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EXISTENCE OF SOLUTIONS TO THE CAHN-HILLIARD/ALLEN-CAHN EQUATION WITH DEGENERATE MOBILITY

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Abstract. This article we study the Cahn-Hilliard/Allen-Cahn equation with degenerate mobility. Under suitable assumptions on the degenerate mobility and the double well potential, we prove existence of weak solutions, which can be obtained by considering the limits of Cahn-Hilliard/Allen-Cahn equations with non-degenerate mobility.

1. Introduction

In this article, we consider a scalar Cahn-Hilliard/Allen-Cahn equation with degenerate mobility

$$u_t = -\nabla[D(u)\nabla(\Delta u - f(u))] + (\Delta u - f(u)), \quad \text{in } Q_T, \tag{1.1}$$

where $Q_T = \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^n$ with a C^3-boundary $\partial\Omega$ and $f(u)$ is the derivative of a double-well potential $F(u)$ with wells ± 1. The mobility $D(u) \in C(\mathbb{R};[0,\infty))$ is in the form

$$\begin{gathered} D(u) = |u|^m, \quad \text{if } |u| < \delta, \\ C_0 \le D(u) \le C_1|u|^m, \quad \text{if } |u| \ge \delta, \end{gathered} \tag{1.2}$$

for some constants $C_0, C_1, \delta > 0$, where $0 < m < \infty$ if $n = 1, 2$ and $\frac{4}{n} < m < \frac{4}{n-2}$ if $n \ge 3$.

Equation (1.1) is supplemented by the boundary conditions

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \quad t > 0, \tag{1.3}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{1.4}$$

Equation (1.1) was introduced as a simplification of multiple microscopic mechanisms model [8] in cluster interface evolution. Equation (1.1) with constant mobility has been intensively studied. Karali and Nagase [9] investigated existence of weak solution to (1.1) with $D(u) \equiv D$ and a quartic bistable potential $F(u) = (1-u^2)^2$. Karali and Nagase [9] only provided existence of the solution for the deterministic case. Then Antonopoulou, Karali and Millet [2] studied the stochastic case. The

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main result of this paper is the existence of a global solution, under a specific sublinear growth condition for the diffusion coefficient. Path regularity in time and in space is also studied. In addition, Karali and Ricciardi [7] constructed special sequences of solutions to a fourth order nonlinear parabolic equation of the Cahn-Hilliard/Allen-Cahn equation, converging to the second order Allen-Cahn equation. They studied the equivalence of the fourth order equation with a system of two second order elliptic equations. Karali and Katsoulakis [8] focus on a mean field partial differential equation, which contains qualitatively microscopic information on particle-particle interactions and multiple particle dynamics, and rigorously derive the macroscopic cluster evolution laws and transport structure. They show that the motion by mean curvature is given by $V = \mu\sigma\kappa$, where κ is the mean curvature, σ is the surface tension and μ is an effective mobility that depends on the presence of the multiple mechanisms and speeds up the cluster evolution. This is in contrast with the Allen-Cahn equation where the velocity equals the mean curvature. Tang, Liu and Zhao [18] proved the existence of global attractor. Liu and Tang [15] obtained the existence of periodic solution for a Cahn-Hilliard/Allen-Cahn equation in two space dimensions.

During the past few years, many authors have paid much attention to the Cahn-Hilliard equation with degenerate mobility. An existence result for the Cahn-Hilliard equation with a degenerate mobility in a one-dimensional situation has been established by Yin [19]. Elliott and Garcke [5] considered the Cahn-Hilliard equation with non-constant mobility for arbitrary space dimensions. Based on Galerkin approximation, they proved the global existence of weak solutions. Dai and Du [4] improved the results of the paper [5]. Liu [12] proved the existence of weak solutions for the convective Cahn-Hilliard equation with degenerate mobility. The relevant equations or inequalities have also been studied in [10, 11, 13, 14].

Motivated by the above works, we prove the existence of weak solution to (1.1)-(1.4) under a more general range of the double-well potential F. In particular, we assume that for $s \in \mathbb{R}$, $F \in C^2(R)$ satisfies

$$k_0(|s|^{r+1} - 1) \le F(s) \le k_1(|s|^{r+1} + 1), \tag{1.5}$$
$$|F'(s)| \le k_2(|s|^r + 1), \tag{1.6}$$
$$|F''(s)| \le k_3(|s|^{r-1} + 1), \tag{1.7}$$

for some constants $k_0, k_1, k_2, k_3 > 0$ where $1 \le r < \infty$ if $n = 1, 2$ and $1 \le r \le \frac{n}{n-2}$ if $n \ge 3$. What's more, we need the assumption on the boundary of $f(u)$,

$$f(u)|_{\partial\Omega} = 0, \quad t > 0. \tag{1.8}$$

We can give examples satisfying the condition (1.8), such as $F(u) = (1 - u^2)^2$ studied by Karali and Nagase [9], the logarithmic function $f(u) = -\theta_c u + \frac{\theta}{2}\ln\frac{1+u}{1-u}$, $u \in (-1, 1)$, $0 < \theta < \theta_c$ [3].

Concerning the Allen-Cahn structure, we rewrite (1.1), (1.3), (1.4) and (1.8) to the form

$$\begin{gathered} u_t = \nabla(D(u)\nabla v) - v, \quad \text{in } Q_T, \\ v = -\Delta u + f(u), \quad \text{in } Q_T, \\ u(x, 0) = u_0(x), \quad \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.9}$$

We consider the free energy functional $E(u)$ defined in [9] given by

$$E(u) := \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u)\Big)\,dx. \tag{1.10}$$

For a pair of solution (u, v) of (1.9) it holds that

$$\frac{d}{dt}E(u) = \int_\Omega v u_t\,dx = \int_\Omega v[\nabla(D(u)\nabla v) - v]\,dx = -\int_\Omega \big(D(u)|\nabla v|^2 + v^2\big)\,dx \le 0.$$

Notation. Define the usual Lebesgue norms and the L^2-inner-product

$$\|u\|_p = \|u\|_{L^p(\Omega)} \quad \text{and} \quad (u, v) = (u, v)_{L^2(\Omega)}.$$

The duality pairing between the space $H^2(\Omega)$ and its dual $(H^2(\Omega))'$ will be denoted using the form $\langle \cdot, \cdot\rangle$. For simplicity, $2^* := \frac{2n}{n-2}$. χ_B denotes the characteristic function of B.

This paper is organized as follows. In Section 2, we use a Galerkin method to give a existence of weak solution for a positive mobility. Section 3 uses a sequence of non-degenerate solutions to approximate the degenerate case (1.9).

2. Existence for positive mobility

In this section, we study the Cahn-Hilliard/Allen-Cahn equation with a non-degenerate mobility $D_\varepsilon(u)$ defined for an ε satisfying $0 < \varepsilon < \delta$ by

$$D_\varepsilon(u) := \begin{cases} |u|^m, & \text{if } |u| > \varepsilon, \\ \varepsilon^m, & \text{if } |u| \le \varepsilon. \end{cases} \tag{2.1}$$

So we consider the problem

$$\begin{gathered} u_t = \nabla(D_\varepsilon(u)\nabla v) - v, \quad \text{in } Q_T, \\ v = -\Delta u + f(u), \quad \text{in } Q_T, \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \\ u = v = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

Theorem 2.1. *Suppose $u_0 \in H^1(\Omega)$, under assumptions (1.2) and (1.5)–(1.7), for any $T > 0$, there exists a pair of functions $(u_\varepsilon, v_\varepsilon)$ such that*

1. *$u_\varepsilon \in L^\infty(0,T; H_0^1(\Omega)) \cap C([0,T]; L^p(\Omega)) \cap L^2(0,T; H^3(\Omega))$, where $1 \le p < \infty$ if $n = 1, 2$ and $2 \le p < \frac{2n}{n-2}$ if $n \ge 3$,*
2. *$\partial_t u_\varepsilon \in L^2(0,T; (H^2(\Omega))')$,*
3. *$u_\varepsilon(x,0) = u_0(x)$ for all $x \in \Omega$,*
4. *$v_\varepsilon \in L^2(0,T; H_0^1(\Omega))$,*

which satisfies equation (2.2) in the following weak sense

$$\begin{aligned} &\int_0^T \langle \partial_t u_\varepsilon, \phi\rangle dt + \iint_{Q_T} \big(-\Delta u_\varepsilon + f(u_\varepsilon)\big)\phi\,dx\,dt \\ &= -\iint_{Q_T} D_\varepsilon(u_\varepsilon)\big(-\nabla\Delta u_\varepsilon + F''(u_\varepsilon)\nabla u_\varepsilon\big)\cdot\nabla\phi\,dx\,dt \end{aligned} \tag{2.3}$$

for all test functions $\phi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$. In addition, u_ε satisfies the energy inequality

$$E(u_\varepsilon) + \int_0^t \int_\Omega \big(D_\varepsilon(u_\varepsilon(x,\tau))|\nabla v_\varepsilon(x,\tau)|^2 + |v_\varepsilon(x,\tau)|^2\big)\,dx\,d\tau \le E(u_0), \tag{2.4}$$

for all $t > 0$.

