\big[2C_4(1+\frac{4}{h_0})\big]^{-2/\alpha}\big\},$$

we have

$$\begin{aligned}&\|\overline{v}_1(\xi,t)-\overline{v}_2(\xi,t)\|_{C([0,h_0]\times[0,T])}+\|\overline{h}_1'\overline{h}_1-\overline{h}_2'\overline{h}_2\|_{C([0,T])}\\&\le T^{(1+\alpha)/2}\|\overline{v}_1-\overline{v}_2\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,T])}+T^{\alpha/2}\|\overline{h}_1'\overline{h}_1-\overline{h}_2'\overline{h}_2\|_{C^{\alpha/2}([0,T])}\\&\le C_4T^{\alpha/2}(\|v_1-v_2\|_{C([0,h_0]\times[0,T])}+\|h_1-h_2\|_{C^1([0,T])})\\&\le C_4T^{\alpha/2}(\|v_1-v_2\|_{C([0,h_0]\times[0,T])}+\frac{2+2T}{h_0}\|h_1'h_1-h_2'h_2\|_{C([0,T])})\\&\le\frac{1}{2}(\|v_1-v_2\|_{C([0,h_0]\times[0,T])}+\|h_1'h_1-h_2'h_2\|_{C([0,T])}).\end{aligned}$$

Thus for this T, $\mathcal{F}$ is a contraction mapping. According the contraction mapping theorem, we can conclude that there exists a unique $(v(\xi,t),h(t))\in\Sigma_T$ such that $\mathcal{F}(v(\xi,t),h(t))=(v(\xi,t),h(t))$. In other words, $(v(\xi,t),h(t))$ is the solution of problem (2.4) and therefore $(u(r,t),h(t))$ is the solution of problem (1.2). Moreover, by using the Schauder estimates, we have additional regularity of the solution, namely, $h(t)\in C^{1+\alpha/2}[0,T]$ and $u\in C^{2+\alpha,1+\alpha/2}((0,h(t))\times(0,T])$, that is, $(u(r,t),h(t))$ is the classical solution of problem (1.2). $\square$

Remark 2.2. If the initial function u_0 is smooth and satisfies the consistency condition

$$-d(u_0''(h_0) + \frac{N-1}{r} u_0'(h_0)) + \mu u_0'(h_0) u_0'(h_0) = u(h_0)(Kau_0(h_0) - b),$$

then the solution $(u, h) \in C^{2+\alpha, 1+\alpha/2}(\overline{D}_T) \times C^{1+\alpha/2}([0, T])$.

Now we give the monotone behavior of the free boundary $h(t)$.

Theorem 2.3. *The free boundary $h(t)$ for problem* (1.2) *is strictly monotone increasing, that is, for any solution in* $(0, T]$*, we have*

$$h'(t) > 0 \quad \text{for } 0 < t \le T.$$

Proof. Applying the Hopf lemma to problem (1.2), we obtain

$$u_r(h(t), t) < 0 \quad \text{for } 0 < t \le T.$$

Thus, combining this inequality with the Stefan condition gives the desired result. □

Next, we give a comparison principle which can be used to estimate both $u(r, t)$ and the free boundary $r = h(t)$.

Lemma 2.4. *Assume that* $T \in (0, \infty)$, $\overline{h} \in C^1([0, T])$, $\overline{u}(r, t) \in C(\overline{D_T^*}) \cap C^{2,1}(D_T^*)$, *with* $D_T^* = \{(r, t) \in \mathbb{R}^2 : 0 < r < \overline{h}(t), 0 < t \le T\}$*, and that*

$$\begin{gathered}
\overline{u}_t - d(\overline{u}_{rr} + \frac{N-1}{r} \overline{u}_r) \ge \overline{u}(Ka\overline{u} - b), \quad 0 < r < \overline{h}(t), \quad 0 < t \le T, \\
\overline{u}_r(0, t) = 0, \quad \overline{u}(\overline{h}(t), t) = 0, \quad 0 < t \le T, \\
\overline{h}'(t) \ge -\mu \overline{u}_r(\overline{h}(t), t), \quad 0 < t \le T.
\end{gathered}$$

If $h_0 \le \overline{h}(0)$ *and* $u_0(r) \le \overline{u}(r, 0)$ *in* $[0, h_0]$*, then the solution* (u, h) *of the free boundary problem* (1.2) *satisfies*

$$h(t) \le \overline{h}(t), \quad t \in (0, T] \quad \text{and} \quad u(r, t) \le \overline{u}(r, t), \quad 0 < r < h(t), 0 < t \le T.$$

Proof. For small $\epsilon > 0$, let (u_ϵ, h_ϵ) denote the unique solution of (1.2) with h_0 replaced by $h_0^\epsilon = h_0(1 - \epsilon)$, with μ replaced by $\mu_\epsilon = \mu(1 - \epsilon)$, and with u_0 replaced by some $u_0^\epsilon \in C^2([0, h_0^\epsilon])$ satisfying

$$0 < u_0^\epsilon(r) \le u_0(r) \quad \text{in } [0, h_0^\epsilon], \quad u_0^\epsilon(0) = u_0^\epsilon(h_0^\epsilon) = 0,$$

and as $\epsilon \to 0$,

$$u_0^\epsilon(\frac{h_0^\epsilon}{h_0} r) \to u_0(r),$$

in the sense of $C^2([0, h_0])$.

