Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 324, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONLINEAR IMPLICIT DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER AT RESONANCE

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Abstract. In this article, we obtain an existence result for periodic solutions to nonlinear implicit fractional differential equations with Caputo fractional derivatives. Our main tools is coincidence degree theory, which was first introduced by Mawhin. Also we present two examples to show the applicability of our results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary orders (non-integer). See, for example, the monographs [1, 2, 4, 5, 22] and the references therein.

In recent years, fractional differential equations arise naturally in various fields such as rheology, fractals, chaotic dynamics, modeling, control theory, signal processing, bioengineering and biomedical applications, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. We refer the reader to the recent works [6, 7, 14, 15, 16, 17, 20, 21, 23, 25] and the references therein.

In this article, we are concerned with the existence of periodic solutions to the nonlinear implicit fractional differential equation (IFDE for short)

$$ {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \quad t \in J := [0, T],\ T > 0,\ 0 < \alpha \le 1, \tag{1.1}$$

$$ y(0) = y(T), \tag{1.2}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Recently, by means of different tools such as the Banach contract principle, Schauder's fixed point, Schaefer's fixed point, the Leray-Schauder nonlinear alternative, Monch's fixed point theorem, and the measure of noncompactness, initial and boundary value problems for implicit fractional differential equations involving Caputo type fractional derivatives have extensively been studied in the books [1, 2] and the papers [8, 9, 10, 11].

This article is motivated by the works [3, 13, 24] where coincidence degree theory is used for some classes of boundary value problem for fractional differential equations of integer as well as noninteger orders. This article is organized as follows. In

2010 *Mathematics Subject Classification.* 34A08, 34B15.
Key words and phrases. Caputo's fractional derivative; fractional integral; existence; implicit fractional differential equations; periodic solutions; coincidence degree theory.

Submitted July 25, 2016. Published December 21, 2016.

Section 2, some notations are introduced and we recall some preliminary concepts about fractional calculus. The proof of our main result is presented in Section 3 by applying the coincidence degree theory of Mawhin. In the last section, we give two examples to illustrate the applicability of our main results. This paper initiates the application of coincidence degree to the study of implicit fractional differential equations.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts that are used throughout this paper. By $C(J,\mathbb{R})$, we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

By $L^1(J)$, we denote the space of Lebesgue-integrable functions $y : J \to \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

Definition 2.1 ([22])**.** The fractional integral of order $\alpha \in \mathbb{R}_+$ of the function $h \in L^1([0,T], \mathbb{R}_+)$ is defined by

$$^cI^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$, $\alpha > 0$.

Definition 2.2 ([16])**.** For a function h given on the interval $[0,T]$, the Caputo fractional derivative of order α of h is defined by

$$(^cD^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3 ([16])**.** *Let $\alpha > 0$ and $n = [\alpha] + 1$; then*

$$^cI^\alpha(^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Lemma 2.4 ([22])**.** *If $\alpha > 0$, the homogeneous differential equation of fractional order*

$$^cD^\alpha h(t) = 0$$

has a solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where c_i, $i = 1, \ldots, n$ are constants and $n = [\alpha] + 1$.

The following definitions and basic lemmas from coincidence degree theory are fundamental in the proof of our main result (see [12, 18]).

Definition 2.5. Let X and Y be normed spaces. A linear operator $L : \operatorname{dom} L \subset X \to Y$ is said to be a Fredholm operator of index zero provided that

1. $\operatorname{img} L$ is a closed subset of Y;
2. $\dim\ker L = \operatorname{codim}\operatorname{img} L < +\infty$.

It follows from Definition refopdeFrdm that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{img} P = \ker L, \quad \ker Q = \operatorname{img} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{img} L \oplus \operatorname{img} Q.$$

This implies that the restriction of L to $\operatorname{dom} L \cap \ker P$, which we will denote by L_P, is an isomorphism onto its image.