To prove the above theorem, we apply a Galerkin approximation. Let $\{\phi_J\}_{j\in N}$ be the eigenfunctions of the Laplace operator on $L^2(\Omega)$ with Dirichlet boundary condition, i.e.,

$$\begin{aligned} -\Delta\phi_J &= \lambda_J\phi_J, \quad \text{in } \Omega, \\ \phi_J &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

The eigenfunctions $\{\phi_J\}_{j=1}^{\infty}$ form an orthogonal basis for $L^2(\Omega), H^1(\Omega)$ and $H^2(\Omega)$. Hence, for initial data $u_0 \in H^1(\Omega)$, we can find sequences of scalars $(u_{N,j}^0;\ j = 1, 2, \dots, N)_{N=1}^{\infty}$ such that

$$\lim_{N\to\infty} \sum_{j=1}^{N} u_{N,j}^0 \phi_J = u_0, \quad \text{in } H^1(\Omega). \tag{2.6}$$

Let V_N denote the linear span of $(\phi_1, \dots, \phi_N)$ and $\mathscr{P}_N$ be the orthogonal projection from $L^2(\Omega)$ to V_N, that is

$$\mathscr{P}_N\phi := \sum_{j=1}^{N} \Big(\int_\Omega \phi\phi_J\, dx \Big) \phi_J.$$

Let $u^N(x,t) = \sum_{j=1}^{N} c_J^N(t)\phi_J(x)$, $v^N(x,t) = \sum_{j=1}^{N} d_J^N(t)\phi_J(x)$ be the approximate solution of (2.2) in V_N; that is, u^N, v^N satisfy the g system of equations

$$\int_\Omega \partial_t u^N \phi_J\, dx = -\int_\Omega D_\varepsilon(u^N)\nabla v^N \cdot \nabla\phi_J\, dx - \int_\Omega v^N \phi_J\, dx, \tag{2.7}$$

$$\int_\Omega v^N \phi_J\, dx = \int_\Omega \nabla u^N \cdot \nabla\phi_J + f(u^N)\phi_J\, dx, \tag{2.8}$$

$$u^N(x,0) = \sum_{j=1}^{N} u_{N,j}^0 \phi_J(x), \tag{2.9}$$

for $j = 1, \dots, N$ and $u_{N,j}^0 = \int_\Omega u_0\phi_J\, dx$.

This gives an initial value problem for a system of ordinary differential equations for $(c_1, \dots, c_N)$

$$\partial_t c_J^N(t) = -\sum_{k=1}^{N} d_k^N(t) \int_\Omega D_\varepsilon\big(\sum_{i=1}^{N} c_i^N(t)\phi_i(x)\big)\nabla\phi_k\nabla\phi_J\, dx - d_J^N(t), \tag{2.10}$$

$$d_J^N(t) = \lambda_J c_J^N(t) + \int_\Omega f\big(\sum_{i=1}^{N} c_i^N(t)\phi_i(x)\big)\phi_J\, dx, \tag{2.11}$$

$$c_J^N(0) = u_{N,j}^0 = (u_0, \phi_J), \tag{2.12}$$

which has to hold for $j = 1, \dots, N$.

Define $\mathbf{X}(t) = \big(c_1^N(t), \dots, c_N^N(t)\big)$, $\mathbf{F}(t, \mathbf{X}(t)) = \big(f_1(t, \mathbf{X}(t)), \dots, f_N(t, \mathbf{X}(t))\big)$, where

$$\begin{aligned} f_J(t, \mathbf{X}(t)) = -\sum_{k=1}^{N} &\int_\Omega D_\varepsilon\Big(\sum_{i=1}^{N} c_i^N(t)\phi_i(x)\Big)\nabla\phi_k\nabla\phi_J\, dx \\ &\times \Big(\lambda_k c_k^N(t) + \int_\Omega f\big(\sum_{i=1}^{N} c_i^N(t)\phi_i(x)\big)\phi_k\, dx\Big) \end{aligned}$$

$$- \lambda_J c_J^N(t) - \int_\Omega f\Big(\sum_{k=1}^N c_k^N(t)\phi_k(x)\Big)\phi_J\,dx$$

for $j = 1, \dots, N$. Then problem (2.10)-(2.12) is equivalent to the problem

$$\mathbf{X}'(t) = \mathbf{F}(t, \mathbf{X}(t)), \quad \mathbf{X}(0) = (u^0_{N,1}, \dots, u^0_{N,N}).$$

Since the right hand side of the above equation is continuous, it follows from the Cauchy-Peano Theorem [16] that the problem (2.10)-(2.12) has a solution $\mathbf{X}(t) \in C^1[0, T_N]$, for some $T_N > 0$, i. e., the system (2.7)-(2.9) has a local solution.

To prove the existence of solutions, we need some a priori estimates on u^N.

Lemma 2.2. *For any $T > 0$, we have*

$$\|u^N\|_{L^\infty(0,T;H_0^1(\Omega))} \le C, \quad \textit{for all } N,$$
$$\|\partial_t u^N\|_{L^2(0,T;(H^2(\Omega))')} \le C, \quad \textit{for all } N,$$

where C independent of N.

Proof. For any fixed $N \in N^+$, we multiply (2.7) by $d_J^N(t)$ and sum over $j = 1, \dots, N$ to obtain

$$\int_\Omega \partial_t u^N v^N\,dx = -\int_\Omega D_\varepsilon(u^N)|\nabla v^N|^2\,dx - \int_\Omega |v^N|^2 dx. \tag{2.13}$$

Multiply (2.8) by $\partial_t c_J^N(t)$ and sum over $j = 1, \dots, N$ to obtain

$$\begin{aligned}\int_\Omega v^N \partial_t u^N\,dx &= \int_\Omega \big(\nabla u^N \partial_t \nabla u^N + f(u^N)\partial_t u^N\big)dx,\\ &= \frac{d}{dt}\int_\Omega \big(\frac{1}{2}|\nabla u^N|^2 + F(u^N)\big)dx.\end{aligned}$$

By (2.13) and the above identity, we have

$$\frac{d}{dt}\int_\Omega \big(\frac{1}{2}|\nabla u^N|^2 + F(u^N)\big)dx = -\int_\Omega D_\varepsilon(u^N)|\nabla v^N|^2 dx - \int_\Omega |v^N|^2\,dx. \tag{2.14}$$

Replacing t by τ in (2.14) and integrating over $\tau \in [0, t]$, by (1.5) and the Sobolev embedding theorem we obtain

$$\begin{aligned}&\int_\Omega \Big(\frac{1}{2}|\nabla u^N(x,t)|^2 + F(u^N(x,t))\Big)\,dx\\ &+ \int_0^t \int_\Omega \big(D_\varepsilon(u^N(x,\tau))|\nabla v^N(x,\tau)|^2 + |v^N(x,\tau)|^2\big)\,dx\,d\tau\\ &= \int_\Omega \Big(\frac{1}{2}|\nabla u^N(x,0)|^2 + F(u^N(x,0))\Big)\,dx\\ &\le \frac{1}{2}\|\nabla u^N(x,0)\|_2^2 + k_1\|u^N(x,0)\|_{r+1}^{r+1} + k_1|\Omega|.\\ &\le \frac{1}{2}\|\nabla u_0\|_2^2 + k_1 C\|u_0\|_{H^1(\Omega)}^{r+1} + k_1|\Omega| \le C.\end{aligned}$$

The last inequality follows from $u_0 \in H^1(\Omega)$. This implies

$$\begin{aligned}&\int_\Omega \big(\frac{1}{2}|\nabla u^N(x,t)|^2 + k_0|u^N|^{r+1}\big)dx\\ &+ \int_0^t \int_\Omega \big(D_\varepsilon(u^N(x,\tau))|\nabla v^N(x,\tau)|^2 + |v^N(x,\tau)|^2\big)\,dx\,d\tau \le C.\end{aligned} \tag{2.15}$$

By (2.15) and Poincaré's inequality we have

$$\|u^N\|_{H^1(\Omega)} \le C, \quad \text{for } t > 0.$$

This estimate implies that the coefficients $\{c_J^N : j = 1, \dots, N\}$ are bounded in time and therefore a global solution to the system (2.7)-(2.9) exists. In addition, for any $T > 0$, we have

$$u^N \in L^\infty(0,T; H_0^1(\Omega)), \quad \|u^N\|_{L^\infty(0,T;H_0^1(\Omega))} \le C, \quad \text{for all } N. \tag{2.16}$$

Inequality (2.15) implies

$$\|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_{L^2(Q_T)} \le C, \quad \text{for all } N, \tag{2.17}$$

$$\|v^N\|_{L^2(Q_T)} \le C, \qquad \text{for all } N. \tag{2.18}$$

By the Sobolev embedding theorem, the growth condition (1.2) and (2.1), for $|u| > \varepsilon$, we obtain

$$\int_\Omega |D_\varepsilon(u^N)|^{n/2}\,dx \le (C_1+1)\int_\Omega |u^N|^{m\cdot\frac{n}{2}}\,dx, \le C\|u^N\|_{H^1(\Omega)}^{mn/2} \le C.$$

If $|u| \le \varepsilon$, obviously we obtain the above estimate. This implies

$$\|D_\varepsilon(u^N)\|_{L^\infty(0,T;L^{n/2}(\Omega))} \le C, \quad \text{for all } N. \tag{2.19}$$

For any $\phi \in L^2(0,T;H^2(\Omega))$, we obtain $\mathscr{P}_N\phi = \sum_{j=1}^N a_J(t)\phi_J$, where $a_J(t) = \int_\Omega \phi\phi_J dx$. Multiplying (2.7) by $a_J(t)$, summing over $j = 1, 2, \dots, N$, by Hölder's inequality, (2.17)-(2.19) and the Sobolev embedding theorem, we have

$$\begin{aligned}
&|\int_0^T\int_\Omega \partial_t u^N \phi\,dx\,dt|\\
&= |\int_0^T\int_\Omega \partial_t u^N \mathscr{P}_N\phi\,dx\,dt|\\
&= |\int_0^T\int_\Omega \big(D_\varepsilon(u^N)\nabla v^N\nabla\mathscr{P}_N\phi + v^N\mathscr{P}_N\phi\big)\,dx\,dt|\\
&\le \int_0^T \|\sqrt{D_\varepsilon(u^N)}\|_n\|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_2\|\nabla\mathscr{P}_N\phi\|_{2^*}\,dt + \int_0^T \|v^N\|_2\|\mathscr{P}_N\phi\|_2\,dt\\
&\le C\int_0^T \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_2\|\phi\|_{H^2} + \|v^N\|_2\|\phi\|_{H^2}\,dt\\
&\le C\big(\|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_{L^2(Q_T)} + \|v^N\|_{L^2(Q_T)}\big)\|\phi\|_{L^2(0,T;H^2(\Omega))}\\
&\le C\|\phi\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