We claim that $h_\epsilon(t) < \overline{h}(t)$ for all $t \in (0, T]$. Clearly, this is true for small $t > 0$. If this does not hold, then we can find a first $t^* \le T$ such that $h_\epsilon(t) < \overline{h}(t)$ for $t \in (0, t^*)$ and $h_\epsilon(t^*) = \overline{h}(t^*)$. It follows that $h_\epsilon'(t^*) \ge \overline{h}'(t^*)$. We now compare u_ϵ and $\overline{u}$ over the region

$$\Omega_{t^*} = \{(r, t) \in \mathbb{R}^2 : 0 < r < h_\epsilon(t), 0 < t \le t^*\}.$$

It follows from the strong maximum principle that $u_\epsilon(r, t) < \overline{u}(r, t)$ in Ω_{t^*}. Hence $w(r, t) = \overline{u}(r, t) - u_\epsilon(r, t) > 0$ in Ω_{t^*} with $w(h_\epsilon(t^*), t^*) = 0$. It follows that $w_r(h_\epsilon(t^*), t^*) \le 0$, from which we deduce, in view of $(u_\epsilon)_r(h_\epsilon(t^*), t^*) < 0$ and

$\mu_\epsilon < \mu$, that $h'_\epsilon(t^*) < \overline{h}'(t^*)$. But this contradicts to $h'_\epsilon(t^*) \geq \overline{h}'(t^*)$, which proves our claim that $h_\epsilon(t) < \overline{h}(t)$ for all $t \in (0, T]$. We may now apply the usual comparison principle over Ω_T to conclude that $u_\epsilon < \overline{u}$ in Ω_T.

Since the unique solution of (1.2) depends continuously on the parameters in (1.2) up to the maximal existence time $T_{\max}$, which can be proven in the same way as in [26, Theorem 2.2], it follows that (u_ϵ, h_ϵ) converges to (u, h) as $\epsilon \to 0$, and that (u, h) is the unique solution of (1.2). The desired result then follows by letting $\epsilon \to 0$ in the inequalities $u_\epsilon < \overline{u}$ and $h_\epsilon < \overline{h}$. The proof is complete. □

3. Finite time blow-up

In this section, we study the blowup property. Firstly, we point out that all solutions that exist for finite time would blow up in the L^∞ sense.

Lemma 3.1. *The solution $u(r, t)$ to (1.2) exists uniquely, and it can be extended to $[0, T_{\max})$, where $T_{\max} \leq \infty$ is the maximal existence time of $u(r, t)$. Furthermore, if $T_{\max} < \infty$, then we have*

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty([0,h(t)])} = \infty. \tag{3.1}$$

Proof. From the comparison principle Lemma 2.4, we know that the solution $u(r, t)$ of (1.2) exists and is unique. Then it is not difficult to obtain from the uniqueness and Zorn's lemma that there exists a number $T_{\max}$ such that $[0, T_{\max})$ is the maximal time interval in which the solution exists. To complete the proof of this lemma, we only have to verify that if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty([0,h(t)])} = \infty.$$

To show this, we use the contradiction argument. Assume that $T_{\max} < \infty$ and that

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty([0,h(t)])} < \infty,$$

then there exist $M_1, M_2 > 0$ such that $T_{\max} < M_1 < \infty$ and $\|u(\cdot, t)\|_{L^\infty([0,h(t)])} < M_2 < \infty$ for all $t \in [0, T_{\max})$.

Next, we prove that $h'(t)$ is uniformly bounded in $(0, T_{\max})$, i.e., $h'(t) \leq M_3$ for all $t \in (0, T_{\max})$ with some M_3 independent of $T_{\max}$. To obtain this, we define

$$\Omega = \Omega_M := \{(r, t) \in \mathbb{R}^2 : h(t) - M^{-1} < r < h(t), 0 < t < T_{\max}\},$$

and construct an auxiliary function

$$w(r, t) := M_2[2M(h(t) - r) - M^2(h(t) - r)^2].$$

We choose M large so that $w(r, t) \geq u(r, t)$ holds in Ω. Direct computations yield that, for $(r, t) \in \Omega$, we have

$$\begin{gathered} w_t = 2M_2Mh'(t)(1 - M(h(t) - r)) \geq 0, \\ w_r = -2M_2M(1 - M(h(t) - r)) \leq 0, \\ -\Delta w = -w_{rr} - \frac{N-1}{r} w_r = 2M_2M^2 - \frac{N-1}{r} w_r, \\ Kau^2 - bu \leq KaM_2^2. \end{gathered}$$

