Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 323, pp. 1–24.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

PARTIAL CONTINUITY FOR A CLASS OF ELLIPTIC SYSTEMS WITH NON-STANDARD GROWTH

JIHOON OK

Abstract. We study partial Hölder continuity of weak solutions to elliptic systems with variable non-standard growth, which are related to the function $\Phi(x,t) := t^{p(x)} \log(e+t)$. We prove that weak solutions are Hölder continuous for any Hölder exponent, except Lebesgue measure zero sets, if systems satisfy certain continuity assumptions. In particular, the variable exponent functions $p(\cdot)$ are assumed to satisfy so-called vanishing log-Hölder continuity.

1. Introduction

The aim of this article is to study partial Hölder continuity of solutions to elliptic systems with non-standard growth conditions, and, in particular, with a certain type of Orlicz growth conditions. Problems with non-standard growth conditions have initially studied in the fundamental papers of Marcellini [31, 32, 33, 34] and have subsequently been the object of intensive investigation. The new methods here will be explained for a special, yet significant case, incorporating both the features of purely Orlicz structures and the ones of variable exponent functionals. The same methods can be considered as a substantial starting point to treat more general structures including both Orlicz structures and the non-autonomous case, that in the case of non-standard growth conditions poses additional difficulties.

Specifically, let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ with $n \geq 2$, and $p(\cdot) : \Omega \to \mathbb{R}$ be a variable exponent function satisfying

$$2 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty. \tag{1.1}$$

Moreover, we assume that $p(\cdot)$ satisfies the so-called vanishing log-Hölder continuity:

$$\lim_{r \to 0} \omega(r) \log\Big(\frac{1}{r}\Big) = 0, \tag{1.2}$$

where $\omega(\cdot) : [0,\infty) \to [0,\infty)$ is the modulus of continuity of $p(\cdot)$, that is, it is a nondecreasing and concave function satisfying that $\omega(0) = 0$ and $|p(x) - p(y)| \leq \omega(|x-y|)$ for every $x, y \in \Omega$. With this $p(\cdot)$, we consider the elliptic system

$$\operatorname{div} \mathbf{a}(x, u, Du) = 0 \quad \text{in } \Omega. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35B65, 35J60.
Key words and phrases. Partial continuity; elliptic system; non-standard growth; variable exponent.

Submitted October 24, 2016. Published December 20, 2016.

Here, $\mathbf{a} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$, $N \geq 1$, satisfies growth and ellipticity conditions:

$$\begin{gathered} |\mathbf{a}(x,\zeta,\xi)| + |\partial \mathbf{a}(x,\zeta,\xi)|(1+|\xi|) \leq \Lambda \Phi_{p(x)-1}(1+|\xi|), \\ \partial \mathbf{a}(x,\zeta,\xi)\eta \cdot \eta \geq \nu \Phi_{p(x)-2}(1+|\xi|)|\eta|^2, \end{gathered} \tag{1.4}$$

and

$$\begin{aligned} &|\mathbf{a}(x_1,\zeta,\xi) - \mathbf{a}(x_2,\zeta,\xi)| \\ &\leq \Lambda \omega(|x_1 - x_2|)[\Phi_{p(x_1)-1}(1+|\xi|) + \Phi_{p(x_2)-1}(1+|\xi|)] \log(1+|\xi|), \end{aligned} \tag{1.5}$$

for every $x, x_1, x_2 \in \Omega$, $\zeta \in \mathbb{R}^N$ and $\xi, \eta \in \mathbb{R}^{nN}$ and for some $0 < \nu \leq \Lambda$, where $\Phi_p : [0,\infty) \to [0,\infty)$, $p > 0$, is denoted by

$$\Phi_p(t) := t^p \log(e+t), \tag{1.6}$$

and $\partial \mathbf{a}(x,\zeta,\xi) := D_\xi \mathbf{a}(x,\zeta,\xi)$. We note from the ellipticity condition, that is the second inequality in (1.4), that one can see the monotonicity

$$\begin{aligned} &(\mathbf{a}(x,\zeta,\xi_1) - \mathbf{a}(x,\zeta,\xi_2)) : (\xi_1 - \xi_2) \\ &\geq \tilde{\nu} \Phi_{p(x)-2}(1+|\xi_1|+|\xi_2|)|\xi_1 - \xi_2|^2 \\ &\geq \frac{\tilde{\nu}}{2}\{\Phi_{p(x)-2}(1+|\xi_1|)|\xi_1-\xi_2|^2 + \Phi_{p(x)}(|\xi_1 - \xi_2|)\}. \end{aligned} \tag{1.7}$$

For a function $p(\cdot)$, we set

$$\Phi(x,t) := \Phi_{p(x)}(t) = t^{p(x)} \log(e+t), \tag{1.8}$$

and define the Musielak-Orlicz space $L^\Phi(\Omega, \mathbb{R}^N)$ by the set of all measurable vector valued functions $f : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(x,|f|)\,dx = \int_\Omega |f|^{p(x)} \log(e+|f|)\,dx < \infty.$$

and $W^{1,\Phi}(\Omega,\mathbb{R}^N)$ by the set of all $f \in W^{1,1}(\Omega,\mathbb{R}^N)$ with $|f|, |Df| \in L^\Phi(\Omega,\mathbb{R}^1)$.

Our purpose in this article is to investigate partial Hölder continuity of weak solutions to (1.3). We say $u \in W^{1,\Phi}(\Omega,\mathbb{R}^N)$ is a weak solution to (1.3) if it satisfies

$$\int_\Omega \mathbf{a}(x,u,Du) : D\varphi\,dx = 0 \quad \text{for every } \varphi \in W_0^{1,\Phi}(\Omega,\mathbb{R}^N),$$

where $W_0^{1,\Phi}(\Omega,\mathbb{R}^N)$ is the closure of $C_0^\infty(\Omega,\mathbb{R}^N)$ in $W^{1,\Phi}(\Omega,\mathbb{R}^N)$. To obtain the desired regularity, we impose continuity assumptions on the nonlinearity $\mathbf{a}$ as follows. There exists a nondecreasing and concave function $\mu : [0,\infty) \to [0,\infty)$ with $\mu(0) = 0$ such that

$$|\mathbf{a}(x,\zeta_1,\xi) - \mathbf{a}(x,\zeta_2,\xi)| \leq \Lambda\, \mu(|\zeta_1 - \zeta_2|)\, \Phi_{p(x)-1}(1+|\xi|), \tag{1.9}$$

$$|\partial \mathbf{a}(x,\zeta,\xi_1) - \partial \mathbf{a}(x,\zeta,\xi_2)| \leq \Lambda \mu\Big(\frac{|\xi_1 - \xi_2|}{1+|\xi_1|+|\xi_2|}\Big) \Phi_{p(x)-2}(1+|\xi|). \tag{1.10}$$

Note that the prototype of $\mathbf{a}$ is

$$\mathbf{a}(x,\zeta,\xi) \equiv \mathbf{a}(x,\xi) = D_\xi[\Phi(x, 1+|\xi|)], \tag{1.11}$$

for which one can see that the nonlinearity $\mathbf{a}$ in (1.11) satisfies (1.10) with $\mu(r) = r^\alpha$ for any $\alpha \in (0,1)$.