Hence,

$$\|\partial_t u^N\|_{L^2(0,T;(H^2(\Omega))')} \le C \quad \text{for all } N. \tag{2.20}$$

The proof is complete. □

Lemma 2.3. *Suppose $u_0 \in H^1(\Omega)$, under assumptions (1.2) and (1.5)-(1.7), for any $T > 0$, there exists a pair of functions $(u_\varepsilon, v_\varepsilon)$ such that*

(1) *$u_\varepsilon \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^p(\Omega))$, where $1 \le p < \infty$ if $n = 1, 2$ and $2 \le p < \frac{2n}{n-2}$ if $n \ge 3$,*
(2) *$\partial_t u_\varepsilon \in L^2(0,T;(H^2(\Omega))')$,*
(3) *$u_\varepsilon(x,0) = u_0(x)$ for all $x \in \Omega$,*

(4) $v_\varepsilon \in L^2(0,T;H_0^1(\Omega))$,

which satisfies

$$\int_0^T \langle \partial_t u_\varepsilon, \phi\rangle\,dt = -\int_0^T\int_\Omega D_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla\phi\,dx\,dt - \int_0^T\int_\Omega v_\varepsilon\phi\,dx\,dt.$$

Proof. Since the embedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega)$ is compact for $1\le p<\infty$ if $n=1,2$ and $1\le p<\frac{2n}{n-2}$ if $n\ge 3$, $L^p(\Omega)\hookrightarrow (H^2(\Omega))'$ is continuous for $p\ge 1$ if $n\le 3$, $p>1$ if $n=4$ and $p\ge\frac{2n}{n+4}$ if $n\ge 5$. Using the Aubin-Lions lemma (Lions [17]), we can find a subsequence which we still denote by u^N and $u_\varepsilon\in L^\infty(0,T;H_0^1(\Omega))$, such that as $N\to\infty$

$$u^N \rightharpoonup u_\varepsilon,\quad \text{weak-* in } L^\infty(0,T;H_0^1(\Omega)), \tag{2.21}$$

$$u^N \to u_\varepsilon,\quad \text{strongly in } C([0,T];L^p(\Omega)), \tag{2.22}$$

$$u^N \to u_\varepsilon,\quad \text{strongly in } L^2(0,T;L^p(\Omega)) \text{ and almost everywehre in } Q_T, \tag{2.23}$$

$$\partial_t u^N \rightharpoonup \partial_t u_\varepsilon,\quad \text{weakly in } L^2(0,T;(H^2(\Omega))'), \tag{2.24}$$

where $2\le p<2^*$ if $n\ge 3$ and $1\le p<\infty$ if $n=1,2$.

By multiplying (2.7) by $a_J(t)$ and integrating (2.7) over $t\in[0,T]$, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t u^N a_J(t)\phi_J\,dx\,dt\\ &= -\int_0^T\int_\Omega D_\varepsilon(u^N)\nabla v^N\cdot a_J(t)\nabla\phi_J\,dx\,dt - \int_0^T\int_\Omega v^N a_J(t)\phi_J\,dx\,dt.\end{aligned} \tag{2.25}$$

To pass to the limit in (2.25), we need the convergence of v^N and $D_\varepsilon(u^N)\nabla v^N$. By (2.17) and $D_\varepsilon(u^N)\ge\varepsilon^m$, then

$$\|\nabla v^N\|_{L^2(Q_T)} \le C\varepsilon^{-\frac{m}{2}} < \infty,\quad \text{for any } \varepsilon>0. \tag{2.26}$$

This implies that $\{\nabla v^N\}$ is a bounded sequence in $L^2(Q_T)$, thus there exists a subsequence, not relabeled, and $\zeta_\varepsilon\in L^2(Q_T)$ such that

$$\nabla v^N \rightharpoonup \zeta_\varepsilon,\quad \text{weakly in } L^2(Q_T). \tag{2.27}$$

By (2.26) and Poincaré's inequality, we have

$$\|v^N\|_{L^2(0,T;H_0^1(\Omega))} \le C\varepsilon^{-\frac{m}{2}} < \infty,\quad \text{for any } \varepsilon>0.$$

Hence we can find a subsequence of v^N, not relabeled, and $v_\varepsilon\in L^2(0,T;H_0^1(\Omega))$ such that

$$v^N \rightharpoonup v_\varepsilon,\quad \text{weakly in } L^2(0,T;H_0^1(\Omega)). \tag{2.28}$$

For any $g\in L^2(0,T;H_0^1(\Omega))$, by (2.26) and (2.27) we have

$$\begin{aligned}&\lim_{N\to\infty}\int_0^T\int_\Omega \nabla v^N g\,dx\,dt = \int_0^T\int_\Omega \zeta_\varepsilon g\,dx\,dt\\ &= \lim_{N\to\infty}\int_0^T\int_\Omega v^N\nabla g\,dx\,dt = \int_0^T\int_\Omega \nabla v_\varepsilon g\,dx\,dt.\end{aligned}$$

Hence $\zeta_\varepsilon=\nabla v_\varepsilon$ almost all in Q_T and

$$\nabla v^N \rightharpoonup \nabla v_\varepsilon,\quad \text{weakly in } L^2(Q_T). \tag{2.29}$$

By (2.18), we can extract a further sequence of v^N, not relabeled, and $\eta_\varepsilon \in L^2(Q_T)$ such that

$$v^N \rightharpoonup \eta_\varepsilon, \quad \text{weakly in } L^2(Q_T). \tag{2.30}$$

By (2.28) and (2.30) for any $g \in L^2(Q_T) \subset L^2(0,T;H^{-1}(\Omega))$, we have

$$\lim_{N\to\infty} \int_0^T \int_\Omega v^N g \,dx\,dt = \int_0^T \int_\Omega v_\varepsilon g \,dx\,dt = \int_0^T \int_\Omega \eta_\varepsilon g \,dx\,dt.$$

This implies $\eta_\varepsilon = v_\varepsilon$ almost all Q_T and

$$v^N \rightharpoonup v_\varepsilon, \quad \text{weakly in } L^2(Q_T). \tag{2.31}$$

Consequently we have the bound

$$\int_{Q_T} |v_\varepsilon|^2 \,dx\,dt \le C. \tag{2.32}$$

For any $t \in [0,T]$, by $D_\varepsilon(u^N) \le C(1+|u^N|^m)$, we have

$$\big(D_\varepsilon(u^N)\big)^{n/2} \le C(1+|u^N|^m)^{n/2} \le (C(1+|u^N|))^{mn/2},$$

where $2 \le \frac{mn}{2} < 2^*$. By (2.22), $C(1+|u^N|) \to C(1+|u_\theta|)$ in $L^{mn/2}(\Omega)$. Since D_ε is continuous and (2.23), we obtain

$$D_\varepsilon(u^N) \to D_\varepsilon(u_\varepsilon), \quad \text{a.e. in } \Omega.$$

The generalized Lebesgue convergence theorem [1] gives

$$D_\varepsilon(u^N) \to D_\varepsilon(u_\varepsilon), \quad \text{in } L^{n/2}(\Omega).$$

This implies

$$\|D_\varepsilon(u^N) - D_\varepsilon(u_\varepsilon)\|_{n/2} \to 0, \quad \text{as } N \to \infty.$$

The above estimate holds for any $t \in [0,T]$, and we can take supremum on both sides of the above estimate to obtain

$$\sup_{t\in[0,T]} \|D_\varepsilon(u^N) - D_\varepsilon(u_\varepsilon)\|_{n/2} \to 0, \quad \text{as } N \to \infty.$$

This implies

$$D_\varepsilon(u^N) \to D_\varepsilon(u_\varepsilon), \quad \text{strongly in } C(0,T;L^{n/2}(\Omega)). \tag{2.33}$$

By $\sqrt{D_\varepsilon(u^N)} \le C(1+|u^N|^{\frac{m}{2}})$, (2.22), (2.23) and the generalized Lebesgue convergence theorem, similarly, we have

$$\sqrt{D_\varepsilon(u^N)} \to \sqrt{D_\varepsilon(u_\varepsilon)}, \quad \text{strongly in } C(0,T;L^n(\Omega)). \tag{2.34}$$

For any $\varphi \in L^2(0,T;L^{2^*}(\Omega))$, by Hölder's inequality we have

$$\begin{aligned}
&\Big|\iint_{Q_T} \Big(\sqrt{D_\varepsilon(u^N)}\nabla v^N\varphi - \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon\varphi\Big)\,dx\,dt\Big| \\
&= \Big|\iint_{Q_T} \Big([\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}]\nabla v^N\varphi + \sqrt{D_\varepsilon(u_\varepsilon)}[\nabla v^N\varphi - \nabla v_\varepsilon\varphi]\Big)\,dx\,dt\Big| \\
&\le \int_0^T \|\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}\|_n \|\nabla v^N\|_2 \|\varphi\|_{2^*}\,dt \\
&\quad + \Big|\iint_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)}\varphi[\nabla v^N - \nabla v_\varepsilon]\,dx\,dt\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{t\in[0,T]} \|\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}\|_n \|\nabla v^N\|_{L^2(Q_T)} \|\varphi\|_{L^2(0,T;L^{2^*}(\Omega))} \\
&\quad + \Big| \iint_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)}\varphi[\nabla v^N - \nabla v_\varepsilon]\,dx\,dt \Big| \\
&\equiv I + II.
\end{aligned}
$$

By (2.29) and (2.34), $I \to 0$ as $N \to \infty$. By Hölder's inequality and (2.34) we have

$$
\begin{aligned}
\iint_{Q_T} |\sqrt{D_\varepsilon(u_\varepsilon)}\varphi|^2\,dx\,dt &\le \int_0^T \Big(\int_\Omega \big(D_\varepsilon(u_\varepsilon)\big)^{n/2}\,dx\Big)^{n/2} \Big(\int_\Omega |\varphi|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\,dt \\
&\le \sup_{t\in[0,T]} \|\sqrt{D_\varepsilon(u_\varepsilon)}\|_n^2 \int_0^T \|\varphi\|_{L^{2^*}(\Omega)}^2\,dt \\
&\le C\|\varphi\|^2_{L^2(0,T;L^{2^*}(\Omega))}.
\end{aligned}
$$