Definition 2.6. Let L be a Fredholm operator of index zero and let $\Omega \subseteq X$ be a bounded set with $\operatorname{dom} L \cap \Omega \neq \emptyset$. The operator $N : \overline{\Omega} \to Y$ is L-compact in $\overline{\Omega}$ if

(1) the mapping $QN : \overline{\Omega} \to Y$ is continuous and $QN(\overline{\Omega}) \subseteq Y$ is bounded, and
(2) the mapping $(L_P)^{-1}(I - Q)N : \overline{\Omega} \to X$ is completely continuous.

Lemma 2.7 ([19]). *Let X and Y be Banach spaces and let $\Omega \subset X$ be a bounded open symmetric set with $0 \in \Omega$. Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero with $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$ and $N : X \to Y$ be an L-compact operator on $\overline{\Omega}$. Assume that*

$$Lx - Nx \neq -\lambda(Lx + N(-x))$$

for all $x \in \operatorname{dom} L \cap \partial\Omega$ and all $\lambda \in (0, 1]$, where $\partial\Omega$ is the boundary of Ω with respect to X. Then the equation $Lx = Nx$ has at least one solution on $\operatorname{dom} L \cap \overline{\Omega}$.

3. Existence of solutions

Let $X = \{y \in C(J, \mathbb{R}) : y(t) = {}^cI^\alpha u(t) : u \in C(J, \mathbb{R}), t \in J\}$ with the norm

$$\|y\|_X = \max\{\|y\|_\infty, \|{}^cD^\alpha y\|_\infty\}$$

and $Y = C(J, \mathbb{R})$ with the norm

$$\|u\|_Y = \sup\{|u(t)| : t \in J\}.$$

Define the linear operator $L : \operatorname{dom} L \subseteq X \to Y$ by

$$Ly := {}^cD^\alpha y, \tag{3.1}$$

where

$$\operatorname{dom} L = \{y \in X : {}^cD^\alpha y \in Y \quad \text{and} \quad y(0) = y(T)\}.$$

Define the operator $N : X \to Y$ by

$$Ny(t) := f(t, y(t), {}^cD^\alpha y(t)), \quad t \in J. \tag{3.2}$$

Then problem (1.1)–(1.2) can be equivalently rewritten as $Ly = Ny$.

Lemma 3.1. *Let L be defined by* (3.1). *Then* $\ker L = \{c : c \in \mathbb{R}\}$ *and*

$$\operatorname{img} L = \Big\{y \in Y : \int_0^T (T - s)^{\alpha-1} y(s) ds = 0\Big\}.$$

Proof. By Lemma 2.4, for $t \in J$, $Ly(t) = {}^cD^\alpha y(t) = 0$ has the solution $y(t) = c$, where $c \in \mathbb{R}$. Then

$$\ker L = \{y(t) = c : c \in \mathbb{R}\}.$$

For $u \in \operatorname{img} L$, there exists $y \in domL$ such that $u = Ly \in Y$. By Lemma 2.3, for each $t \in J$ we have

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s) ds.$$

Since $y \in \operatorname{dom} L$, u satisfies

$$\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} u(s) ds = 0.$$

On the other hand, suppose $u \in Y$ satisfies

$$\int_0^T (T-s)^{\alpha-1} u(s) ds = 0.$$

Let $y(t) = {}^c I^\alpha u(t)$; then $u(t) = {}^c D^\alpha y(t)$ and so $y \in \operatorname{dom} L$. Hence, $u \in \operatorname{img} L$, so

$$\operatorname{img} L = \left\{ y \in Y : \int_0^T (T-s)^{\alpha-1} y(s) ds = 0 \right\},$$

which completes the proof. $\square$

Lemma 3.2. *Let L be defined by* (3.1)*. Then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$Py = y(0), \quad Qu(t) = \frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1} u(s) ds.$$

Furthermore, the operator $L_P^{-1} : \operatorname{img} L \to X \cap \ker P$ satisfies

$$L_P^{-1}(u)(t) = {}^c I^\alpha u(t).$$

Proof. Clearly, $\operatorname{img} P = \ker L$ and $P^2 = P$. It follows that for each $y \in X$, $y = (y - Py) + Py$, that is, $X = \ker P + \ker L$. A simple calculation shows that $\ker P \cap \ker L = 0$. Therefore, $X = \ker P \oplus \ker L$. A similar argument shows that for each $u \in Y$, $Q^2 u = Qu$ and $u = (u - Q(u)) + Q(u)$, where $(u - Q(u)) \in \ker Q = \operatorname{img} L$.