Then we have

$$w_t - d\Delta w \geq 2dM_2M^2 \geq Kau^2 - bu, \quad (r, t) \in \Omega,$$

if we take $M^2 \geq \frac{KaM_2}{2d}$. On the other hand, we have

$$w(h(t)-M^{-1},t)=M_2 \geq u(h(t)-M^{-1},t), \quad w(h(t),t)=0=u(h(t),t).$$

Thus, if we can choose M large such that $u_0(r) \leq w(r,0)$ for $r \in [h_0-M^{-1},h_0]$, then we can apply the maximum principle to $w-u$ over Ω to deduce that $u(r,t) \leq w(r,t)$ for $(r,t) \in \Omega$. It would then follow that

$$u_r(h(t),t) \geq w_r(h(t),t) = -2MM_2, h'(t) = -\mu u_r(h(t),t) \leq M_3 := 2MM_2\mu.$$

Now, we aim to finding some M independent of $T_{\max}$ such that $u_0(r) \leq w(r,0)$ for $r \in [h_0-M^{-1},h_0]$. By some calculations, we see

$$w_r(r,0) = -2M_2M[1-M(h_0-r)] \leq -M_2M \quad \text{for } r \in [h_0-(2M)^{-1},h_0].$$

Therefore, upon choosing $M := \max\big\{\sqrt{\frac{KaM_2}{2d}}, \frac{4\|u_0\|_{C^1([0,h_0])}}{3M_2}\big\}$, we have $w_r(r,0) \leq u_0'(r)$ for $r \in [h_0-(2M)^{-1},h_0]$, which implies $w(r,0) \geq u_0(r)$ for $r \in [h_0-(2M)^{-1},h_0]$ because $w(h_0,0)=u_0(h_0)=0$.

Moreover, for $r \in [h_0-M^{-1},h_0-(2M)^{-1}]$, we have

$$w(r,0) \geq \frac{3}{4}M_2,\ u_0(r) \leq \|u_0\|_{C^1([0,h_0])}M^{-1} \leq \frac{3}{4}M_2.$$

Hence $u_0(r) \leq w(r,0)$ for $r \in [h_0-M^{-1},h_0]$, which tells us $h'(t) \leq M_3$ in $[0,T_{\max})$, with $M_3 = 2MM_2\mu$ independent of $T_{\max}$.

Now, we fix $\delta_0 \in (0,T_{\max})$. By standard L^p estimates, the Sobolev embedding theorem and the Hölder estimates for parabolic equations, we can find $L^* > 0$ depending only on M_1, M_2 and M_3 such that $\|u(\cdot,t)\|_{C^2([0,h(t)])} \leq L^*$ for all $t \in [\delta_0,T_{\max})$. Using Theorem 2.1 again, we conclude that there exists a $\tau>0$ depending on M_3 and L^* such that the solution to (1.2) with the initial time $T_{\max}-\frac{\tau}{2}$ can be extended uniquely to the time $T_{\max}-\frac{\tau}{2}+\tau$, which is a contradiction to the hypothesis. Thus the proof is completed. □

Let $(\phi_1(x),\lambda_1)$ be the first pair of eigenfunction-eigenvalue of the eigenvalue problem

$$\begin{gathered} -d\Delta\phi(x) = \lambda\phi(x), \quad x \in B_{h_0}, \\ \phi(x) = 0, \quad x \in \partial B_{h_0}, \end{gathered} \tag{3.2}$$

where B_{h_0} is the ball with radius h_0, then $\phi_1(x)$ is positive and symmetric in B_{h_0}, that is, $\phi_1(x)=\phi_1(r)(r=|x|)$, and we can assume that $\int_0^{h_0}\phi_1(r)dr=1$. Now, by using the convexity argument, we give some sufficient conditions under which blowup occurs for problem (1.2), which produce a positive effect when establishing the existence of global slow solution in Section 4.

Theorem 3.2. *Let $u(r,t)$ be the solution of problem* (1.2)*, if the initial datum $u_0(r)$ is sufficiently large such that*

$$\int_0^{h_0} u_0(r)\phi_1(r)dr > \frac{\lambda_1+b}{Ka},$$

then $u(r,t)$ blows up in finite time.

Proof. First, we consider the auxiliary problem

$$\begin{gathered} v_t - d\Delta v = Kav^2(r,t) - bv(r,t), \quad 0<r<h_0, t>0, \\ v_r(0,t) = v(h_0,t) = 0, \quad t>0, \\ v(r,0) = u_0(r), \quad 0<r<h_0. \end{gathered} \tag{3.3}$$

It follows from the comparison principle that $u(r,t) \geq v(r,t)$ for $0 \leq r \leq h_0$ and $t \geq 0$.