This implies

$$
\sqrt{D_\varepsilon(u_\varepsilon)}\varphi \in L^2(Q_T). \tag{2.35}
$$

Thus $II \to 0$ as $N \to \infty$ by (2.29). Hence

$$
\sqrt{D_\varepsilon(u^N)}\nabla v^N \rightharpoonup \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon, \quad \text{weakly in } L^2(0,T;L^{\frac{2n}{n+2}}(\Omega)). \tag{2.36}
$$

Next we consider the convergence of $D_\varepsilon(u^N)\nabla v^N$. By (2.17), (2.36) and $L^2(Q_T) \subset L^2(0,T;L^{\frac{2n}{n+2}}(\Omega))$, we can extract a further sequence, not relabeled, such that

$$
\sqrt{D_\varepsilon(u^N)}\nabla v^N \rightharpoonup \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon, \quad \text{weakly in } L^2(Q_T). \tag{2.37}
$$

By Hölder's inequality and (2.17), we have

$$
\begin{aligned}
&\iint_{Q_T} \sqrt{D_\varepsilon(u^N)}\nabla v^N \cdot \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon\,dx\,dt \\
&\le \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_{L^2(Q_T)} \|\sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon\|_{L^2(Q_T)} \\
&\le C\|\sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon\|_{L^2(Q_T)},
\end{aligned} \tag{2.38}
$$

where C is independent of ε. Taking the limit of (2.38) on both sides, by (2.37) we have

$$
\|\sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon\|_{L^2(Q_T)} \le C. \tag{2.39}
$$

For any $\varphi \in L^2(0,T;L^{2^*}(\Omega))$, by Hölder's inequality we obtain

$$
\begin{aligned}
&\Big| \iint_{Q_T} \big(D_\varepsilon(u^N)\nabla v^N\varphi - D_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\varphi\big)\,dx\,dt \Big| \\
&\le \Big| \iint_{Q_T} [\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}]\sqrt{D_\varepsilon(u^N)}\nabla v^N\varphi\,dx\,dt \Big| \\
&\quad + \Big| \iint_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)}[\sqrt{D_\varepsilon(u^N)}\nabla v^N\varphi - \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon\varphi]\,dx\,dt \Big| \\
&\le \int_0^T \|\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}\|_n \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_2 \|\varphi\|_{2^*}\,dt \\
&\quad + \Big| \iint_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)}\varphi[\sqrt{D_\varepsilon(u^N)}\nabla v^N - \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon]\,dx\,dt \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup_{t\in[0,T]} \|\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}\|_n \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_{L^2(Q_T)} \|\varphi\|_{L^2(0,T;L^{2^*}(\Omega))} \\
&\quad + \Big| \iint_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)}\varphi[\sqrt{D_\varepsilon(u^N)}\nabla v^N - \sqrt{D_\varepsilon(u_\varepsilon)}\nabla v_\varepsilon]\,dx\,dt\Big| \\
&= I + II.
\end{aligned}
$$

By (2.34) and (2.37), $I \to 0$ as $N \to \infty$. By (2.35) and (2.37), we have $II \to 0$ as $N \to \infty$. Thus

$$
D_\varepsilon(u^N)\nabla v^N \rightharpoonup D_\varepsilon(u_\varepsilon)\nabla v_\varepsilon, \quad \text{weakly in } L^2(0,T;L^{\frac{2n}{n+2}}(\Omega)). \tag{2.40}
$$

For any $\phi \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, we obtain $\mathscr{P}_n\phi = \sum_{j=1}^n a_J(t)\phi_J$, where $a_J(t) = \int_\Omega \phi\phi_J dx$, then $\mathscr{P}_n\phi$ converges strongly to ϕ in $L^2(0,T;H^2\cap H_0^1(\Omega))$ and $a_J(t) \in L^2(0,T)$. For $\phi_J \in H^2(\Omega)$, by Sobolev embedding theorem, we obtain

$$
\|\nabla\phi_J\|_{2^*} \le C\|\nabla\phi_J\|_{H^1(\Omega)} \le C.
$$

Thus $a_J(t)\nabla\phi_J \in L^2(0,T;L^{2^*})$ and

$$
a_J(t)\phi_J \in L^2(0,T;H^2\cap H_0^1(\Omega)) \subset L^2(0,T;H^{-1}(\Omega)).
$$

Taking the limit as $N \to \infty$ on both sides of (2.25), by (2.24), (2.40) and (2.28), we have

$$
\begin{aligned}
&\int_0^T \langle \partial_t u_\varepsilon, a_J(t)\phi_J\rangle\,dt \\
&= -\int_0^T\int_\Omega D_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot a_J(t)\nabla\phi_J\,dx\,dt - \int_0^T\int_\Omega v_\varepsilon a_J(t)\phi_J\,dx\,dt,
\end{aligned} \tag{2.41}
$$

for all $j \in N$.

Then we sum over $j = 1,2,\dots,n$ on both sides (2.41) to get

$$
\begin{aligned}
&\int_0^T \langle \partial_t u_\varepsilon, \mathscr{P}_n\phi\rangle\,dt \\
&= -\int_0^T\int_\Omega D_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla\mathscr{P}_n\phi\,dx\,dt - \int_0^T\int_\Omega v_\varepsilon\mathscr{P}_n\phi\,dx\,dt.
\end{aligned} \tag{2.42}
$$

Since $\mathscr{P}_n\phi$ converges strongly to ϕ in $L^2(0,T;H^2(\Omega))$, thus as $n \to \infty$,

$$
\begin{aligned}
\int_0^T \|\nabla\mathscr{P}_n\phi - \nabla\phi\|_{2^*}^2\,dt &\le \int_0^T \|\nabla\mathscr{P}_n\phi - \nabla\phi\|_{H^1}^2\,dt \\
&\le \int_0^T \|\mathscr{P}_n\phi - \phi\|_{H^2}^2\,dt \to 0.
\end{aligned}
$$

This implies that $\nabla\mathscr{P}_n\phi$ converges strongly to $\nabla\phi$ in $L^2(0,T;L^{2^*}(\Omega))$. Thus we obtain

$$
\mathscr{P}_n\phi \rightharpoonup \phi, \quad \text{weakly in } L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \tag{2.43}
$$

$$
\nabla\mathscr{P}_n\phi \rightharpoonup \nabla\phi, \quad \text{weakly in } L^2(0,T;L^{2^*}(\Omega)). \tag{2.44}
$$

By $L^2(0,T;H_0^1(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$, we take the limit as $n \to \infty$ on both sides (2.42), then obtain

$$
\int_0^T \langle \partial_t u_\varepsilon, \phi\rangle\,dt = -\int_0^T\int_\Omega D_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla\phi\;dx\,dt - \int_0^T\int_\Omega v_\varepsilon\phi\;dx\,dt. \tag{2.45}
$$

As for the initial value, by (2.9) as $N \to \infty$,

$$u^N(x,0) \to u_0(x) \quad \text{in } L^2(\Omega).$$

By (2.22), $u_\varepsilon(x,0) = u_0(x)$ in $L^2(\Omega)$. The proof is complete. □

Proof of Theorem 2.1. We need only to check that $u_\varepsilon \in L^2(0,T;H^3(\Omega))$, $v_\varepsilon = -\Delta u_\varepsilon + f(u_\varepsilon)$ and $\nabla v_\varepsilon = -\nabla\Delta u_\varepsilon + F''(u_\varepsilon)\nabla u_\varepsilon$. First we consider the convergence of ∇u^N and $f(u^N)$. By (2.21), we have

$$\int_0^T \|\nabla u^N\|_2^2 dt \le C.$$

Hence we can find a subsequence of u^N, not relabeled, and $\upsilon \in L^2(Q_T)$, such that

$$\nabla u^N \rightharpoonup \upsilon \ \text{ weakly in } L^2(Q_T). \tag{2.46}$$

For any $\phi \in L^2(0,T;H_0^1(\Omega))$, by integration by parts we have

$$\lim_{N\to\infty}\int_0^T\int_\Omega \nabla u^N \phi \,dx\,dt = \lim_{N\to\infty}\int_0^T\int_\Omega u^N \nabla\phi \,dx\,dt.$$

By (2.21), (2.46) and $\nabla\phi \in L^2(Q_T) \subset L^1(0,T;H^{-1}(\Omega))$ we have

$$\int_0^T\int_\Omega \upsilon\phi \,dx\,dt = \int_0^T\int_\Omega u_\varepsilon\nabla\phi \,dx\,dt = \int_0^T\int_\Omega \nabla u_\varepsilon\phi \,dx\,dt.$$

Hence $\upsilon = \nabla u_\varepsilon$ almost all in $\Omega \times [0,T]$ and

$$\nabla u^N \rightharpoonup \nabla u_\varepsilon \quad \text{weakly in } L^2(Q_T). \tag{2.47}$$

By $|F'(u^N)| \le C(1 + |u^N|^r)$, (2.22), (2.23) and the general dominated convergence theorem, similarly, we have

$$F'(u^N) \to F'(u_\varepsilon) \quad \text{strongly in } C(0,T;L^q(\Omega)), \tag{2.48}$$

for $1 \le q < \infty$ if $n = 1,2$ and $2 \le q < \frac{2n}{r(n-2)}$ if $n \ge 3$.

By the growth condition (1.6) and the Sobolev embedding theorem, we obtain

$$\begin{aligned}
\|f(u^N)\|_{L^2(\Omega)}^2 &= \int_\Omega (F'(u^N))^2\,dx \\
&\le C\int_\Omega (|u^N|^r + 1)^2\,dx \\
&\le 2C\int_\Omega |u^N|^{2r}\,dx + 2C|\Omega| \\
&\le C\|u^N\|_{H^1(\Omega)}^{2r} + C.
\end{aligned}$$

Thus there exists a $w \in L^\infty(0,T;L^2(\Omega))$ such that

$$F'(u^N) \rightharpoonup w \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)).$$

This implies

$$\lim_{N\to\infty}\int_0^T\int_\Omega F'(u^N)g\,dx\,dt = \int_0^T\int_\Omega wg\,dx\,dt, \tag{2.49}$$

for any $g \in L^1(0,T;L^2(\Omega))$.