It follows from $\operatorname{img} L = \ker Q$ and $Q^2 = Q$, that $\operatorname{img} Q \cap \operatorname{img} L = 0$. Then, we have $Y = \operatorname{img} L \oplus \operatorname{img} Q$. Thus,

$$\dim \ker L = \dim \operatorname{img} Q = \operatorname{codim} \operatorname{img} L.$$

This means that L is a Fredholm operator of index zero.

To prove that L_P^{-1} is the inverse of $L|_{\operatorname{dom} L \cap \ker P}$, let $u \in \operatorname{img} L$. Then

$$L L_P^{-1}(u) = {}^c D^\alpha({}^c I^\alpha u) = u. \tag{3.3}$$

Moreover, for $y \in \operatorname{dom} L \cap \ker P$, we obtain that

$$L_P^{-1}(L(y(t))) = {}^c I^\alpha({}^c D^\alpha y(t)) = y(t) - y(0).$$

Since $y \in \operatorname{dom} L \cap \ker P$, we know that $y(0) = 0$. Therefore

$$L_P^{-1}(L(y(t))) = y(t). \tag{3.4}$$

Combining (3.3) and (3.4) shows that $L_P^{-1} = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. This proves the lemma. $\square$

In the sequel we use of the following assumption.

(H1) There exist constants $K, \overline{K} > 0$ with $K + \overline{K} < \min\left\{1, \frac{\Gamma(\alpha+1)}{T^\alpha}\right\}$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le K|u - \bar{u}| + \overline{K}|v - \bar{v}| \quad \text{for } t \in J \text{ and } u, \bar{u}, v, \bar{v} \in \mathbb{R}.$$

Lemma 3.3. *Assume* (H1) *holds. Then the operator N is L-compact on any bounded open set $\Omega \subset X$.*

Proof. Define the bounded open set $\Omega = \{y \in X : \|y\|_X < M\}$, where M is a positive constant. The proof will be given in a sequence of claims.

Claim 1: QN is continuous. The continuity of QN follows from the conditions on f and the Lebesgue dominated convergence theorem.

Claim 2: $QN(\overline{\Omega})$ is bounded. For each $y \in \overline{\Omega}$ and $t \in J$, we have

$$\begin{aligned}
|QN(y)(t)| &\leq \frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}|f(s,y(s),{}^cD^\alpha y(s))|ds \\
&\leq \frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}|f(s,y(s),{}^cD^\alpha y(s)) - f(s,0,0)|ds \\
&\quad + \frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}|f(s,0,0)|ds \\
&\leq f^* + \frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}(K|y(s)| + \overline{K}|{}^cD^\alpha y(s)|)ds \\
&\leq f^* + M(K+\overline{K}),
\end{aligned}$$

where $f^* = \sup_{t\in J}|f(t,0,0)|$. Thus,

$$\|QN(y)\|_Y \leq f^* + M(K+\overline{K}) := R.$$

This shows that $QN(\overline{\Omega}) \subseteq Y$ is bounded.

Claim 3: $L_P^{-1}(I-Q)N : \overline{\Omega} \to X$ is completely continuous.

In view of the Ascoli-Arzelà theorem, we need to prove that $L_P^{-1}(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous and bounded. First, for each $y \in \overline{\Omega}$ and $t \in J$, we have

$$\begin{aligned}
&L_P^{-1}(I-Q)Ny(t) \\
&= L_P^{-1}(Ny(t) - QNy(t)) \\
&= {}^cI^\alpha\Big[f(t,y(t),{}^cD^\alpha y(t)) - \frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}f(s,y(s),{}^cD^\alpha y(s))\Big]ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s),{}^cD^\alpha y(s))ds \\
&\quad - \frac{t^\alpha}{T^\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s,y(s),{}^cD^\alpha y(s))ds.
\end{aligned}$$