Next, we prove that $v(r,t)$ blows up at a finite time. Observing that $\phi_1'(0) = 0$, we multiply the first equation in (3.3) by $\phi_1(r)$ and integrate the resulting equation over $(0, h_0)$, then we have

$$F'(t) + (\lambda_1 + b)F(t) = Ka \int_0^{h_0} v^2(r,t)\phi_1(r)dr,$$

where $F(t) = \int_0^{h_0} v(r,t)\phi_1(r)dr$. Since v^2 is convex and $\phi_1(r)$ is positive within $(0, h_0)$ and $\int_0^{h_0} \phi_1(r)dr = 1$, we have

$$\int_0^{h_0} v^2(r,t)\phi_1(r)dr \geq F^2(t).$$

Hence

$$F'(t) + (\lambda_1 + b)F(t) \geq KaF^2(t).$$

Since $\int_0^{h_0} u_0(r)\phi_1(r)dr > \frac{\lambda_1+b}{Ka}$, we see that $KaF^2(0) > (\lambda_1 + b)F(0)$, which implies that $v(r,t)$ would blow up in finite time. Then the desired result holds since $v(r,t)$ is a lower solution. □

Remark 3.3. From Theorem 3.2, we know that if $u_0(r)$ is in the form of $\delta\phi_1(r)$, then the solution $u(r,t)$ to (1.2) will grow to infinity provided that δ is sufficiently large.

4. Global fast solution and slow solution

In this section, we investigate the long time behavior of the global solution of (1.2). By constructing an upper solution, we first give the existence of the global fast solution.

Theorem 4.1 (Global fast solution)**.** *Let u be a solution to* (1.2)*. If u_0 is small in the sense that*

$$\|u_0(\cdot)\|_{L^\infty([0,h_0])} \leq \frac{d}{16} \min\big\{\frac{1}{2Kah_0^2}, \frac{1}{\mu}\big\}, \tag{4.1}$$

then $T_{\max} = \infty$. Moreover, $h_\infty = \lim_{t\to\infty} h(t) < \infty$ and there exist two constants $C, \beta > 0$ depending on u_0 such that

$$\|u(\cdot,t)\|_{L^\infty([0,h(t)])} \leq Ce^{-\beta t} \quad \text{for } t \geq 0.$$

Proof. It suffices to construct a suitable global supersolution. Let

$$\begin{gathered}\sigma(t) = 2h_0(2 - e^{-\gamma t}), \quad t \geq 0,\\ V(y) = 1 - y^2, \quad -1 \leq y \leq 1,\\ w(r,t) = \epsilon e^{-\beta t} V(r/\sigma(t)), \quad 0 \leq r \leq \sigma(t),\ t \geq 0,\end{gathered}$$

where γ, β and $\epsilon > 0$ are to be determined later. Direct calculations yield that for all $t > 0$ and $0 < r < \sigma(t)$,

$$\begin{aligned}&w_t - d\Delta w - w(Kaw - b)\\ &= \epsilon e^{-\beta t}\big[-\beta V - r\sigma'\sigma^{-2}V' - d\sigma^{-2}V'' - d\frac{N-1}{r\sigma}V' - V(Ka\epsilon e^{-\beta t}V - b)\big]\end{aligned}$$

$$\geq \epsilon e^{-\beta t}[-\beta + \frac{d}{8h_0^2} - Ka\epsilon].$$

On the other hand, we can easily obtain that $\sigma'(t) = 2\gamma h_0 e^{-\gamma t} > 0$, $w_r(0,t) = w(\sigma(t),t) = 0$ and $-w_r(\sigma(t),t) = 2\epsilon e^{-\beta t}/\sigma(t)$.

Setting $\epsilon = 2\|u_0(r)\|_{L^\infty[0,h_0]}$, then one can see that $u_0(r) < w(r,0)$ for $0 \leq r \leq h_0$. If we further choose $\gamma = \beta = \dfrac{d}{16h_0^2}$, then (w,σ) satisfies

$$\begin{gathered}
w_t - d\Delta w - w(Kaw - b) \geq 0, \quad 0 < r < \sigma(t),\ t > 0,\\
\sigma'(t) > -\mu w_r(\sigma(t),t), \quad t > 0,\\
w_r(0,t) = w(\sigma(t),t) = 0, \quad t > 0,\\
\sigma(0) = 2h_0 > h_0,
\end{gathered}$$

by (4.1). By applying the comparison principle Lemma 2.4, we deduce that $h(t) \leq \sigma(t)$ and $u(r,t) \leq w(r,t)$ for $0 \leq r \leq h(t)$, as long as u exists. In particular, it follows from the continuation property (3.1) that u exists globally, which completes the proof. $\square$

Remark 4.2. It is easy to see that if the initial datum $u_0(r)$ is in the form of $\delta\phi_1(r)$ and $\delta \leq \frac{d}{16\max_{r\in[0,h_0]}\phi_1(r)}\min\{\frac{1}{2Kah_0^2}, \frac{1}{\mu}\}$, then the above theorem holds. From the above theorem, we also know that as t tends to infinity, the free boundary $h(t)$ converges to a finite limit and the solution $u(r,t)$ decays uniformly to 0 at an exponential rate, so, we call it global fast solution. In a later result (Theorem 4.5), one can see that the free boundary will grow up to infinity as t goes to infinity, and hence the latter solution is called global slow solution.