By Hölder's inequality, (2.48) and (2.49), we have as $N \to \infty$

$$\Big|\iint_{Q_T} \big(F'(u_\varepsilon) - w\big)g\,dx\,dt\Big|$$

$$\begin{aligned}
&\le \iint_{Q_T} |F'(u_\varepsilon) - F'(u^N)||g|\,dx\,dt + \Big| \iint_{Q_T} [F'(u^N) - w]g\,dx\,dt \Big| \\
&\le \int_0^T \|F'(u_\varepsilon) - F'(u^N)\|_2 \|g\|_2 dt + \Big| \iint_{Q_T} [F'(u^N) - w]g\,dx\,dt \Big| \le 0,
\end{aligned}$$

for any $g \in L^1(0,T;L^2(\Omega))$. Hence $F'(u_\varepsilon) = w$ a.e. in Q_T and

$$F'(u^N) \rightharpoonup F'(u_\varepsilon) \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)). \tag{2.50}$$

Multiplying (2.8) by $a_J(t)$ and integrating (2.8) over $t \in [0,T]$, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega v^N a_J(t)\phi_J\,dx\,dt \\
&= \int_0^T \int_\Omega \big(\nabla u^N \cdot a_J(t)\nabla\phi_J + F'(u^N)a_J(t)\phi_J\big)\,dx\,dt.
\end{aligned} \tag{2.51}$$

For any $\phi \in L^2(0,T;H_0^1(\Omega))$, we obtain $\mathscr{P}_n\phi = \sum_{j=1}^n a_J(t)\phi_J$, where $a_J(t) \in L^2(0,T)$. Thus $a_J(t)\phi_J \in L^2(0,T;H_0^1(\Omega))$ and $a_J(t)\nabla\phi_J \in L^2(Q_T)$. By (2.28), (2.47) and (2.50), we take the limit as $N \to \infty$ on both sides of (2.51) to get

$$\int_0^T \int_\Omega v_\varepsilon a_J(t)\phi_J\,dx\,dt = \int_0^T \int_\Omega (\nabla u_\varepsilon a_J(t)\nabla\phi_J + F'(u_\varepsilon)a_J(t)\phi_J)\,dx\,dt, \tag{2.52}$$

for all $j \in N$.

Then we sum over $j = 1,\dots,n$ on both sides (2.52), and obtain

$$\int_0^T \int_\Omega v_\varepsilon \mathscr{P}_n\phi\,dx\,dt = \int_0^T \int_\Omega (\nabla u_\varepsilon \cdot \nabla\mathscr{P}_n\phi + F'(u_\varepsilon)\mathscr{P}_n\phi)\,dx\,dt. \tag{2.53}$$

Since $\mathscr{P}_n\phi$ converges strongly to ϕ in $L^2(0,T;H_0^1(\Omega))$, we have as $n \to \infty$

$$\int_0^T \|\nabla\mathscr{P}_n\phi - \nabla\phi\|_2^2\,dt \le \int_0^T \|\mathscr{P}_n\phi - \phi\|_{H_0^1}^2\,dt \to 0.$$

This implies that $\nabla\mathscr{P}_n\phi$ converges strongly to $\nabla\phi$ in $L^2(Q_T)$. Thus we obtain

$$\mathscr{P}_n\phi \rightharpoonup \phi \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)) \tag{2.54}$$

$$\nabla\mathscr{P}_n\phi \rightharpoonup \nabla\phi \quad \text{weakly in } L^2(Q_T). \tag{2.55}$$

By $L^2(0,T;H_0^1(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$ and $L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$, we take the limit as $n \to \infty$ on both sides (2.53), and we obtain

$$\iint_{Q_T} v_\varepsilon\phi\,dx\,dt = \iint_{Q_T} (\nabla u_\varepsilon \cdot \nabla\phi + F'(u_\varepsilon)\phi)\,dx\,dt.$$

Since $F'(u_\varepsilon) \in L^\infty(0,T;L^2(\Omega))$ and $v_\varepsilon \in L^2(0,T;H_0^1(\Omega))$, it follows from regularity theory [6] that $u_\varepsilon \in L^2(0,T;H^2(\Omega))$. Hence

$$v_\varepsilon = -\Delta u_\varepsilon + F'(u_\varepsilon) \quad \text{almost everywhere in } Q_T. \tag{2.56}$$

Next we show $F'(u_\varepsilon) \in L^2(0,T;H^1(\Omega))$. By Hölder's inequality, the Sobolev embedding theorem and (1.7), we have

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla F'(u_\varepsilon)|^2\,dx\,dt &= \int_0^T \int_\Omega |F''(u_\varepsilon)|^2 |\nabla u_\varepsilon|^2\,dx\,dt \\
&\le \int_0^T \Big(\int_\Omega |F''(u_\varepsilon)|^{2\times\frac{n}{2}}\,dx\Big)^{2/n} \Big(\int_\Omega |\nabla u_\varepsilon|^{2\times\frac{n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^T \Big(\int_\Omega (1+|u_\varepsilon|^{r-1})^n\,dx\Big)^{2/n}\|\nabla u_\varepsilon\|_{\frac{2n}{n-2}}^2\,dt\\
&\le C\int_0^T \Big(1+\int_\Omega |u_\varepsilon|^{(r-1)n}\,dx\Big)^{2/n}\|\nabla u_\varepsilon\|_{H^1(\Omega)}^2\,dt\\
&\le C\int_0^T \big(1+\|u_\varepsilon\|_{\frac{2n}{n-2}}^{\frac{4}{n-2}}\big)\|u_\varepsilon\|_{H^2(\Omega)}^2\,dt\\
&\le C\big(1+\|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}^{\frac{4}{n-2}}\big)\int_0^T \|u_\varepsilon\|_{H^2(\Omega)}^2\,dt\\
&\le C\big(1+\|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}^{\frac{4}{n-2}}\big)\|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))}^2 \le C.
\end{aligned}
$$

Thus $\nabla F'(u_\varepsilon) \in L^2(Q_T)$ and $F'(u_\varepsilon) \in L^2(0,T;H^1(\Omega))$. Combined with $v_\varepsilon \in L^2(0,T;H_0^1(\Omega))$, by (2.56) and regularity theory we have $u_\varepsilon \in L^2(0,T;H^3(\Omega))$ and

$$
\nabla v_\varepsilon = -\nabla\Delta u_\varepsilon + F''(u_\varepsilon)\nabla u_\varepsilon, \quad \text{almost everywhere in } Q_T. \tag{2.57}
$$

By (2.45), (2.56) and (2.57), we obtain

$$
\begin{aligned}
&\int_0^T \langle \partial_t u_\varepsilon, \phi\rangle\,dt + \int_0^T\int_\Omega (-\Delta u_\varepsilon + F'(u_\varepsilon))\phi\,dx\,dt\\
&= -\int_0^T\int_\Omega D_\varepsilon(u_\varepsilon)(-\nabla\Delta u_\varepsilon + F''(u_\varepsilon)\nabla u_\varepsilon)\cdot\nabla\phi\,dx\,dt,
\end{aligned} \tag{2.58}
$$

for all $\phi \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

Last we show that a weak solution u_ε to (2.2) satisfies energy inequality (2.4). Replacing t by τ in (2.14) and integrating over $\tau\in[0,T]$, we have

$$
\begin{aligned}
&E(u^N(x,t)) + \int_0^t\int_\Omega D_\varepsilon(u^N(x,\tau))|\nabla v^N(x,\tau)|^2\,dx\,d\tau\\
&+\int_0^t\int_\Omega |v^N(x,\tau)|^2\,dx\,d\tau = E(u^N(x,0)).
\end{aligned} \tag{2.59}
$$

Next, we pass to the limit in (2.59). First, by mean value theorem and (1.6) we have

$$
\begin{aligned}
&\Big|\int_\Omega \big(F(u^N(t)) - F(u_\varepsilon(t))\big)\,dx\Big|\\
&\le \int_\Omega |F'(\xi)||u^N(t)-u_\varepsilon(t)|\,dx\\
&\le \int_\Omega C(|u^N(t)|^r + |u_\varepsilon(t)|^r + 1)|u^N(t)-u_\varepsilon(t)|\,dx,
\end{aligned} \tag{2.60}
$$

for $1\le r<\infty$ if $n=1,2$ and $1\le r\le \frac{n}{n-2}$ if $n\ge 3$, $\xi = \lambda u^N(t) + (1-\lambda)u_\varepsilon(t)$ for some $\lambda\in(0,1)$. By Hölder's inequality, we have

$$
\int_\Omega |u^N(t)|^r|u^N(t)-u_\varepsilon(t)|\,dx \le \|u^N(t)-u_\varepsilon(t)\|_2\|u^N(t)\|_{2r}^r. \tag{2.61}
$$

Since the Sobolev embedding theorem says that $H_0^1(\Omega)\hookrightarrow L^p(\Omega)$ for $1\le p\le 2^*$ and the embedding is compact if $1\le p<2^*$, by (2.21), then for a subsequence, not relabeled, we have $u^N\to u_\varepsilon$ strongly in $L^\infty(0,T;L^2(\Omega))$ and u^N is bounded in $L^\infty(0,T;L^{2r}(\Omega))$. Hence, it follows from (2.61) that

$$
\int_\Omega |u^N(t)|^r|u^N(t)-u_\varepsilon(t)|\,dx \to 0, \tag{2.62}
$$

as $N \to \infty$, for almost all $t \in [0, T]$.