On one hand, for each $y \in \overline{\Omega}$ and $t \in J$, we have

$$\begin{aligned}
&|L_P^{-1}(I-Q)Ny(t)| \\
&\leq \frac{2}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y(s),{}^cD^\alpha y(s)) - f(t,0,0)|ds \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(t,0,0)|ds \\
&\leq [f^* + M(K+\overline{K})]\frac{2T^\alpha}{\Gamma(\alpha+1)} := B_1,
\end{aligned}$$

so

$$\|L_P^{-1}(I-Q)Ny\|_\infty \leq B_1. \tag{3.5}$$

On the other hand,

$$\begin{aligned} &{}^cD^\alpha(L_P^{-1}(I-Q)Ny(t)) \\ &= f(t,y(t),{}^cD^\alpha y(t)) - \frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1} f(s,y(s),{}^cD^\alpha y(s))ds, \end{aligned} \tag{3.6}$$

which implies that for each $y \in \overline{\Omega}$ and $t \in J$,

$$|{}^cD^\alpha(L_P^{-1}(I-Q)Ny(t))| \le 2f^* + 2M(K+\overline{K}) := B_2,$$

so that

$$\|{}^cD^\alpha(L_P^{-1}(I-Q)Ny)\|_\infty \le B_2. \tag{3.7}$$

From inequalities (3.5) and (3.7), we have

$$\|L_P^{-1}(I-Q)Ny\|_X \le \max\{B_1, B_2\},$$

which shows that $L_P^{-1}(I-Q)N(\overline{\Omega})$ is uniformly bounded in X.

To prove that $L_P^{-1}(I-Q)N(\overline{\Omega})$ is equicontinuous, notice that for $0 \le t_1 \le t_2 \le T$ and $y \in \overline{\Omega}$, we have

$$\begin{aligned} &|L_P^{-1}(I-Q)Ny(t_2) - L_P^{-1}(I-Q)Ny(t_1)| \\ &\le \frac{f^* + M(K+\overline{K})}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds + \int_0^{t_1}|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|ds\Big] \\ &\quad + \Big[\frac{M(K+\overline{K})+f^*}{\Gamma(\alpha+1)}\Big](t_2^\alpha - t_1^\alpha). \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Now from (3.6), we have

$$\begin{aligned} &|{}^cD^\alpha(L_P^{-1}(I-Q)Ny)(t_2) - {}^cD^\alpha(L_P^{-1}(I-Q)Ny)(t_1)| \\ &\le |f(t_2,y(t_2),{}^cD^\alpha y(t_2)) - f(t_1,y(t_1),{}^cD^\alpha y(t_1))|. \end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality also tends to zero. Thus, $L_P^{-1}(I-Q)N(\overline{\Omega})$ is equicontinuous in X. By the Ascoli-Arzelà theorem, $L_P^{-1}(I-Q)N(\overline{\Omega})$ is relatively compact. As a consequence of Claims 1 to 3, we can conclude that the operator N is L-compact in $\overline{\Omega}$, and this completes the proof. □

Lemma 3.4. *If condition* (H1) *holds, then there exists a positive number A, not depending on λ, such that, if*

$$L(y) - N(y) = -\lambda[L(y) + N(-y)], \quad \lambda \in (0,1], \tag{3.8}$$

then $\|y\|_X \le A$.

Proof. Assume (H1) holds and that $y \in X$ satisfies (3.8). Then

$$L(y) - N(y) = -\lambda L(y) - \lambda N(-y),$$

so

$$L(y) = \frac{1}{1+\lambda}N(y) - \frac{\lambda}{1+\lambda}N(-y). \tag{3.9}$$

Using the definitions of the operators L and N (see (3.1) and (3.2)), for each $t \in J$, we obtain