Before giving the existence of the global slow solution, we need the following lemma, which provides a priori estimate for the global solution.

Lemma 4.3. *Let u be a solution to* (1.2) *with $u_0(r)$ in the form of $\delta\phi_1(r)$. If $T_{\max} = \infty$, then there exists a constant $C = C(\|u_0\|_{C^2}, h_0, 1/h_0)$, such that*

$$\sup_{t\geq 0}\|u(\cdot,t)\|_{L^\infty([0,h(t)])} \leq C,$$

where C remains bounded for $\|u_0\|_{C^2}, h_0$, and $1/h_0$ bounded.

Proof. First, from the continuous dependence upon the data and coefficients (see [9, Theorem 2]), we know for each $M > 1$ that there exists an $\eta > 0$ such that, if $\|u_0\|_{C^{1+\alpha}} < M$ and $1/M < h_0 < M$, then $\|u(\cdot,t)\|_{L^\infty} < 2M$ on $[0,\eta]$. In what follows, we use the contradiction argument. Suppose that the result is not true, then there exist an $M > 0$ and a sequence of global solutions (u_n, h_n) of (1.2) such that

$$\begin{gathered}
1/M < h_n(0) = h_0 < M, \quad \|u_n(r,0)\|_{C^{1+\alpha}[0,h_0]} < M,\\
\sup_{t\geq 0}\|u_n(\cdot,t)\|_{L^\infty([0,h_n(t)])} \to \infty \quad \text{as } n \to \infty.
\end{gathered}$$

Then there exists a sequence (r_n, t_n) with $r_n \in [0, h_n(t_n))$ and $t_n \to \infty$ as $n \to \infty$ such that

$$\sup_{t\in[0,t_n]}\|u_n(r,t)\|_{L^\infty([0,h_n(t)])} = u_n(r_n,t_n) \to \infty \quad \text{as } n \to \infty.$$

We make an assertion that $r_n = 0$. Before showing this result, we first prove that $\phi_1'(r) \le 0$. Actually, from $\phi_1(r)$ being the first eigenvalue of (3.2), with $\int_0^{h_0} \phi_1(x)\,dx = 1$, we can easily see that $\phi_1'(0) = 0$. Since $-d[r^{N-1}\phi_1'(r)]' = r^{N-1}\lambda_1\phi_1(r) > 0$ in $(0, h_0)$, we have $\phi_1'(r) < 0$ in $(0, h_0)$. Next, we set $v(r,t) = u_{nr}(r,t)$, then $v(r,t)$ satisfies

$$\begin{gathered} v_t - d\big[\frac{N-1}{r}v + v_r\big]_r = (2Kau_n - b)v, \quad 0 < r < h_n(t),\ t > 0, \\ v(0,t) = u_{nr}(0,t) = 0, \quad v(h_n(t),t) = u_{nr}(h_n(t),t) < 0, \quad t > 0, \\ v(r,0) = u_n'(r,0) = \delta_n\phi_1'(r) \le 0, \quad 0 \le r \le h_0. \end{gathered} \tag{4.2}$$

According to the maximum principle, we see that $v(r,t) = u_{nr}(r,t) \le 0$. Hence $u_n(r,t) \le u_n(0,t)$ over $\{(r,t) \in \mathbb{R}^2 : 0 \le r \le h_n(t), 0 \le t < +\infty\}$. Thus $r_n = 0$, which implies that

$$\sup_{t\in[0,t_n]} \|u_n(r,t)\|_{L^\infty(0,h_n(t))} = u_n(0,t_n) =: \rho_n.$$

Setting $\lambda_n = \rho_n^{-1/2}$, then it is evident to see that $\lambda_n \to 0$ as $n \to \infty$. Extend $u_n(\cdot,t)$ by 0 on $(h_n(t),\infty)$ and define the rescaled function

$$v_n(s,\tau) = \lambda_n^2 u_n(\lambda_n s, t_n + \lambda_n^2\tau) \quad \text{for } (s,\tau) \in \tilde{D}_n, \tag{4.3}$$

where $\tilde{D}_n = \{(s,\tau) \in \mathbb{R}^2 : 0 < s < \infty,\ -\lambda_n^{-2}t_n < \tau \le 0\}$. Also, we denote

$$\begin{gathered} s_1 = 0, \quad s_2(\tau) = \lambda_n^{-1}h_n(t_n + \lambda_n^2\tau), \\ D_n = \{(s,\tau) \in \mathbb{R}^2 : s_1 < s < s_2(\tau), -\lambda_n^{-2}t_n < \tau \le 0\}, \end{gathered}$$

which corresponds to the domain $\{(r,t) \in \mathbb{R}^2 : 0 < r < h_n(t), 0 < t \le t_n\}$. Then the function v_n satisfies $v_n(0,0) = 1$, $0 \le v_n \le 1$ and

$$\partial_\tau v_n - d(v_{nss} + \frac{N-1}{s}v_{ns}) = v_n(Kav_n - b_n), \quad (s,\tau) \in D_n, \tag{4.4}$$