Similarly, we can prove that

$$\int_\Omega (|u_\varepsilon(t)|^r + 1)|u^N(t) - u_\varepsilon(t)|\, dx \to 0, \tag{2.63}$$

as $N \to \infty$, for almost all $t \in [0, T]$, by (2.60), (2.62) and (2.63), we have

$$\lim_{N\to\infty} \int_\Omega F(u^N(t))\, dx = \int_\Omega F(u_\varepsilon(t))\, dx. \tag{2.64}$$

Since $u^N(x, 0) \to u_0(x)$ strongly in $L^2(\Omega)$, we obtain

$$\lim_{N\to\infty} \int_\Omega F(u^N(0))\, dx = \int_\Omega F(u_0(x))\, dx. \tag{2.65}$$

By (2.47), (2.64), (2.37), (2.29), (2.59) and the weak lower semicontinuity of the L^p norms [3]. Then

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{2}|\nabla u_\varepsilon(x,t)|^2 + F(u_\varepsilon(x,t))\Big)dx \\ &+ \int_0^t \int_\Omega \big(D_\varepsilon(u_\varepsilon(x,\tau))|\nabla v_\varepsilon(x,\tau)|^2 + |v_\varepsilon(x,\tau)|^2\big)\, dx\, d\tau \\ &\le \lim_{N\uparrow\infty} \inf \int_\Omega \left(\frac{1}{2}|\nabla u^N(x,t)|^2 + F(u^N(x,t))\right) dx \\ &\quad + \lim_{N\uparrow\infty} \inf \iint_{Q_t} \big(D_\varepsilon(u^N(x,\tau))|\nabla v^N(x,\tau)|^2 + |v^N(x,\tau)|^2\big)\, dx\, d\tau \\ &= \lim_{N\uparrow\infty} \inf E(u^N(x,0)). \end{aligned} \tag{2.66}$$

Since $u^N(x, 0) \to u_0(x)$ strongly in $H^1(\Omega)$, by (2.65) we have

$$\lim_{N\to\infty} E(u^N(x,0)) = \int_\Omega \Big(\frac{1}{2}|\nabla u_0(x)|^2 + F(u_0(x))\Big)dx. \tag{2.67}$$

Combining (2.66) with (2.67) gives the energy inequality (2.4). The proof is complete. □

3. Degenerate mobility

This section is devoted to the existence of weak solutions to the equations (1.9). Here we consider the limit of approximate solutions u_{ε_i} defined in section 2. The limiting value u does exist and solves the degenerate Allen-Cahn/Cahn-Hilliard equation in the weak sense.

Theorem 3.1. *Suppose $u_0 \in H^1(\Omega)$, under assumptions* (1.2) *and* (1.5)-(1.7)*, for any $T > 0$, problem* (1.9) *has a weak solution $u : Q_T \to R$ satisfying*

1. *$u \in L^\infty(0, T; H_0^1(\Omega)) \cap C([0, T]; L^p(\Omega)) \cap L^2(0, T; H^2(\Omega))$, where $1 \le p < \infty$ if $n = 1, 2$ and $2 \le p < \frac{2n}{n-2}$ if $n \ge 3$,*
2. *$\partial_t u \in L^2(0, T; (H^2(\Omega))')$,*
3. *$u(x, 0) = u_0(x)$ for all $x \in \Omega$,*

which satisfies (1.9) *in the following weak sense:*

(1) *Define P as the set where $D(u)$ is non-degenerate, that is*

$$P := \{(x,t) \in Q_T : |u| \neq 0\}.$$

There exists a set $A \subset Q_T$ with $|Q_T \setminus A| = 0$ and a function $\zeta : Q_T \to R^n$ satisfying $\chi_{A\cap P} D(u)\zeta \in L^2(0,T; L^{\frac{2n}{n+2}}(\Omega))$, such that

$$\begin{aligned}&\int_0^T \langle \partial_t u, \phi\rangle\,dt \\ &= -\int_0^T \int_{A\cap P} D(u)\zeta\cdot\nabla\phi\,dx\,dt - \int_0^T\int_\Omega(-\Delta u + f(u))\phi\,dx\,dt\end{aligned} \tag{3.1}$$

for all test functions $\phi \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

(2) *For each $j \in N$, there exists $E_J := \{(x,t) \in Q_T; u_i \to u$ uniformly, $|u| > \delta_J$ for $\delta_J > 0\} = T_J \times S_J$ such that*

$$u \in L^2(T_J; H^3(S_J)),$$
$$\zeta = -\nabla\Delta u + F''(u)\nabla u, \quad in \ \ E_J.$$

In addition, u satisfies the energy inequality

$$\begin{aligned}&E(u) + \iint_{Q_t\cap A\cap P} D(u(x,\tau))|\zeta(x,\tau)|^2\,dx\,d\tau \\ &+ \iint_{Q_t} |-\Delta u + f(u)|^2\,dx\,d\tau \le E(u_0),\end{aligned} \tag{3.2}$$

for all $t > 0$.

Proof. We consider a sequence of positive numbers ε_i monotonically decreasing to 0 as $i \to \infty$. Fix $u_0 \in H^1(\Omega)$, for any fixed ε_i, here, for the sake of simplicity, we write $u_i := u_{\varepsilon_i}$ and $D_i(u_i) := D_{\varepsilon_i}(u_{\varepsilon_i})$. By Theorem 2.1, there exists a function u_i such that

(1) $u_i \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^p(\Omega)) \cap L^2(0,T;H^3(\Omega))$, where $1 \le p < \infty$ if $n = 1,2$ and $2 \le p < \frac{2n}{n-2}$ if $n \ge 3$,

(2) $\partial_t u_i \in L^2(0,T;(H^2(\Omega))')$,

$$\int_0^T \langle \partial_t u_i, \phi\rangle\,dt = -\int_0^T\int_\Omega D_i(u_i)\nabla v_i\cdot\nabla\phi\,dx\,dt - \int_0^T\int_\Omega v_i\phi\,dx\,dt \tag{3.3}$$

for all test functions $\phi \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, where

$$v_i = -\Delta u_i + f(u_i), \text{almost everywhere in } Q_T. \tag{3.4}$$

By the arguments in the proof of Theorem 2.1, the bounds on the right hand side of (2.16), (2.20), (2.39) and (2.32) depend only on the growth conditions of the mobility and potential, so there exists a constant $C > 0$ independent of ε_i such that

$$\|u_i\|_{L^\infty(0,T;H_0^1(\Omega))} \le C, \tag{3.5}$$
$$\|\partial_t u_i\|_{L^2(0,T;(H^2(\Omega))')} \le C, \tag{3.6}$$
$$\|\sqrt{D_i(u_i)}\nabla v_i\|_{L^2(Q_T)} \le C, \tag{3.7}$$
$$\|v_i\|_{L^2(Q_T)} \le C. \tag{3.8}$$

Similar to the proof of Theorem 2.1, the above boundedness of $\{u_i\}$ and $\{\partial_t u_i\}$ enable us to find a subsequence, not relabeled, and $u \in L^\infty(0,T;H_0^1(\Omega))$ such that as $i \to \infty$,

$$u_i \rightharpoonup u, \quad \text{weak-* in } L^\infty(0,T;H_0^1(\Omega)), \tag{3.9}$$

$$u_i \to u, \quad \text{strongly in } C(0,T;L^p(\Omega)), \tag{3.10}$$

$$u_i \to u, \quad \text{strongly in} L^2(0,T;L^p(\Omega)) \text{ and almost all in } Q_T, \tag{3.11}$$

$$\partial_t u_i \rightharpoonup \partial_t u, \quad \text{weakly in } L^2(0,T;(H^2(\Omega))'), \tag{3.12}$$

where $1 \le p < \infty$ if $n = 1,2$ and $2 \le p < \frac{2n}{n-2}$ if $n \ge 3$.

By (3.7) and (3.8), there exists $\xi, \eta \in L^2(Q_T)$ such that

$$\sqrt{D_i(u_i)}\nabla v_i \rightharpoonup \xi, \quad \text{weakly in } L^2(Q_T), \tag{3.13}$$

$$v_i \rightharpoonup \eta, \quad \text{weakly in } L^2(Q_T). \tag{3.14}$$

Next we show the convergence of $D_i(u_i)\nabla v_i$ and $\eta = -\Delta u + f(u)$ a.e. Q_T. Similar to having (2.33) and (2.34), by the uniform convergence of $D_i \to D$, we obtain

$$D_i(u_i) \to D(u), \quad \text{strongly in } C(0,T;L^{n/2}(\Omega)), \tag{3.15}$$

$$\sqrt{D_i(u_i)} \to \sqrt{D(u)}, \quad \text{strongly in } C(0,T;L^n(\Omega)). \tag{3.16}$$

For any $\varphi \in L^2(0,T;L^{2^*}(\Omega))$, by Hölder's inequality, we have

$$\begin{aligned}
&\Big|\iint_{Q_T} \Big(D_i(u_i)\nabla v_i\varphi - \sqrt{D(u)}\xi\varphi\Big)\,dx\,dt\Big| \\
&\le \Big|\iint_{Q_T} [\sqrt{D_i(u_i)} - \sqrt{D(u)}]\sqrt{D_i(u_i)}\nabla v_i\varphi\,dx\,dt\Big| \\
&\quad + \Big|\iint_{Q_T} \sqrt{D(u)}[\sqrt{D_i(u_i)}\nabla v_i\varphi - \xi\varphi]\,dx\,dt\Big| \\
&\le \int_0^T \|\sqrt{D_i(u_i)} - \sqrt{D(u)}\|_n \|\sqrt{D_i(u_i)}\nabla v_i\|_2 \|\varphi\|_{2^*}\,dt \\
&\quad + \Big|\iint_{Q_T} \sqrt{D(u)}\varphi[\sqrt{D_i(u_i)}\nabla v_i - \xi]\,dx\,dt\Big| \\
&\le \sup_{t\in[0,T]} \|\sqrt{D_i(u_i)} - \sqrt{D(u)}\|_n \|\sqrt{D_i(u_i)}\nabla v_i\|_{L^2(Q_T)} \|\varphi\|_{L^2(0,T;L^{2^*}(\Omega))} \\
&\quad + \Big|\iint_{Q_T} \sqrt{D(u)}\varphi[\sqrt{D_i(u_i)}\nabla v_i - \xi]\,dx\,dt\Big| \\
&=: I + II.
\end{aligned}$$