The following theorem is the main result in this article; while the notation used here will be introduced in the next section.

Theorem 1.1. *Let $u \in W^{1,\Phi}(\Omega, \mathbb{R}^N)$ be a weak solution to* (1.3)*. Then there exists $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$ and $u \in C^\alpha(\Omega_u, \mathbb{R}^N)$ for every $\alpha \in (0,1)$. Moreover,*

$$\Omega \setminus \Omega_u \subset \Big\{ x_0 \in \Omega : \liminf_{r \downarrow 0} \fint_{B_r(x_0)} |Du - (Du)_{x_0,r}| \, dx > 0 \Big\} \\ \cup \Big\{ x_0 \in \Omega : \limsup_{r \downarrow 0} \fint_{B_r(x_0)} \Phi(x, |Du|) \, dx = \infty \Big\}.$$

Partial Hölder continuity for elliptic systems is one of classical regularity issues, which was first presented by Campanato [6, 7], after several papers where partial regularity has been proved for the gradient (rather than for the solution itself) under stronger assumptions. Note that there has been reported various systems, for which there exists a weak solution having a singularity, see for example [35], to which we also refer for results concerning regularity and vectorial problems. For systems with standard p-growth, Foss and Mingione [18] proved it when $p \geq 2$ and $\mathbf{a}$ is continuous for the space variable x, see also [4] for the parabolic counterpart. On the other hand, Habermann [21] considered systems with $p(x)$-growth and also p-growth with $1 < p < 2$. In addition, Bögelein, Duzaar, Harbermann and Scheven [5] considered p-growth systems with VMO coefficients. Finally, it is worth mentioning the papers [26, 27], which treat elliptic system with measure data.

As mentioned at the beginning the problems considered here are significant since they do combine features of problems with non-standard growth typical of the Orlicz setting (see for instance [8, 9]) with those coming from variable exponent spaces (see [14] and references) and more in general of functionals with non-standard growth conditions of non-autonomous type. The main point of the energies considered here is that the type of growth with respect to the gradient variable strongly depends on the variable x; this type of phenomenon poses new problems, leads to new techniques and conditions for regularity. This is a recent direction that only now starts being exploited in detail [2, 3, 10, 11, 12, 16, 17, 22, 23, 36, 37, 38, 39]. The main common feature of all these examples is the analysis of functionals

$$v \mapsto \int_\Omega f(x, Dv), dx \tag{1.12}$$

with Ψ-growth, i.e.,

$$f(x, Dv) \approx \Psi(x, |Dv|)$$

and for every fixed x, $z \mapsto \Psi(x, |z|)$ is an Orlicz function. In many cases, this maps has non-standard growth conditions of polynomial type, as for instance considered in [31, 32, 28, 29].

In this paper, we consider systems with Φ-growth, with Φ being the function introduced in (1.8); anyway, as already mentioned at the beginning, some of the methods considered can be extended to more general choices of Φ. These systems are related to functionals as (1.12) via the Euler-Lagrange equation. For the single equation case, i.e. $N = 1$, and $\mathbf{a}(x, \zeta, \xi) \equiv \mathbf{a}(x, \xi)$, in view of [19, 37], one can deduce that $u \in C^{0,\alpha}$ for all $\alpha \in (0,1)$ if $p(\cdot)$ satisfies (1.2) without any further continuity assumption like (1.10). In addition, For systems with $p(x)$-growth, partial Hölder continuity was proved by Harbermann [21], in which the assumptions on $p(\cdot)$ and $\mathbf{a}$ are the exactly same to ones in this paper (of course, those are modified in the setting of $p(x)$-growth).

However, we point out that compared with [18, 5, 21], our case has a different behavior. Note that the function Φ is vary with respect to the position x. Moreover, even on fixed positions, we have $\Phi(x_0, t) = \Phi_{p(x_0)}(t) = t^{p(x_0)} \log(e + t)$, which does not have standard p-growth structure. In fact, Φ_p, $1 < p < \infty$, is the typical example of the Orlicz function. We refer to [30, 13] and related references for problems with Orlicz growth. This makes obtaining partial Hölder continuity difficult. We point out that although the paper [21] considers systems with variable growth, estimates in there are obtained in space with fixed exponent, i.e., classical Lebesgue space.

The first difficulty is the lack of normalization property, and the second one is that we do not expect the integral version of Hölder's continuity. If $G(t) = t^p$, then it is trivial that $G(st) = G(s)G(t)$ for every $s, t > 0$, and

$$\int_U |fg|\, dx = \Big(\int_U G(|f|)\, dx \Big)^{1/p} \Big(\int_U |g|^{p'}\, dx \Big)^{1/p'}.$$

Note that these properties have been used over all in [18, 5, 21]. However, for $G \equiv \Phi_p$, those properties do not hold anymore. In fact, they are not true for general Orlicz function. It seems that those difficulties are the reasons why the partial Hölder continuity for Orlicz growth has not been proved yet. Therefore, in our best knowledge, our result with $p(\cdot) \equiv p$ is the first result for the partial Hölder continuity for systems with Orlicz growth.

The technical starting point of our proof is the approach in [18]. We also refer to [5, 21]. However, we need a lot of new ideas to overcome the difficulties mentioned above. We also point out that in the proof of [21], when analyzing on a ball, the author froze the variable exponent $p(\cdot)$ as a maximum value in a ball. This makes computations complicated since we consider the sequence of shrinking balls hence the frozen exponent is continuously changed. On the other hand, in this paper we freeze the variable exponent as a center point of balls. Therefore, the exponent is not changed with respect to shrinking balls. This makes our argument somehow clearer.

We organize this paper as follows. In the next section, we introduce notation and ingredients of the proof of Theorem 1.3 such as higher integrability, affine functions and the $\mathcal{A}$-harmonic approximation lemma. In Section 3, we prove Theorem 1.3. To do that we first derive a Caccioppoli type inequality, compare weak solutions with $\mathcal{A}$-harmonic maps, investigate a decay of a suitable excess functional E, see (3.3), and finally obtain estimates of Du in the Morrey space which implies the desired Hölder continuity.