$$\begin{aligned} |Ly(t)| = |{}^cD^\alpha y(t)| &\le \frac{1}{1+\lambda}\,|f(t,y(t),{}^cD^\alpha y(t))| + \frac{\lambda}{1+\lambda}\,|f(t,-y(t),-{}^cD^\alpha y(t))| \\ &\le \frac{1}{1+\lambda}[|f(t,y(t),{}^cD^\alpha y(t)) - f(t,0,0)| + f^*] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda}{1+\lambda}[|f(t,-y(t),-{}^cD^\alpha y(t)) - f(t,0,0)| + f^*] \\
&\le \Big(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\Big) f^* + \frac{1}{1+\lambda}[K|y(t)| + \overline{K}|{}^cD^\alpha y(t)|] \\
&+ \frac{\lambda}{1+\lambda}[K|-y(t)| + \overline{K}|-{}^cD^\alpha y(t)|] \\
&= f^* + \Big(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\Big)[K|y(t)| + \overline{K}|{}^cD^\alpha y(t)|] \\
&= f^* + K|y(t)| + \overline{K}|{}^cD^\alpha y(t)| \\
&\le f^* + K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty,
\end{aligned}
$$

which implies

$$
\|{}^cD^\alpha y\|_\infty \le f^* + K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty. \tag{3.10}
$$

By (3.9), for each $t \in J$, we have

$$
y(t) = \frac{1}{1+\lambda} L_p^{-1} N y(t) - \frac{\lambda}{1+\lambda} L_p^{-1} N(-y(t)),
$$

and so

$$
\begin{aligned}
|y(t)| &\le \frac{1}{(1+\lambda)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,y(s),{}^cD^\alpha y(s)) - f(s,0,0)|\, ds \\
&\quad + \frac{\lambda}{(1+\lambda)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,-y(s),-{}^cD^\alpha y(s)) - f(s,0,0)|\, ds \\
&\quad + \frac{f^* T^\alpha}{(1+\lambda)\Gamma(\alpha+1)} + \frac{\lambda f^* T^\alpha}{(1+\lambda)\Gamma(\alpha+1)} \\
&\le \Big(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\Big) \frac{T^\alpha}{\Gamma(\alpha+1)} (K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty) \\
&\quad + \Big(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\Big) \frac{f^* T^\alpha}{\Gamma(\alpha+1)} \\
&= \frac{T^\alpha}{\Gamma(\alpha+1)} (K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty) + \frac{f^* T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Hence,

$$
\|y\|_\infty \le [f^* + K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty] \frac{T^\alpha}{\Gamma(\alpha+1)}. \tag{3.11}
$$

Using the definition of the norm $\|\cdot\|_X$, we see that if $\|y\|_X = \|{}^cD^\alpha y\|_\infty$, then, by (3.10), we have

$$
\begin{aligned}
\|y\|_X &\le f^* + K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty \\
&\le f^* + K\|y\|_X + \overline{K}\|y\|_X \\
&= f^* + (K + \overline{K})\|y\|_X,
\end{aligned}
$$

and thus

$$
\|y\|_X \le \frac{f^*}{1 - (K + \overline{K})} := A_1.
$$

On the other hand, if $\|y\|_X = \|y\|_\infty$, then (3.11) implies

$$
\|y\|_X \le [f^* + K\|y\|_\infty + \overline{K}\|{}^cD^\alpha y\|_\infty] \frac{T^\alpha}{\Gamma(\alpha+1)}
$$

$$
\begin{aligned}
&\le [f^* + K\|y\|_X + \overline{K}\|y\|_X]\frac{T^\alpha}{\Gamma(\alpha+1)} \\
&= [f^* + (K+\overline{K})\|y\|_X]\frac{T^\alpha}{\Gamma(\alpha+1)},
\end{aligned}
$$

and so

$$
\|y\|_X \le \frac{f^*}{\frac{\Gamma(\alpha+1)}{T^\alpha} - (K+\overline{K})} := A_2.
$$

Therefore

$$
\|y\|_X \le \max\{A_1, A_2\} := A,
$$

and this completes the proof. □

Lemma 3.5. *If condition* (H1) *is satisfied, then there is a bounded open set* $\Omega \subset X$ *such that*

$$
L(y) - N(y) \ne -\lambda[L(y) + N(-y)], \tag{3.12}
$$

for all $y \in \partial\Omega$ *and all* $\lambda \in (0,1]$.