By (3.16) and (3.7), $I \to 0$ as $N \to \infty$. By Hölder's inequality and the boundedness of $D(u)$ in $C(0,T;L^{n/2}(\Omega))$ we have

$$\begin{aligned}
\iint_{Q_T} |\sqrt{D(u)}\varphi|^2\,dx\,dt &\le \int_0^T \Big(\int_\Omega \big(D(u)\big)^{n/2}\,dx\Big)^{n/2} \Big(\int_\Omega |\varphi|^{2^*}\,dx\Big)^{\frac{n-2}{n}}\,dt \\
&\le \sup_{t\in[0,T]} \|D(u)\|_{n/2} \int_0^T \|\varphi\|^2_{L^{2^*}(\Omega)}\,dt \\
&\le C\|\varphi\|^2_{L^2(0,T;L^{2^*}(\Omega))}.
\end{aligned}$$

This implies

$$\sqrt{D(u)}\varphi \in L^2(Q_T). \tag{3.17}$$

By (3.13), thus $II \to 0$ as $N \to \infty$, this implies

$$D_i(u_i)\nabla v_i \rightharpoonup \sqrt{D(u)}\xi \quad \text{weakly in } L^2(0,T;L^{\frac{2n}{n+2}}(\Omega)). \tag{3.18}$$

By (3.4), for any $\phi \in L^2(0,T;H_0^1(\Omega)) \subset L^2(Q_T)$ we have

$$\begin{aligned} \iint_{Q_T} v_i\phi\,dx\,dt &= -\iint_{Q_T} \Delta u_i\phi\,dx\,dt + \iint_{Q_T} f(u_i)\phi\,dx\,dt \\ &= \iint_{Q_T} \nabla u_i\nabla\phi\,dx\,dt + \iint_{Q_T} f(u_i)\phi\,dx\,dt. \end{aligned} \tag{3.19}$$

Recalling that the convergence of ∇u_i and $f(u_i)$ are similar to get (2.47) and (2.50), we have

$$\nabla u_i \rightharpoonup \nabla u, \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)), \tag{3.20}$$

$$f(u_i) \rightharpoonup f(u), \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)). \tag{3.21}$$

By (3.20), (3.21) and $L^2(0,T;H_0^1(\Omega)) \subset L^1(0,T;L^2(\Omega))$, taking the limits of (3.19) on both sides, we have

$$\iint_{Q_T} \eta\phi\,dx\,dt = \iint_{Q_T} \nabla u\nabla\phi\,dx\,dt + \iint_{Q_T} f(u)\phi\,dx\,dt.$$

Since $f(u) \in L^\infty(0,T;L^2(\Omega))$ and $\eta \in L^2(Q_T)$, by regularity theory we see that $u \in L^2(0,T;H^2(\Omega))$ and

$$\eta = -\Delta u + f(u), \quad \text{almost everywhere in } Q_T. \tag{3.22}$$

By (3.12), (3.18) and (3.22), taking the limits of (3.3), we have

$$\int_0^T \langle \partial_t u, \phi\rangle\,dt = -\int_0^T\int_\Omega \sqrt{D(u)}\xi\nabla\phi\,dx\,dt - \int_0^T\int_\Omega [-\Delta u + f(u)]\phi\,dx\,dt. \tag{3.23}$$

As for the initial value, since $u_i(x,0) = u_0(x)$ in $L^2(\Omega)$, by (3.10) we have $u(x,0) = u_0(x)$.

Now we consider the weak convergence of ∇v_i. Choose a sequence of positive numbers δ_J that monotonically decreases to 0 as $j \to \infty$. By (3.11) and Egorov's theorem, for every $\delta_J > 0$, there exists a subset $B_J \subset Q_T$ with $|Q_T \backslash B_J| < \delta_J$ such that

$$u_i \to u, \quad \text{uniformly in } B_J.$$

Define $A_1 = B_1, A_2 = B_1 \cup B_2, \dots, A_J = B_1 \cup B_2 \cup \dots \cup B_J$. Then

$$A_1 \subset A_2 \subset \dots \subset A_J \subset A_{j+1} \subset \dots \subset Q_T. \tag{3.24}$$

Thus the limit of $\{A_J\}$ exists, then we have $\lim_{j\to\infty} A_J = \cup_{j=1}^\infty A_J := A$ and $|Q_T\backslash A| = 0$.

Define $P_J := \{(x,t) \in Q_T; |u| > \delta_J\}$. Then

$$P_1 \subset P_2 \subset \dots \subset P_J \subset P_{j+1} \subset \dots \subset Q_T. \tag{3.25}$$

Thus the limit of $\{P_J\}$ exists, then we have $\lim_{j\to\infty} P_J = \cup_{j=1}^\infty P_J := P$. For each j, we define

$$\begin{gathered} E_J := A_J \cap P_J, \quad \text{where } |u| > \delta_J \text{ and } u_i \to u \text{ uniformly}, \\ G_J := A_J \backslash P_J, \quad \text{where } |u| \le \delta_j \text{ and } u_i \to u \text{ uniformly}. \end{gathered}$$

Thus we obtain $A_J = E_J \cup G_J$. By (3.24) and (3.25), we have

$$E_1 \subset E_2 \subset \cdots \subset E_J \subset E_{j+1} \subset \cdots \subset Q_T.$$

Thus the limit of $\{E_J\}$ exists, then we have $\lim_{j\to\infty} E_J = \cup_{j=1}^{\infty} E_J = A \cap P := E$.

For any $\psi \in L^2(0,T;L^{2^*}(\Omega))$,

$$\begin{aligned} &\iint_{Q_T} D_i(u_i)\nabla v_i \psi \,dx\,dt \\ &= \iint_{Q_T\setminus A_J} D_i(u_i)\nabla v_i\psi\,dx\,dt + \iint_{G_J} D_i(u_i)\nabla v_i\psi\,dx\,dt \\ &\quad + \iint_{E_J} D_i(u_i)\nabla v_i\psi\,dx\,dt. \end{aligned} \tag{3.26}$$

As $i\to\infty$, by (3.18) we obtain

$$\lim_{i\to\infty}\iint_{Q_T} D_i(u_i)\nabla v_i\psi\,dx\,dt = \iint_{Q_T}\sqrt{D(u)}\xi\psi\,dx\,dt, \tag{3.27}$$

$$\lim_{i\to\infty}\iint_{Q_T\setminus A_J} D_i(u_i)\nabla v_i\psi\,dx\,dt = \iint_{Q_T\setminus A_J}\sqrt{D(u)}\xi\psi\,dx\,dt. \tag{3.28}$$

By $|Q_T\setminus A| = 0$, taking the limit of (3.28) as $j\to\infty$, we have

$$\lim_{j\to\infty}\lim_{i\to\infty}\iint_{Q_T\setminus A_J} D_i(u_i)\nabla v_i\psi\,dx\,dt = 0. \tag{3.29}$$

To analyze the second and third terms of (3.26), we write $u_{j-1,i} := u_i$ and $v_{j-1,i} := v_i$ in A_J, then we have

$$u_{j-1,i}\to u, \quad \text{uniformly in } A_J \text{ for all } j\in N.$$

This implies that there exists an index $N_J \in N^+$ such that for all $i \ge N_J$,

$$|u_{j-1,i} - u| < \frac{\delta_J}{2}.$$

We can easily get the following result:

$$\begin{aligned} &|u_{j-1,i}| \ge \frac{\delta_J}{2}, \quad \text{if } (x,t)\in E_J,\\ &|u_{j-1,i}| \le 2\delta_J, \quad \text{if } (x,t)\in G_J. \end{aligned} \tag{3.30}$$

Considering the limit of the second term of (3.26), by Hölder's inequality and (3.7) we have

$$\begin{aligned} &\Big|\iint_{G_J} D_{j-1,i}(u_{j-1,i})\nabla v_{j-1,i}\psi\,dx\,dt\Big| \\ &\le \sup_{(x,t)\in G_J}\sqrt{D_{j-1,i}(u_{j-1,i})}\iint_{Q_T}|\sqrt{D_{j-1,i}(u_{j-1,i})}\nabla v_{j-1,i}||\psi|\,dx\,dt \\ &\le \sup_{(x,t)\in G_J}\sqrt{D_{j-1,i}(u_{j-1,i})}\|\sqrt{D_{j-1,i}(u_{j-1,i})}\nabla v_{j-1,i}\|_{L^2(Q_T)}\|\psi\|_{L^2(Q_T)} \\ &\le C\sup_{(x,t)\in G_J}\sqrt{D_{j-1,i}(u_{j-1,i})}|\Omega|^{1/n}\|\psi\|_{L^2(0,T;L^{2^*}(\Omega))} \\ &\le C\max\{(2\delta_J)^{m/2}, \varepsilon_{j-1,i}^{m/2}\}. \end{aligned} \tag{3.31}$$

Taking the limits of (3.31) as $i \to \infty$ and $j \to \infty$, we have

$$\begin{aligned} &\lim_{j\to\infty}\lim_{i\to\infty} \Big| \iint_{G_J} D_{j-1,i}(u_{j-1,i})\nabla v_{j-1,i}\psi \,dx\,dt\Big| \\ &\le C \lim_{j\to\infty}\lim_{i\to\infty} \max\{(2\delta_J)^{m/2}, \varepsilon_{j-1,i}^{m/2}\} = 0. \end{aligned} \tag{3.32}$$

By (3.7) and (3.30), we obtain

$$\begin{aligned} (\frac{\delta_J}{2})^m \iint_{D_J} |\nabla v_{j-1,i}|^2 \,dx\,dt &\le \iint_{D_J} D_{j-1,i}(u_{j-1,i})|\nabla v_{j-1,i}|^2 \,dx\,dt \\ &\le \iint_{Q_T} D_{j-1,i}(u_{j-1,i})|\nabla v_{j-1,i}|^2 \,dx\,dt \ \le C. \end{aligned}$$