2. Preliminaries

2.1. **Orlicz function Φ_p and modulus of continuity.** Let $y \in \mathbb{R}^n$, $r > 0$ and U is a bounded domain in $\mathbb{R}^n$. $B_r(y)$ is a standard ball in $\mathbb{R}^n$ centered at y with radius r. For a locally integrable vector valued function f in $\mathbb{R}^n$, $(f)_{y,r}$ is the integral average of f in $B_r(y)$ such that

$$(f)_{y,r} = \fint_{B_r(y)} f\, dx = \frac{1}{|B_r(y)|} \int_{B_r(y)} f\, dx.$$

Note that if the center point is not important or well understood, we will use the brief notation $B_r = B_r(y)$ and $(f)_r = (f)_{y,r}$. For matrix values $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{nN}$, we define an inner product by $A : B := \sum a_{ij} b_{ij}$. $P : \mathbb{R}^n \to \mathbb{R}^N$ is always an affine function; that is, $P(x) = Ax + b$ for some $A \in \mathbb{R}^{nN}$ and $b \in \mathbb{R}^N$. For simplicity, we shall write $\log^\beta t := [\log t]^\beta$, where $\beta > 0$ and $t \geq 1$.

Suppose $1 < p < \infty$, and recall the function $\Phi_p(t) = t^p \log(e + t)$. We define the conjugate function of Φ_p, $\Phi_p^* : [0, \infty) \to [0, \infty)$, by

$$\Phi_p^*(\tau) := \sup_{t \geq 0} (\tau t - \Phi_p(t)).$$

We then state elementary properties of Φ_p and Φ_p^*. We refer to [37] for their proofs, see also [14].

Proposition 2.1. *Let $t, \tau > 0$, $1 < p < \infty$, $s > 1$ and $0 < \theta < 1$.*

(1) $\Phi_p(st) \leq s^{p+1} \Phi_p(t)$ *and* $\Phi_p(\theta t) \leq \theta^p \Phi_p(t)$.

(2) $\Phi_p^*(s\tau) \leq s^{\frac{p}{p-1}} \Phi_p^*(\tau)$ *and* $\Phi_p^*(\theta\tau) \leq \theta^{\frac{p+1}{p}} \Phi_p^*(\tau)$.

(3) $\Phi_p(t + \tau) \leq \frac{1}{2}(\Phi_p(2t) + \Phi_p(2\tau)) \leq 2^p(\Phi_p(t) + \Phi_p(\tau))$.

(4) *(Young's inequality) For any $\kappa \in (0, 1]$, we have*

$$t\tau \leq \Phi_p(\kappa^{\frac{1}{p}} t) + \Phi_p^*(\kappa^{-\frac{1}{p}} \tau) \leq \kappa \Phi_p(t) + \kappa^{-\frac{1}{p-1}} \Phi_p^*(\tau), \tag{2.1}$$

$$t\tau \leq \Phi_p^*(\kappa^{\frac{p-1}{p}} \tau) + \Phi_p(\kappa^{-\frac{p-1}{p}} t) \leq \kappa \Phi_p^*(\tau) + \kappa^{-\frac{p^2-1}{p}} \Phi_p(t). \tag{2.2}$$

(5) *There exists $c = c(p) > 1$ such that*

$$\frac{1}{c} \Phi_p(t) \leq \Phi_p^*\big(\Phi_p(t) t^{-1}\big) \leq c \Phi_p(t). \tag{2.3}$$

Moreover, if $1 < \gamma_1 \leq p \leq \gamma_2 < \infty$, then the constant c depends only on γ_1 and γ_2, instead of p.

Note that (1)–(3) of the previous proposition are simple results, hence, we will use them without any comment except the cases that they are crucially used.

We recall the concave functions $\omega(\cdot)$ and $\mu(\cdot)$ introduced in Section 1. Then, since $\omega(0) = \mu(0) = 0$, we see that for $r > 0$, $0 < \theta < 1$ and $s \geq 1$,

$$\theta\omega(r) \leq \omega(\theta r), \quad \theta\mu(r) \leq \mu(\theta r), \quad \omega(sr) \leq s\omega(r), \quad \mu(sr) \leq s\mu(r). \tag{2.4}$$

Moreover, we will ultimately consider sufficiently small value r in $\omega(r)\log(1/r)$ and $\mu(r)$ in this paper. Hence, in view of (1.2) and the monotonicity of μ, we shall assume without loss of generality that

$$\omega(r) \log\big(\frac{1}{r}\big) \leq 1 \quad \text{and} \quad \mu(r) \leq 1 \quad \text{for every } r > 0. \tag{2.5}$$

We fix a weak solution $u \in W^{1,\Phi}(\Omega, \mathbb{R}^N)$ to (1.3), and then define

$$M := \int_\Omega [\Phi(x, Du) + 1] \, dx + 1. \tag{2.6}$$

For the rest of this article, we write letter $c > 0$ for any constant depending on the structure constants $n, N, \gamma_1, \gamma_2, \nu, \Lambda$.

2.2. **Basic inequalities.** First, we recall Jensen's inequality such that for any convex function $G : [0, \infty) \to [0, \infty)$,

$$G\Big(\fint_U |f| \, dx \Big) \leq \fint_U G(|f|) \, dx,$$

which will be used frequently in this paper. For any vector valued function f, constant vector A and $1 < p < \infty$, we have

$$\begin{gathered}\fint_{B_r(y)} |f-(f)_{y,r}|^2\,dx \le 2\fint_{B_r(y)} |f-A|^2\,dx,\\ \fint_{B_r(y)} \Phi_p(|f-(f)_{y,r}|)\,dx \le 2^{p+1}\fint_{B_r(y)} \Phi_p(|f-A|)\,dx.\end{gathered}$$

Note that the second inequality follows from (3) of Proposition 2.1 and Jensen's inequality.

The next inequality related to the embedding property from $L\log^\beta L(U)$ space to $L^q(U)$ space, where $\beta > 0$ and $1 < q < \infty$. We refer to [24], see also [1].

$$\fint_U |f|\log^\beta\Big(e+\frac{|f|}{\fint_U |f|\,dx}\Big)\,dx \le c(q,\beta)\Big(\fint_U |f|^q\,dx\Big)^{1/q}. \tag{2.7}$$

Here, the constant $c(q,\beta) > 0$ is continuously changed with respect to q and β. The next lemma is the Sobolev-Poincaré type inequality for the function Φ_p, which can be easily obtained by using the result in [13, Theorem 7].

Lemma 2.2. *Let $1 < \gamma_1 \le p \le \gamma_2 < \infty$ and $f \in W^{1,1}(B_r,\mathbb{R}^N)$. Then there exists $\sigma_s = \sigma_s(n,N,\gamma_1,\gamma_2) > 0$ such that*

$$\fint_{B_r}\Big[\Phi_p\Big(\frac{|f-(f)_r|}{r}\Big)\Big]^{1+\sigma_s}\,dx \le c\Big(\fint_{B_r}\Phi_p(|Df|)\,dx\Big)^{1+\sigma_s}. \tag{2.8}$$

for some $c = c(n,N,\gamma_1,\gamma_2) > 0$.