with $b_n = \lambda_n^2 b$. As in [10, Lemma 2.1], we can derive local L^2 estimates for the derivatives of v_n, then according to the compact embeddings $H^1(Q_m) \subset L^p(Q_m)$ and $H^1([0,m]) \subset C([0,m])$, where $Q_m = [0,m]\times[-m,0]$ and $m > 0$, it follows that some subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converges in $L^p_{\rm loc}([0,+\infty)\times(-\infty,0])$ to some function $w(s,\tau) \in L^p([0,+\infty)\times(-\infty,0])$ and that $\{v_{n_k}(s,0)\}$ converges in $C_{\rm loc}([0,+\infty))$ to some function $z(s) \in C([0,+\infty))$ with $z(0) = 1$. Moreover, similarly to [10, Lemmas 2.2 and 2.3], we also have $w_\tau = 0$ in $D'((0,+\infty)\times(-\infty,0))$. In a word, we have obtained a function $w(s)$ that is nonnegative, bounded and continuous on $[0,\infty)$ and satisfies $-d(w_{ss} + \frac{N-1}{s}w_s) = Kaw^2$. Observing that $w(0) = z(0) = 1$, we see that w is further governed by

$$\begin{gathered} -d(w_{ss} + \frac{N-1}{s}w_s) = Kaw^2, \quad s > 0, \\ w(0) = 1. \end{gathered} \tag{4.5}$$

If $N = 2$, let $U(z) = w(s)$, $z = -\ln s$, if $N > 2$, let $U(z) = w(s)$, $z = \frac{1}{N-2}s^{2-N}$, then problem (4.5) can be reduced to

$$\begin{gathered} -dU_{zz} = Kas^{2N-2}U^2, \quad z \in (0,+\infty), \\ U(+\infty) = 1. \end{gathered} \tag{4.6}$$

We can easily see from the equation in (4.6) that $U \in C^2([0,+\infty))$ and that U is nonnegative, concave and bounded. Therefore $U \equiv 0$, which leads a contradiction to the fact that $U(+\infty) = 1$, and the desired result follows. $\square$

The above result indicates that the global solution is uniformly bounded while the next result shows that such a solution will decay uniformly to 0 as t approaches infinity.

Lemma 4.4. *Assume the conditions in Lemma 4.3 hold, then the solution of problem* (1.2) *satisfies*

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty([0,h(t)])} = 0.$$

Proof. We first consider the case $h_\infty = \infty$. Assume that

$$l := \limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty([0,h(t)])} > 0$$

by contradiction. From Lemma 4.3, we see that $l < +\infty$. Choose $t_0 > 0$ such that $l/2 \le \sup_{t\in[t_0,\infty)} \|u(\cdot,t)\|_{L^\infty([0,h(t)])} \le 3l/2$. Then there exist an $\epsilon_0 > 0$ and a sequence $t_n \to +\infty$ such that

$$\sigma_n := \|u(\cdot,t_n)\|_{L^\infty([0,h(t_n)])} \ge \frac{3}{4} l \ge \frac{1}{2} \sup_{t\in[t_0,\infty)} \|u(\cdot,t)\|_{L^\infty([0,h(t)])} \ge \epsilon_0.$$

Pick $r_n \in [0, h(t_n))$ such that $\sigma_n = u(r_n,t_n)$. As in Lemma 4.3, we can prove that $r_n = 0$, thus $\sigma_n := \|u(\cdot,t_n)\|_{L^\infty([0,h(t_n)])} = u(0,t_n)$.

Set $\lambda_n = \sigma_n^{-1/2} \le \epsilon_0^{-1/2}$, extend $u(\cdot,t)$ by 0 on $(h(t),\infty)$ and define the rescaled function $v_n(s,\tau)$ as in Lemma 4.3:

$$v_n(s,\tau) = \lambda_n^2 u(\lambda_n s, t_n + \lambda_n^2 \tau), \quad \text{for } (s,\tau) \in \tilde{D}_n, \tag{4.7}$$

where $\tilde{D}_n = \{(s,\tau) \in \mathbb{R}^2 : 0 < s < \infty, \lambda_n^{-2}(t_0 - t_n) < \tau \le 0\}$. Also, we denote

$$s_1 = 0, \quad s_2(\tau) = \lambda_n^{-1} h(t_n + \lambda_n^2 \tau),$$
$$D_n = \{(s,\tau) : s_1 < s < s_2(\tau), \lambda_n^{-2}(t_0 - t_n) < \tau \le 0\}.$$