This implies

$$\iint_{D_J} |\nabla v_{j-1,i}|^2 \,dx\,dt \ \le C(\delta_J)^{-m}.$$

So $\nabla v_{j-1,i}$ is bounded in $L^2(E_J)$, thus there exists a subsequence, labeled as $\{\nabla v_{j,i}\}$, and $\zeta_J \in L^2(E_J)$ such that

$$\nabla v_{j,i} \rightharpoonup \zeta_J, \quad \text{weakly in } L^2(E_J). \tag{3.33}$$

By $E_{j-1} \subset E_J$, for any $g \in L^2(E_J)$, we have $g \in L^2(E_{j-1})$ and $\nabla v_{j-1,i} = \nabla v_{j,i}$ in E_{j-1}. By (3.33) we have

$$\begin{aligned} \lim_{i\to\infty} \iint_{E_{j-1}} \nabla v_{j,i} g \,dx\,dt &= \lim_{i\to\infty} \iint_{E_{j-1}} \nabla v_{j-1,i} g \,dx\,dt \\ &= \iint_{E_{j-1}} \zeta_J g \,dx\,dt = \iint_{E_{j-1}} \zeta_{j-1} g \,dx\,dt. \end{aligned}$$

Thus $\zeta_J = \zeta_{j-1}$ almost everywhere in E_{j-1}. we define

$$\omega_J := \begin{cases} \zeta_J, & \text{if } (x,t) \in E_J, \\ 0, & \text{if } (x,t) \in E \backslash E_J. \end{cases}$$

So for almost every $(x,t) \in E$, there exists a limit of $\omega_J(x,t)$ as $j \to \infty$. We write

$$\zeta(x,t) = \lim_{j\to\infty} \omega_J(x,t), \quad \text{almost everywhere in } E.$$

Clearly $\zeta(x,t) = \zeta_J(x,t)$ for almost all $(x,t) \in E_J$ for all j. Using a standard diagonal argument, we can extract a subsequence such that

$$\nabla v_{k,N_k} \rightharpoonup \zeta, \quad \text{weakly in } L^2(E_J) \text{ for all } j. \tag{3.34}$$

For any $\varphi \in L^2(0,T;L^{2^*}(\Omega))$, by Hölder's inequality we have

$$\begin{aligned} &\Big| \iint_{Q_T} \Big(\chi_{E_J}\sqrt{D_{k,N_k}(u_{k,N_k})}\nabla v_{k,N_k}\varphi - \chi_{E_J}\sqrt{D(u)}\zeta\varphi\Big) \,dx\,dt\Big| \\ &\le \Big| \iint_{Q_T} \chi_{E_J}[\sqrt{D_{k,N_k}(u_{k,N_k})} - \sqrt{D(u)}]\nabla v_{k,N_k}\varphi \,dx\,dt\Big| \\ &\quad + \Big| \iint_{Q_T} \chi_{E_J}\sqrt{D(u)}[\nabla v_{k,N_k}\varphi - \zeta\varphi] \,dx\,dt\Big| \\ &\le \sup_{t\in[0,T]} \|\sqrt{D_{k,N_k}(u_{k,N_k})} - \sqrt{D(u)}\|_n \int_0^T \|\chi_{E_J}\nabla v_{k,N_k}\|_2 \|\varphi\|_{\frac{2n}{n-2}} \,dt \end{aligned}$$

$$
\begin{aligned}
&+\Big|\iint_{E_J}\sqrt{D(u)}\varphi[\nabla v_{k,N_k}-\zeta]\,dx\,dt\Big|\\
&\le \sup_{t\in[0,T]}\|\sqrt{D_{k,N_k}(u_{k,N_k})}-\sqrt{D(u)}\|_n\|\nabla v_{k,N_k}\|_{L^2(E_J)}\|\varphi\|_{L^2(0,T;L^{2^*}(\Omega))}\\
&\quad+\Big|\iint_{E_J}\sqrt{D(u)}\varphi[\nabla v_{k,N_k}-\zeta]\;dx\,dt\Big|\\
&=: I+II.
\end{aligned}
$$

By (3.16) and (3.34), $I\to 0$ as $N\to\infty$. By (3.17) and (3.34), we have $II\to 0$ as $N\to\infty$. Thus

$$
\chi_{E_J}\sqrt{D_{k,N_k}(u_{k,N_k})}\nabla v_{k,N_k}\rightharpoonup \chi_{E_J}\sqrt{D(u)}\zeta,\quad \text{weakly in } L^2(0,T;L^{\frac{2n}{n+2}}(\Omega)),
$$

for all j.

From $L^2\subset L^{\frac{2n}{n+2}}$ and (3.13), we see that $\xi=\sqrt{D(u)}\zeta$ in every E_J and

$$
\xi=\sqrt{D(u)}\zeta\quad \text{in } E. \tag{3.35}
$$

Consequently, by (3.18),

$$
\chi_E D_{k,N_k}(u_{k,N_k})\nabla v_{k,N_k}\rightharpoonup \chi_E D(u)\zeta,\quad \text{weakly in } L^2(0,T;L^{\frac{2n}{n+2}}(\Omega)).
$$

Thus by Taking the limits of third term of (3.26), we have

$$
\begin{aligned}
&\lim_{j\to\infty}\lim_{k\to\infty}\iint_{E_J} D_{k,N_k}(u_{k,N_k})\nabla v_{k,N_k}\psi\,dx\,dt\\
&=\lim_{j\to\infty}\iint_{E_J} D(u)\zeta\psi\,dx\,dt=\iint_E D(u)\zeta\psi\,dx\,dt.
\end{aligned} \tag{3.36}
$$

By (3.27), (3.29), (3.32) and (3.36), we have

$$
\iint_{Q_T}\sqrt{D(u)}\xi\psi\,dx\,dt=\iint_E D(u)\zeta\psi\,dx\,dt. \tag{3.37}
$$

By (3.23) and (3.37), we find that u and ζ solve equation (1.9) in the following weak sense

$$
\int_0^T\langle\partial_t u,\phi\rangle\,dt=-\iint_E D(u)\zeta\nabla\phi\,dx\,dt-\int_0^T\int_\Omega[-\Delta u+f(u)]\phi\,dx\,dt, \tag{3.38}
$$

for all $\phi\in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

From (3.14) and (3.34), we notice that v_i is bounded in $L^2(T_J;H^1(S_J))$, where $E_J=T_J\times S_J$. So we can extract a further sequence, not relabeled, and

$$
\begin{gathered}
v\in L^2(T_J;H^1(S_J)),\\
v_i\rightharpoonup v\quad \text{weakly in } L^2(T_J;H^1(S_J)).
\end{gathered} \tag{3.39}
$$

Similar to show $F'(u_\varepsilon)\in L^2(0,T;H^1(\Omega))$ and (3.22). Hence, we have $F'(u)\in L^2(0,T;H^1(\Omega))$ and $v=-\Delta u+f(u)$, a.e. in E_J, By $v\in L^2(T_J;H^1(S_J))$ we have $u\in L^2(T_J;H^3(S_J))$ and

$$
\nabla v=-\nabla\Delta u+F''(u)\nabla u,\quad \text{almost everywhere in } E_J. \tag{3.40}
$$

Obviously we have $\eta=v$, $\zeta=\nabla v$, a. e. inE_J. So we obtain the desired relation between ζ and u:

$$
\zeta=-\nabla\Delta u+F''(u)\nabla u,\quad \text{in } E_J.
$$

Finally, we show that a weak solution u to (1.9) satisfies energy inequality (3.2). By (2.4) we have

$$\begin{aligned}&\int_\Omega \Big(\frac{1}{2}|\nabla u_{k,N_k}(x,t)|^2 + F(u_{k,N_k}(x,t))\Big)dx \\ &+ \iint_{Q_t\cap E} D_{k,N_k}(u_{k,N_k}(x,\tau))|\nabla v_{k,N_k}(x,\tau)|^2\,dx\,d\tau \\ &+ \iint_{Q_t} |v_{k,N_k}(x,\tau)|^2\,dx\,d\tau \\ &\le \int_\Omega \Big(\frac{1}{2}|\nabla u_0|^2 + F(u_0)\Big)dx.\end{aligned} \tag{3.41}$$

By having (2.47) and (2.66), similarly we have

$$\nabla u_{k,N_k} \rightharpoonup \nabla u, \quad \text{weakly in } L^2(Q_T), \tag{3.42}$$

$$\lim_{N\to\infty} \int_\Omega F(u_{k,N_k}(t))\,dx = \int_\Omega F(u(t))\,dx. \tag{3.43}$$

By (3.42), (3.43), (3.13), (3.35), (3.14), (3.22), (3.41) and the weak lower semicontinuity of the L^p norms. Then

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}|\nabla u(x,t)|^2 + F(u(x,t))\right)dx + \iint_{Q_t\cap E} D(u(x,\tau))|\zeta(x,\tau)|^2 dx d\tau \\ &\quad + \iint_{Q_t} |-\Delta u + f(u)|^2\,dx\,d\tau \\ &\le \liminf_{N\uparrow\infty} \int_\Omega \Big(\frac{1}{2}|\nabla u_{k,N_k}(x,t)|^2 + F(u_{k,N_k}(x,t))\Big)dx \\ &\quad + \liminf_{N\uparrow\infty} \iint_{Q_t\cap E} D_{k,N_k}(u_{k,N_k}(x,\tau))|\nabla v_{k,N_k}(x,\tau)|^2\,dx\,d\tau \\ &\quad + \liminf_{N\uparrow\infty} \iint_{Q_t} |v_{k,N_k}(x,\tau)|^2 dx d\tau \\ &\le \int_\Omega \Big(\frac{1}{2}|\nabla u_0|^2 + F(u_0)\Big)dx.\end{aligned}$$

This gives the energy inequality (3.2). The proof is complete. □

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