We finally state the iteration lemma, see [20, Lemma 7.3] and [18, Lemma 2.3].

Lemma 2.3. *Let $\phi : (0,\rho] \to \mathbb{R}$ be a positive and nondecreasing function satisfying*

$$\phi(\theta^{k+1}\rho) \le \theta^q\phi(\theta^k\rho) + c_0(\theta^k\rho) \quad \text{for } k = 0,1,2,\dots,$$

where $\theta \in (0,1)$, $q \in (0,n)$ and $c_0 > 0$. Then there exists $\tilde c = \tilde c(n,\theta,\gamma) > 0$ such that

$$\phi(t) \le \tilde c\big\{(\frac{t}{\rho})^q\phi(\rho) + c_0t^q\big\} \quad \text{for } t \in (0,\rho).$$

2.3. **Higher integrability.** We introduce one of the basic regularity properties of weak solutions to (1.3), which is the higher integrability. For the scalar case, i.e. $N = 1$, with

$$\mathbf{a}(x,\xi) = D_\xi(\Phi(x,|\xi|)),$$

this property was proved in [36, Theorem 3.9]. We point out that one can also prove the same result for the system (1.3) in the same way. Therefore, we shall omit to prove the following result here.

Theorem 2.4. *Suppose that $p(\cdot) : \Omega \to \mathbb{R}$ with (1.1) is vanishing log-Hölder continuous and satisfies the first inequality (2.5), $\mathbf{a} : \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to \mathbb{R}^{nN}$ only satisfies the growth and ellipticity conditions in (1.4), and $u \in W^{1,\Phi}(\Omega,\mathbb{R}^N)$ is a weak solution to (1.3). Then, there exist small constants $\delta_0 = \delta_0(n,N,\gamma_1,\gamma_2,\nu,\Lambda,\omega(\cdot)) > 0$ and $\sigma_h = \sigma_h(n,N,\gamma_1,\gamma_2,\nu,\Lambda) > 0$ such that for any $B_{2r}(x_0) \subset \Omega$ with $r \le \delta_0M^{-1}$, where $M > 1$ is denoted in (2.6), and any $\sigma \le \sigma_h$ we have*

$$\fint_{B_r(x_0)}[\Phi(x,|Du|)]^{1+\sigma}\,dx \le c\Big(\fint_{B_{2r}(x_0)}\Phi(x,|Du|)\,dx + 1\Big)^{1+\sigma}. \tag{2.9}$$

for some $c = c(n, N, \gamma_1, \gamma_2, \nu, \Lambda) \geq 1$*. Moreover, for any* $\delta \in (0, 1]$ *we also have*

$$\fint_{B_r(x_0)} [\Phi(x, |Du|)]^{1+\sigma}\, dx \leq c(\delta)\Big\{\Big(\fint_{B_{2r}(x_0)} [\Phi(x, |Du|)]^{\delta}\, dx\Big)^{\frac{1+\sigma}{\delta}} + 1\Big\}. \tag{2.10}$$

for some $c(\delta) = c(n, N, \gamma_1, \gamma_2, \nu, \Lambda, \delta) \geq 1$.

Note that estimates (2.10) follows from (2.9) by using the argument in [20, Remark 6.12].

From the above lemma we can observe the following. Consider $B_{4r}(x_0) \subset \Omega$ with $x_0 \in \Omega$ and $r > 0$ satisfying

$$2r \leq \delta_0 M^{-1} \quad \text{and} \quad \omega(2r) \leq \min\{\frac{\sigma_h}{8}, \frac{1}{2}\}, \tag{2.11}$$

and set

$$p_0 := p(x_0), \quad p_1 := \inf_{B_{4r}(x_0)} p(\cdot), \quad p_2 := \sup_{B_{4r}(x_0)} p(\cdot).$$

We first note from the fact $r \leq M^{-1}$ and the first inequality in (2.5) that

$$\Big(\fint_{B_{2r}(x_0)} \Phi(x, |Du|)\, dx + 1\Big)^{\omega(r)} \leq c(r^{-n}M)^{\omega(r)} \leq cr^{-(n+1)\omega(r)} \leq c. \tag{2.12}$$

Then since $p_0 \leq p(x)(1 + \omega(r)) \leq p(x)(1 + \sigma_h/8)$ for $x \in B_r(x_0)$, applying (2.9) with $\sigma = \omega(r)$ and (2.12), we have

$$\begin{aligned}
\fint_{B_r(x_0)} \Phi_{p_0}(|Du|)\, dx &\leq \fint_{B_{2r}(x_0)} [\Phi(x, |Du|)]^{1+\omega(r)}\, dx + 1\\
&\leq c\Big(\fint_{B_{2r}(x_0)} \Phi(x, |Du|)\, dx + 1\Big)^{1+\omega(r)}\\
&\leq c\Big(\fint_{B_{2r}(x_0)} \Phi(x, |Du|)\, dx + 1\Big).
\end{aligned}$$

Note that this implies $\Phi_{p_0}(|Du|) \in L^1_{loc}(\Omega)$. In the same way, since $p_0(1 + \sigma_h/2) \leq p(x)(1 + \omega(r))(1 + \sigma_h/2) \leq p(x)(1 + \sigma_h)$ for $x \in B_r(x_0)$, we also have

$$\fint_{B_r(x_0)} [\Phi_{p_0}(|Du|)]^{1+\frac{\sigma_h}{2}}\, dx \leq c\Big\{\Big(\fint_{B_{2r}(x_0)} \Phi(x, |Du|)\, dx\Big)^{1+\frac{\sigma_h}{2}} + 1\Big\}. \tag{2.13}$$

On the other hand, by taking $\delta = 1/2 \leq p_1/p_2$ from (2.10), Hölder's inequality and (2.12) with (2.4) we have

$$\begin{aligned}
&\fint_{B_{2r}(x_0)} \Phi(x, |Du|)\, dx\\
&\leq c\Big\{\Big(\fint_{B_{4r}(x_0)} [\Phi(x, |Du|)]^{\frac{p_1}{p_2}}\, dx\Big)^{p_2/p_1} + 1\Big\}\\
&\leq c\Big(\fint_{B_{4r}(x_0)} \Phi_{p_0}(|Du|)\, dx + 1\Big)\Big(\fint_{B_{4r}(x_0)} \Phi(x, |Du|)\, dx + 1\Big)^{\omega(8r)}\\
&\leq c\Big(\fint_{B_{4r}(x_0)} \Phi_{p_0}(|Du|)\, dx + 1\Big).
\end{aligned} \tag{2.14}$$