Then v_n satisfies

$$\partial_\tau v_n - d(v_{nss} + \frac{N-1}{s} v_{ns}) = v_n(Kav_n - b_n), \ (s,\tau) \in D_n, \tag{4.8}$$

with $b_n = \lambda_n^2 b$, $v_n(0,0) = 1$, $0 \le v_n \le 2$ and $\lim_{s\to+\infty} v_n(s,\tau) = 0$. Noticing that $b_n \le b(\frac{3}{4}l)^{-1}$, we know that there exist a sequence $\{b_{n_k}\}$ and $b^* \le b(\frac{3}{4}l)^{-1}$ such that $b_{n_k} \to b^*$ as $k \to \infty$. Also we must be mentioned here that $D_n \to [0,\infty)\times(-\infty,0]$ as $n \to \infty$. Similarly as [10, Lemmas 2.1-2.3], also we can obtain a function $w(s) \ge 0$, which is bounded, continuous on $[0,\infty)$ and satisfies that

$$\begin{gathered} -d(w_{ss} + \frac{N-1}{s} w_s) = Kaw^2 - b^* w, \quad s > 0, \\ \lim_{s\to+\infty} w(s) = 0. \end{gathered} \tag{4.9}$$

Similarly as in Lemma 4.3, by introducing a transformation we can show that $w \equiv 0$ or $w \equiv \frac{b^*}{Ka}$. If $w \equiv 0$, there is a contradiction to the fact that $w(0) = 1$. If $w \equiv \frac{b^*}{Ka}$, there is also a contradiction to the fact that $\lim_{s\to+\infty} w(s) = 0$. So in the case $h_\infty = \infty$, the conclusion in this lemma holds.

Now, we consider the case that $h_\infty < \infty$. Arguing indirectly, we suppose that $\delta := \limsup_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty([0,h(t)])} > 0$. Then there exists a sequence $\{(r_k,t_k)\} \subset$

$\{(r,t) \in \mathbb{R}^2 : 0 < r < h(t)), 0 < t < \infty\}$ with $t_k \to \infty$ such that $u(r_k, t_k) \geq \delta/2$ for all $k \in \mathbb{N}$. Since $0 < r_k < h_\infty < +\infty$, we then have a subsequence of $\{r_k\}$ converging to $r_0 \in [0, h_\infty)$. Without loss of generality, we assume that $r_k \to r_0$ as $k \to \infty$.

Define

$$u_k(r,t) = u(r, t + t_k) \quad \text{for } r \in [0, h(t+t_k)],\ t \in (-t_k, \infty).$$

According to the above lemma, we know that u_k is bounded, hence from the parabolic regularity it follows that $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ such that $u_{k_i} \to \overline{u}$ as $i \to \infty$ and $\overline{u}$ satisfies

$$\overline{u}_t - d(\overline{u}_{rr} + \frac{N-1}{r}\overline{u}_r) = \overline{u}(Ka\overline{u} - b) \quad \text{for } r \in (0, h_\infty),\ t \in (-\infty, +\infty).$$

Note that $\overline{u}(r_0, 0) \geq \delta/2 > 0$, hence $\overline{u} > 0$ in $(0, h_\infty) \times (-\infty, +\infty)$. Applying the Hopf lemma to the above equation of $\overline{u}$ at the point $(h_\infty, 0)$ yields $\overline{u}_r(h_\infty, 0) \leq -\sigma < 0$ for some $\sigma > 0$.

On the other hand, noticing that $h(t)$ is increasing and bounded, for any $\alpha \in (0,1)$, there exists a constant $\tilde{C}$ depending on α, h_0, $\|u_0\|_{C^2[0,h_0]}$ and h_∞ such that

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_\infty)} + \|h\|_{C^{1+\alpha/2}([0,\infty))} \leq \tilde{C}, \tag{4.10}$$

where $\overline{D}_\infty = \{(r,t) \in \mathbb{R}^2 : 0 < r < h(t),\ t > 0\}$. In fact, we let

$$y = \frac{h_0 r}{h(t)}, \quad v(y,t) = u(r,t). \tag{4.11}$$

Then $v(y,t)$ satisfies

$$\begin{gathered}
v_t - Av_y - Bv_{yy} = v(Kav - b), \quad 0 < y < h_0, t > 0, \\
v_y(0,t) = v(h_0, t) = 0, \quad t > 0, \\
v(y,0) = v_0(y) := u_0(y) \geq 0, \quad 0 \leq y \leq h_0,
\end{gathered} \tag{4.12}$$

where $A = \frac{h'(t)}{h(t)}y + d\frac{h_0^2}{h^2(t)}\frac{N-1}{y}$ and $B = d\frac{h_0^2}{h^2(t)}$. Also transformation (4.11) changes the free boundary $r = h(t)$ to the fixed line $y = h_0$, but at the expense of making the equation more complex. Next, we can use exactly the same arguments as in Lemma 3.1 to deduce that $h'(t)$ is uniformly bounded for all $t > 0$. Observe that v is bounded in the sense of L^∞, hence by standard L^p theory and the Sobolev imbedding theorem we can find $\tilde{C}^*$ depending on α, h_0, $\|u_0\|_{C^2[0,h_0]}$, h_∞ such that

$$\|v\|_{C^{1+\alpha,(1+\alpha)/2}([0,h_0]\times[0,\infty))} \leq \tilde{C}^*,$$

which immediately leads to (4.10). Combining $\|h\|_{C^{1+\alpha/2}[0,+\infty)} \leq \tilde{C}$ with the fact that $h(t)$ is bounded, then we have $h'(t) \to 0$ as $k \to \infty$; i.e., $u_r(h(t_k), t_k) \to 0$ as $k \to \infty$, in view of the Stefan condition. Furthermore,