Proof. By (H1) and Lemma 3.4, there exists a positive constant A that does not depend on λ such that, if y satisfies

$$
L(y) - N(y) = -\lambda[L(y) + N(-y)],\ \lambda \in (0,1],
$$

then $\|y\|_X \le A$. Thus, if

$$
\Omega = \{y \in X : \|y\|_X < B\} \tag{3.13}
$$

where $B > A$, we have

$$
L(y) - N(y) \ne -\lambda[L(y) - N(-y)]
$$

for every $y \in \partial\Omega = \{y \in X : \|y\|_X = B\}$ and $\lambda \in (0,1]$. □

We are now ready to prove the main result in our paper.

Theorem 3.6. *If* (H1) *holds, then problem* (1.1)–(1.2) *has at least one solution.*

Proof. From (H1) it is clear that the set Ω defined by (3.13) is symmetric, $0 \in \Omega$, and $X \cap \overline{\Omega} = \overline{\Omega} \ne \emptyset$. Furthermore, it follows from Lemma 3.5 that if condition (H1) is satisfied, then

$$
L(y) - N(y) \ne -\lambda[L(y) - N(-y)]
$$

for all $y \in X \cap \partial\Omega = \partial\Omega$ and all $\lambda \in (0,1]$. This together with Lemma 2.7 imply that problem (1.1)–(1.2) has at least one solution, and this completes the proof. □

4. Examples

In this section we give two examples to illustrate our theorem.

Example 4.1. Consider the problem for non-linear implicit fractional differential equations

$$
D^{1/2}y(t) = \frac{e^{-t}}{(11+e^t)}\Big[\frac{|y(t)|}{1+|y(t)|} - \frac{|D^{1/2}y(t)|}{1+|D^{1/2}y(t)|}\Big], \quad t \in [0,1], \tag{4.1}
$$

$$
y(0) = y(1). \tag{4.2}
$$

Here we have

$$
f(t,u,v) = \frac{e^{-t}}{(11+e^t)}\Big(\frac{u}{1+u} - \frac{v}{1+v}\Big), \quad t \in [0,1],\ u,v \in [0,+\infty),
$$

and clearly the function f is jointly continuous. For each u, $\bar{u}$, v, $\bar{v} \in [0, +\infty)$ and $t \in [0, 1]$,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{e^{-t}}{(11+e^t)}[|u-\bar{u}| + |v-\bar{v}|] \le \frac{1}{12}[|u-\bar{u}| + |v-\bar{v}|].$$

Hence, condition (H1) is satisfied with $K = \overline{K} = 1/12$ and

$$K + \overline{K} = \frac{1}{6} < \min\big\{1, \frac{\Gamma(\alpha+1)}{T^\alpha}\big\} = \min\big\{1, \frac{\sqrt{\pi}}{2}\big\} = \frac{\sqrt{\pi}}{2}.$$

It follows from Theorem 3.6 that problem (4.1)–(4.2) has at least one solution.

Example 4.2. Consider the problem

$$D^{1/2}y(t) = \frac{t}{3}\sin y(t) + \frac{1}{100}\sin D^{1/2}y(t) + \frac{1}{2}, \quad t \in [0,1], \tag{4.3}$$

$$y(0) = y(1). \tag{4.4}$$

Here,

$$f(t,u,v) = \frac{t}{3}\sin u + \frac{1}{100}\sin v + \frac{1}{2}, \quad t \in [0,1],\ u, v \in \mathbb{R}.$$

which is jointly continuous. For any u, $\bar{u}$, v, $\bar{v} \in \mathbb{R}$ and $t \in [0, 1]$,

$$\begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\le \frac{|t|}{3}|\sin u - \sin\bar{u}| + \frac{1}{100}|\sin v - \sin\bar{v}| \\ &\le \frac{1}{3}|u-\bar{u}| + \frac{1}{100}|v-\bar{v}|. \end{aligned}$$

Hence, condition (H1) is satisfied with $K = 1/3$, $\overline{K} = 1/100$, and

$$K + \overline{K} = \frac{103}{300} < \min\big\{1, \frac{\Gamma(\alpha+1)}{T^\alpha}\big\} = \frac{\sqrt{\pi}}{2}.$$

It follows from Theorem 3.6 that problem (4.3)–(4.4) has at least one solution on J.

Acknowledgments. The authors would like to thank the anonymous referee for the very helpful and clearly stated recommendations for changes to this article.

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