Moreover, since

$$\frac{p_2 - 1}{p_0 - 1}\Big(1 + \frac{\sigma_h}{4}\Big)p_0 \leq p(x)(1 + \omega(r))^2\Big(1 + \frac{\sigma_h}{4}\Big) \leq p(x)\Big(1 + \frac{7}{8}\sigma_h\Big),$$

applying (2.9) with $\sigma = \frac{p_2-1}{p_0-1}(1+\omega(r))$, or

$$\sigma = \frac{p_2-1}{p_0-1}\big(1+\frac{\sigma_h}{4}\big)(1+\omega(r)),$$

(2.12) and (2.14), we have

$$\begin{aligned} \fint_{B_r} \Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}}\,dx &\le c\Big(\fint_{B_{2r}} \Phi(x,|Du|)\,dx+1\Big)^{(1+\omega(r))^2} \\ &\le c\Big(\fint_{B_{4r}} \Phi_{p_0}(|Du|)\,dx+1\Big) \end{aligned} \tag{2.15}$$

and

$$\fint_{B_r} \Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}\left(1+\frac{\sigma_h}{4}\right)}\,dx \le c\Big(\fint_{B_{4r}} \Phi_{p_0}(|Du|)\,dx+1\Big)^{1+\frac{\sigma_h}{4}}. \tag{2.16}$$

2.4. **Affine functions.** For a given $u \in L^2(B_r(x_0),\mathbb{R}^N)$, we define an affine function $P_{x_0,r}$ by the minimizer of the functional

$$P \mapsto \fint_{B_r(x_0)} |u-P|^2\,dx.$$

Then we can easily see that $P_{x_0,r} = DP_{x_0,r}(x-x_0)+(u)_{x_0,r}$, where

$$DP_{x_0,r} := \frac{n+2}{r^2}\fint_{B_r(x_0)} u\otimes(x-x_0)\,dx.$$

Then we have the following lemma.

Lemma 2.5. *Let $2<p<\infty$ and $r>0$.*

(1) *For any $u\in L^{\Phi_p}(B_r(x_0))$ and $\theta\in(0,1)$, we have*

$$\Phi_p(|DP_{x_0,r}-DP_{x_0,\theta r}|) \le c\fint_{B_{\theta r}(x_0)} \Phi_p\Big(\frac{|u-P_{x_0,r}|}{\theta r}\Big)\,dx. \tag{2.17}$$

(2) *For any $u\in W^{1,\Phi_p}(B_r(x_0))$, we have*

$$\Phi_p\big(|DP_{x_0,r}-(Du)_{B_r(x_0)}|\big) \le c\fint_{B_r(x_0)} \Phi_p(|Du-(Du)_{B_r(x_0)}|)\,dx. \tag{2.18}$$

Here, the constants c depend only on n,N,p.

Proof. By [25, Lemma 2], we have

$$|DP_{x_0,r}-DP_{x_0,\theta r}|^2 \le c\fint_{B_{\theta r}(x_0)} \frac{|u-P_{x_0,r}|^2}{(\theta r)^2}\,dx,$$

$$|DP_{x_0,r}-(Du)_{B_r(x_0)}|^2 \le c\fint_{B_r(x_0)} |Du-(Du)_{B_r(x_0)}|^2\,dx.$$

Using these and Jensen's inequality for the convex map $t\mapsto \Phi_p(\sqrt{t})$, we obtain

$$\begin{aligned} \Phi_p(|DP_{x_0,r}-DP_{x_0,\theta r}|) &\le c\Phi_p\Big(\fint_{B_{\theta r}(x_0)} \frac{|u-P_{x_0,r}|^2}{(\theta r)^2}\,dx\Big)^{1/2} \\ &\le c\fint_{B_{\theta r}(x_0)} \Phi_p\Big(\frac{|u-P_{x_0,r}|}{\theta r}\Big)\,dx, \end{aligned}$$

which yileds (2.17). In the same way we obtain (2.18). $\square$

2.5. **$\mathcal{A}$-harmonic approximation.** Let $\mathcal{A}$ be a bilinear form in $\mathbb{R}^{nN}$. Moreover $\mathcal{A}$ satisfies the so-called Legendre-Hadamard condition: there exist $0 < \nu \le \lambda$ such that

$$\nu|\xi|^2|\eta|^2 \le \xi \otimes \eta : \xi \otimes \eta \le \Lambda|\xi|^2|\eta|^2 \tag{2.19}$$

for every $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$. For this $\mathcal{A}$ we say that $h \in W^{1,2t}(\Omega, \mathbb{R}^N)$ is $\mathcal{A}$-harmonic if

$$\int_\Omega \mathcal{A}Dh : D\varphi = 0$$

for every $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$. We then state the so-called $\mathcal{A}$-harmonic approximation lemma, see [15, Lemma 3.3].

Lemma 2.6. *Let $\epsilon > 0$ and $0 < \nu \le \Lambda$. Assume that there exists small $\delta = \delta(n, N, \nu, \Lambda, \epsilon) > 0$ such that if the bilinear form $\mathcal{A}$ on $\mathbb{R}^{nN}$ satisfies* (2.19)*, $r > 0$ and $w \in W^{1,2}(B_r, \mathbb{R}^N)$ satisfies*

$$\fint_{B_r} |Dw|^2\, dx \le 1,$$

$$\Big|\fint_{B_\rho} \mathcal{A}Dw : D\varphi\, dx\Big| \le \delta\|D\varphi\|_{L^\infty(B_r)} \quad \textit{for every } \varphi \in C_0^1(B_r, \mathbb{R}^N),$$

then there exists $\mathcal{A}$-harmonic map h such that

$$\fint_{B_r} |Dw|^2\, dx \le 1, \quad r^{-2}\fint_{B_r} |w - h|^2\, dx \le \epsilon.$$

3. Partial continuity

Let us first consider $\rho > 0$ satisfying

$$\rho \le \rho_0 := \sup\big\{r > 0 : 4r \le \delta_0 M^{-1},\ \omega(4r) \le \min\{\frac{\sigma_h}{8}, \frac{1}{2}\}\big\}, \tag{3.1}$$

where M is denoted in (2.6) and δ_0, σ_h are determined in Theorem 2.4. Then we see that ρ satisfies (2.11) with r replaced by 2ρ. We then consider $B_{4\rho}(x_0) \subset \Omega$ with $x_0 \in \Omega$ and $\rho \le \rho_0$ and set

$$p_0 := p(x_0), \quad p_1 := \inf_{B_{2\rho}(x_0)} p(\cdot), \quad p_2 := \sup_{B_{2\rho}(x_0)} p(\cdot),$$

$$C(x_0, \rho, P) := \fint_{B_\rho(x_0)} \Big[\frac{|Du - DP|^2}{(1 + |DP|)^2} + \frac{\Phi_{p_0}(|Du - DP|)}{\Phi_{p_0}(1 + |DP|)}\Big]\, dx \tag{3.2}$$

and

$$E(x_0, \rho, P) := C(x_0, \rho, P) + \Big[\mu\Big(\fint_{B_\rho} |u - P(x_0)|^2\, dx\Big) + \omega(\rho)\log(\frac{1}{\rho})\Big]^{\frac{1}{2p_0 - 1}}. \tag{3.3}$$

Here, $P : \mathbb{R}^n \to \mathbb{R}^N$ is any affine function. In this setting we derive a Caccioppoli type inequality.