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}([0,h(t)]\times[0,\infty))} \leq \tilde{C}$$

implies $u_r(h(t_k), 0 + t_k) = (u_k)_r(h(t_k), 0) \to \overline{u}_r(h_\infty, 0)$ as $k \to \infty$, which produces a contradiction to the fact that $\overline{u}_r(h_\infty, 0) \leq -\sigma < 0$. Thus the desired result follows. □

Theorem 4.5 (Global slow solution)**.** *Let $\phi_1(r)$ be the first eigenfunction of problem* (3.2) *with $\int_0^{h_0} \phi_1(r)dr = 1$. Then there exists a $\lambda > 0$ such that the solution to* *problem* (1.2) *with initial datum* $u_0 = \lambda\phi_1$ *is a global slow solution, which satisfies that* $h_\infty = \infty$.

Proof. To emphasize the dependence of u on the initial data when necessary, we denote the solution to (1.2) by $u(u_0;\cdot)$. So do $h(t)$, h_∞ and the maximal existence time $T_{\max}$. Motivated by [10], we define

$$\Sigma = \{\lambda > 0 : T_{\max}(\lambda\phi_1) = \infty \text{ and } h_\infty(\lambda\phi_1) < \infty\}.$$

According to Theorem 4.1, when λ is small we know that $\lambda \in \Sigma$, so Σ is not empty. Conversely, when λ is large enough, from Theorem 3.2 and Remark 3.3 we know that the corresponding solution will blow up, i.e., $T_{\max}(\lambda\phi_1) < \infty$, hence Σ is bounded.

Now, we prove that $\Sigma = (0, \lambda^*)$, where $\lambda^* := \sup\Sigma \in (0,\infty)$. From the definition of λ^*, we know that $\Sigma \subseteq (0,\lambda^*]$. Let $v = u(\lambda^*\phi_1;\cdot)$, $\sigma = h(\lambda^*\phi_1;\cdot)$, and $\tau = T_{\max}(\lambda^*\phi_1)$, then we can show that $\tau = \infty$. In fact, by continuous dependence, we know that for each fixed $t \in [0,\tau)$, $u(\lambda\phi_1;\cdot,t)$ converges to $v(\cdot,t)$ in $L^\infty(0,\infty)$ and $h(\lambda\phi_1;t) \to \sigma(t)$ as $\lambda \to \lambda^*$(note $u(r,t) = 0$ on $(h(t),\infty)$). It follows from Lemma 4.3 that $\|v(\cdot,t)\|_{L^\infty([0,\sigma(t)])} \le C$ for all $t \in [0,\tau)$ because $T_{\max}(\lambda\phi_1) = \infty$ for all $\lambda \in (0,\lambda^*)$. Thus $\tau = \infty$ since nonglobal solutions should satisfy $\limsup_{t\to T_{\max}} \|u(\cdot,t)\|_{L^\infty(0,h(t))} = \infty$.

Next we show that $\sigma = \infty$. We assume for contradiction that $\sigma < \infty$, from Lemma 4.4, we see that $\|v(\cdot,t)\|_{L^\infty([0,h(t)])} \to 0$ as $t \to \infty$, hence we can choose t_0 sufficiently large such that $\|v(r,t_0)\|_{L^\infty([0,h(t_0)])} < \frac{d}{16}\min\{\frac{1}{2Kah_0^2}, \frac{1}{\mu}\}$. By continuous dependence, we can deduce that

$$\|u(\lambda\phi_1;\cdot,t_0)\|_{L^\infty([0,h(t_0)])} \le \frac{d}{16}\min\big\{\frac{1}{2Kah_0^2}, \frac{1}{\mu}\big\}$$

for $\lambda > \lambda^*$ sufficiently close to λ^*. But this implies that $T_{\max}(\lambda\phi_1) = \infty$ and $h_\infty(\lambda\phi_1) < \infty$ by Theorem 4.1, which is a contradiction to the definition of λ^*. So $\sigma_\infty = \infty$ and $\lambda^* \notin \Sigma$, we can further find that $\Sigma = (0,\lambda^*)$. Suppose that $\Sigma \subset (0,\lambda^*)$ for a contradiction, i.e., there exists a number $\lambda_0 \in (0,\lambda^*)$ with $\lambda_0 \notin \Sigma$. Then there must be some number $\lambda_1 \in (\lambda_0,\lambda^*)$ such that $\lambda_1 \in \Sigma$, otherwise we can deduce a contradiction that $\lambda^* := \sup\Sigma \le \lambda_0 < \lambda^*$. On the other hand, according to the comparison principle, the fact that $\lambda_0 < \lambda_1 \in \Sigma$ indicates that $\lambda_0 \in \Sigma$, a contradiction to the assumption. Thus $\Sigma = (0,\lambda^*)$ holds. The proof is complete. □

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