Lemma 3.1. *Assume that*

$$C(x_0, 8\rho, P) \le 1. \tag{3.4}$$

Then

$$\begin{aligned} C(x_0,\rho,P) \le c \fint_{B_{2\rho}(x_0)} \Big[\frac{|u-P|^2}{(2\rho)^2(1+|DP|)^2}\,dx + \frac{\Phi_{p_0}(|u-P|/(2\rho))}{\Phi_{p_0}(1+|DP|)}\Big]\,dx \\ + c\Big\{\mu\Big(\fint_{B_{2\rho}(x_0)} |u-P(x_0)|^2\,dx\Big) + \omega(2\rho)\log(\frac{1}{2\rho})\Big\} \end{aligned} \tag{3.5}$$

for some $c = c(n,N,\gamma_1,\gamma_2,\nu,\Lambda) > 0$.

Proof. Set $\varphi = \eta^{p_0+1}(u-P)$, where $\eta \in C_0^\infty(B_{2\rho})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on B_ρ and $|D\eta| \le c(n)/\rho$. Taking φ as a test function in (1.3), we have

$$\fint_{B_{2\rho}} \eta^{p_0+1}\mathbf{a}(x,u,Du) : D(u-P)\,dx = -(p_0+1)\fint_{B_{2\rho}} \eta^{p_0}\mathbf{a}(x,u,Du) : D\eta\otimes(u-P)\,dx.$$

This and the trivial identity

$$\fint_{B_{2\rho}} \mathbf{a}(x_0,P(x_0),DP) : D\varphi\,dx = 0$$

imply

$$\begin{aligned} I_1 &:= \fint_{B_{2\rho}} \eta^{p_0+1}(\mathbf{a}(x_0,u,Du) - \mathbf{a}(x_0,u,DP)) : (Du-DP)\,dx \\ &= \fint_{B_{2\rho}} (\mathbf{a}(x_0,u,Du) - \mathbf{a}(x_0,u,DP)) : D\varphi\,dx \\ &\quad - (p_0+1)\fint_{B_{2\rho}} \eta^{p_0}(\mathbf{a}(x_0,u,Du) - \mathbf{a}(x_0,u,DP)) : D\eta\otimes(u-P)\,dx \\ &= \fint_{B_{2\rho}} (\mathbf{a}(x_0,u,Du) - \mathbf{a}(x,u,Du)) : D\varphi\,dx \\ &\quad + \fint_{B_{2\rho}} (\mathbf{a}(x_0,P(x_0),DP) - \mathbf{a}(x_0,u,DP)) : D\varphi\,dx \\ &\quad - (p_0+1)\fint_{B_{2\rho}} \eta^{p_0}(\mathbf{a}(x_0,u,Du) - \mathbf{a}(x_0,u,DP)) : D\eta\otimes(u-P)\,dx \\ &=: I_2 + I_3 - I_4. \end{aligned} \tag{3.6}$$

Estimate for I_1. From (1.7) we have

$$\fint_{B_\rho} \eta^{p_0+1}\{\Phi_{p_0-2}(1+|DP|)|Du-DP|^2 + \Phi_{p_0}(|Du-DP|)\}\,dx \le cI_1. \tag{3.7}$$

Estimate for I_2. Using (1.5) and (2.1), for $\kappa \in (0,1)$, we have

$$\begin{aligned} |I_2| &\le c\fint_{B_{2\rho}} \omega(2\rho)\log(e+1+|Du|)\,\Phi_{p_2-1}(1+|Du|)\Big(|Du-DP| + \frac{|u-P|}{\rho}\Big)\,dx \\ &\le \kappa\fint_{B_{2\rho}} \big[\Phi_{p_0}(|Du-DP|) + \Phi_{p_0}\big(\frac{|u-P|}{\rho}\big)\big]\,dx \\ &\quad + c(\kappa)\fint_{B_{2\rho}} \Phi_{p_0}^*\big(\omega(2\rho)\log(e+1+|Du|)\Phi_{p_2-1}(1+|Du|)\big)\,dx. \end{aligned}$$

Now we estimate the last integral above:

$$\tilde{I}_2 := \fint_{B_{2\rho}} \Phi_{p_0}^*\left(\omega(2\rho)\log(e+1+|Du|)\Phi_{p_2-1}(1+|Du|)\right)\,dx. \tag{3.8}$$

Using (2) and (5) of Proposition 2.1 and that $1+|Du| \le \Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}}$, we have

$$
\begin{aligned}
\tilde{I}_2 &\le \fint_{B_{2\rho}} [\omega(2\rho)\log(e+1+|Du|)]^{i(p_0)}(1+|Du|)^{(p_2-p_0)\frac{p_0}{p_0-1}}\Phi_{p_0}(1+|Du|)\,dx \\
&\le c\Big[\omega(2\rho)\log\Big(e+\big(\Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}}\big)_{2\rho}\Big)\Big]^{i(p_0)} \fint_{B_{2\rho}} [\Phi_{p_0}(1+|Du|)]^{\frac{p_2-1}{p_0-1}}\,dx \\
&\quad + c[\omega(2\rho)]^{\frac{p_0+1}{p_0}} \fint_{B_{2\rho}} \log^{\frac{p_0}{p_0-1}}\Big(e+\frac{\Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}}}{(\Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}})_{2\rho}}\Big) \\
&\quad \times \big[\Phi_{p_0}(1+|Du|)\big]^{\frac{p_2-1}{p_0-1}}\,dx,
\end{aligned}
$$

where

$$
i(p_0)=\begin{cases} \frac{p_0}{p_0-1} & \text{if } \omega(2\rho)\log(e+1+|Du|)>1,\\ \frac{p_0+1}{p_0} & \text{if } \omega(2\rho)\log(e+1+|Du|)\le 1. \end{cases}
$$

Then, applying (2.5), (2.7) and two inequalities concerning the higher integrability: (2.15) and (2.16) with $r=2\rho$, we obtain

$$
\begin{aligned}
\tilde{I}_2 &\le c\big[\omega(2\rho)\log(\frac{1}{\rho})\big]^{i(p_0)} \fint_{B_{8\rho}} \Phi_{p_0}(1+|Du|)\,dx \\
&\quad + [\omega(2\rho)]^{\frac{p_0+1}{p_0}}\Big(\fint_{B_{2\rho}} \Phi_{p_0}(1+|Du|)^{\frac{p_2-1}{p_0-1}\left(1+\frac{\sigma_h}{4}\right)}\,dx\Big)^{\frac{1}{1+\frac{\sigma_h}{4}}} \\
&\le c\big[\omega(2\rho)\log(\frac{1}{2\rho})\big]^{\frac{p_0+1}{p_0}} \fint_{B_{8\rho}} \Phi_{p_0}(1+|Du|)\,dx \\
&\le c\big[\omega(2\rho)\log(\frac{1}{2\rho})\big]^{\frac{p_0+1}{p_0}} \Phi_{p_0}(1+|DP|).
\end{aligned} \tag{3.9}
$$

In the last equality above, we have used the assumption (3.4) so that

$$
\begin{aligned}
\fint_{B_{8\rho}} \Phi_{p_0}(1+|Du|)\,dx &\le c\Big(\fint_{B_{8\rho}} \Phi_{p_0}(|Du-DP|)\,dx + \Phi_{p_0}(1+|DP|)\,dx\Big) \\
&\le c\Phi_{p_0}(1+|DP|).
\end{aligned} \tag{3.10}
$$

Therefore,

$$
\begin{aligned}
|I_2| &\le \kappa \fint_{B_{2\rho}} \big[\Phi_{p_0}(|Du-DP|)+\Phi_{p_0}\big(\frac{|u-P|}{\rho}\big)\big]\,dx \\
&\quad + c(\kappa)\,\omega(2\rho)\log(\frac{1}{2\rho})\Phi_{p_0}(1+|DP|).
\end{aligned} \tag{3.11}
$$

Estimate for I_3. Applying (1.9),

$$
|I_3| \le c\fint_{B_{2\rho}} \mu\left(|u-P(x_0)|^2\right)\Phi_{p_0-1}(1+|DP|)\Big(\eta^{p_0+1}|Du-DP|+\frac{|u-P|}{\rho}\Big)\,dx.
$$

Using (2.1), (2.5) and Jensen's inequality, we see that for $\kappa \in (0,1)$,

$$
\begin{aligned}
|I_3| &\leq \kappa \fint_{B_{2\rho}} \Big[\eta^{p_0+1}\Phi_{p_0}(|Du-DP|) + \Phi_{p_0}\big(\frac{|u-P|}{\rho}\big)\Big]\,dx \\
&\quad + c(\kappa)\fint_{B_{2\rho}} \mu\left(|u-P(x_0)|^2\right)^{\frac{p_0+1}{p_0}}\Phi_{p_0}(1+|DP|)\,dx \\
&\leq \kappa \fint_{B_{2\rho}} \eta^{p_0+1}\Phi_{p_0}(|Du-DP|)\,dx + \fint_{B_{2\rho}} \Phi_{p_0}\big(\frac{|u-P|}{\rho}\big)\,dx \\
&\quad + c(\kappa)\,\mu\Big(\fint_{B_{2\rho}} |u-P(x_0)|^2\,dx\Big)\Phi_{p_0}(1+|DP|).
\end{aligned}
\tag{3.12}
$$

Estimate for I_4. From the first inequality in (1.4), Young's inequality and (2.2), we have that for $\kappa \in (0,1)$,

$$
\begin{aligned}
&|I_4| \\
&\leq c\fint_{B_{2\rho}} \Big(\int_0^1 |\partial \mathbf{a}(x_0,u,t(Du-DP)+DP)|\,dt\Big)|Du-DP|\eta^{p_0}\frac{|u-P|}{\rho}\,dx \\
&\leq c\fint_{B_{2\rho}} \Phi_{p_0-2}(1+|DP|+|Du-DP|)|Du-DP|\eta^{p_0}\frac{|u-P|}{\rho}\,dx \\
&\leq c\Phi_{p_0-2}(1+|DP|)\fint_{B_{2\rho}} \eta^{\frac{p_0+1}{2}}|Du-DP|\frac{|u-P|}{\rho}\,dx \\
&\quad + c\fint_{B_{2\rho}} \Phi_{p_0-1}(|Du-DP|)\eta^{p_0}\frac{|u-P|}{\rho}\,dx \\
&\leq \kappa\fint_{B_{2\rho}} \eta^{p_0+1}\left[\Phi_{p_0-2}(1+|DP|)|Du-DP|^2 + \Phi_{p_0}(|Du-DP|)\right]\,dx \\
&\quad + c(\kappa)\Big(\Phi_{p_0-2}(1+|DP|)\fint_{B_{2\rho}} \frac{|u-P|^2}{\rho^2}\,dx + \fint_{B_{2\rho}} \Phi_{p_0}\big(\frac{|u-P|}{\rho}\big)\,dx\Big).
\end{aligned}
\tag{3.13}
$$

Consequently, inserting (3.7)-(3.13) into (3.6) and choosing κ sufficiently small, we get the estimate (3.5). □

Next, we define

$$
\mathcal{A} = \mathcal{A}(x_0,P) := \frac{\partial \mathbf{a}(x_0,P(x_0),DP)}{\Phi_{p(x_0)-2}(1+|DP|)}, \quad w := \frac{u-P}{(1+|DP|)\sqrt{E(x_0,\rho,P)}}. \tag{3.14}
$$

Then one can see from (1.4) that $\mathcal{A}$ satisfies (2.19). The next lemma implies that if $E(x_0,\rho,P)$ is sufficiently small then one can apply Lemma 3.2 to $\mathcal{A}$ and w denoted above.

Lemma 3.2. *Suppose that* (3.4) *holds. Then there exists* $c = c(n,N,\gamma_1,\gamma_2,\nu,\Lambda) > 0$ *such that*

$$
\Big|\fint_{B_\rho(x_0)} \mathcal{A}Dw : D\varphi\,dx\Big| \leq c[\mu(\sqrt{E(x_0,\rho,P)}) + E(x_0,\rho,P)]^{1/2}\sup_{B_\rho(x_0)}|D\varphi| \tag{3.15}
$$

for every $\varphi \in C_0^\infty(B_\rho(x_0))$.

Proof. We first consider $\varphi \in C_0^\infty(B_\rho(x_0))$ with $\sup_{B_\rho(x_0)} |D\varphi| = 1$, and set $v := u - P$, i.e., $v = (1+|DP|)\sqrt{E(x_0,\rho,P)}\, w$, and $B_\rho = B_\rho(x_0)$. Then

$$
\begin{aligned}
&\Phi_{p_0-2}(1+|DP|) \fint_{B_\rho} \mathcal{A}Dv : D\varphi \, dx \\
&= \fint_{B_\rho} \int_0^1 (\partial \mathbf{a}(x_0, P(x_0), DP) - \partial \mathbf{a}(x_0, P(x_0), DP + sDv))Dv : D\varphi \, dt \, dx \\
&\quad + \fint_{B_\rho} \int_0^1 \partial \mathbf{a}(x_0, P(x_0), DP + sDv)Dv : D\varphi \, dt \, dx \\
&=: I_5 + I_6.
\end{aligned} \tag{3.16}
$$

We first estimate I_5. Applying (1.10), we have

$$
\begin{aligned}
|I_5| &\le c \fint_{B_\rho} \int_0^1 \mu\Big(\frac{s|Dv|}{1+|DP|}\Big) \Phi_{p_0-2}(1+|DP|+|DP+sDv|)|Dv| \, dt \, dx \\
&\le c \fint_{B_\rho} \mu\Big(\frac{|Du-DP|}{1+|DP|}\Big) (\Phi_{p_0-2}(1+|DP|) + \Phi_{p_0-2}(|Du-DP|))|Du-DP| \, dx \\
&= c\Phi_{p_0-1}(1+|DP|) \times \fint_{B_\rho} \mu\Big(\frac{|Du-DP|}{1+|DP|}\Big) \\
&\quad \times \Big(\frac{|Du-DP|}{1+|DP|} + \frac{\Phi_{p_0-1}(|Du-DP|)}{\Phi_{p_0-1}(1+|DP|)}\Big) \, dx.
\end{aligned}
$$

Set $B_\rho^+ := \{x \in B_\rho : |Du(x) - DP| > 1 + |DP|\}$ and $B_\rho^- := B_\rho \setminus B_\rho^+$. Then, by (2.4) and (2.5), in B_ρ^+,

$$
\begin{aligned}
&\mu\Big(\frac{|Du-DP|}{1+|DP|}\Big)\Big(\frac{|Du-DP|}{1+|DP|} + \frac{\Phi_{p_0-1}(|Du-DP|)}{\Phi_{p_0-1}(1+|DP|)}\Big) \\
&\le \frac{|Du-DP|^2}{(1+|DP|)^2} + \frac{\Phi_{p_0}(|Du-DP|)}{\Phi_{p_0}(1+|DP|)}.
\end{aligned}
$$

On the other hand, in B_ρ^-,

$$
\begin{aligned}
&\mu\Big(\frac{|Du-DP|}{1+|DP|}\Big)\Big(\frac{|Du-DP|}{1+|DP|} + \frac{\Phi_{p_0-1}(|Du-DP|)}{\Phi_{p_0-1}(1+|DP|)}\Big) \\
&\le \mu\Big(\frac{|Du-DP|}{1+|DP|}\Big)\frac{|Du-DP|}{1+|DP|} + \mu\Big(\frac{|Du-DP|}{1+|DP|}\Big)\Big(\frac{\Phi_{p_0}(|Du-DP|)}{\Phi_{p_0}(1+|DP|)}\Big)^{\frac{p_0-1}{p_0}}.
\end{aligned}
$$

Using Hölder's inequality, (2.4) and Jensen's inequality for the concave function μ^{-1}, we have from the pervious two inequalities that

$$
\begin{aligned}
&\frac{|I_5|}{\Phi_{p_0-1}(1+|DP|)} \\
&\le c \fint_{B_\rho} \Big(\frac{|Du-DP|}{1+|DP|} + \frac{\Phi_{p_0}(|Du-DP|)}{\Phi_{p_0}(1+|DP|)}\Big) \, dx \\
&\quad + c\Big[\mu\Big(\fint_{B_\rho} \frac{|Du-DP|}{1+|DP|} \, dx\Big)\Big]^{1/2} \Big(\fint_{B_\rho} \frac{|Du-DP|^2}{1+|DP|^2} \, dx\Big)^{1/2} \\
&\quad + c\Big[\mu\Big(\fint_{B_\rho} \frac{|Du-DP|}{1+|DP|} \, dx\Big)\Big]^{1/p_0} \Big(\fint_{B_\rho} \frac{\Phi_{p_0}(|Du-DP|)}{\Phi_{p_0}(1+|DP|)} \, dx\Big)^{\frac{p_0-1}{p_0}}.
\end{aligned}
$$

Therefore, recalling the definitions of $C(x_0,\rho,P)$ and $E(x_0,\rho,P)$, and using Young's inequality, we obtain

$$
\begin{aligned}
&\frac{|I_5|}{\Phi_{p_0-1}(1+|DP|)}\\
&\le c\Big\{C(x_0,\rho,P)+\big[\mu(\sqrt{C(x_0,\rho,P)})\big]^{1/2}[C(x_0,\rho,P)]^{1/2}\\
&\quad+\big[\mu(\sqrt{C(x_0,\rho,P)})\big]^{1/p_0}[C(x_0,\rho,P)]^{\frac{p_0-1}{p_0}}\Big\}\\
&\le c\Big\{E(x_0,\rho,P)+\big[\mu(\sqrt{E(x_0,\rho,P)})\big]^{1/2}[E(x_0,\rho,P)]^{1/2}\Big\}.
\end{aligned}
\tag{3.17}
$$

Next we estimate I_6. Applying (1.5) and (1.9), we have

$$
\begin{aligned}
I_6 &= \fint_{B_\rho}(\mathbf{a}(x_0,P(x_0),Du)-\mathbf{a}(x_0,P(x_0),DP)):D\varphi\,dx\\
&=\fint_{B_\rho}(\mathbf{a}(x_0,P(x_0),Du)-\mathbf{a}(x,P(x_0),Du)):D\varphi\,dx\\
&\quad+\fint_{B_\rho}(\mathbf{a}(x,P(x_0),Du)-\mathbf{a}(x,u,Du)):D\varphi\,dx\\
&\le c\fint_{B_\rho}\omega(\rho)\log(e+1+|Du|)\Phi_{p_2-1}(1+|Du|)\,dx\\
&\quad+c\fint_{B_\rho}\mu(|u-P(x_0)|^2)\Phi_{p_0-1}(1+|Du|)\,dx\\
&=: c(I_{6a}+I_{6b}).
\end{aligned}
$$

For I_{6a}, from Jensen's inequality for the convex function $\Phi^*_{p_0}$ we have

$$
I_{6a}\le(\Phi^*_{p_0})^{-1}\Big(\fint_{B_\rho}\Phi^*_{p_0}\left(\omega(\rho)\log(e+1+|Du|)\Phi_{p_2-1}(1+|Du|)\right)\,dx\Big).
$$

Note that the integration on the right hand side above is exactly same to $\tilde{I}_2$ denoted in (3.8) with 2ρ replaced by ρ, hence it follows from (3.9) and (5) of Proposition 2.1 that

$$
\begin{aligned}
I_{6a}&\le(\Phi^*_{p_0})^{-1}\Big(\big[\omega(\rho)\log(\frac{1}{\rho})\big]^{\frac{p_0+1}{p_0}}\Phi_{p_0}(1+|DP|)\Big)\\
&\le\big[\omega(\rho)\log(\frac{1}{\rho})\big]^{\frac{p_0^2-1}{p_0}}\Phi_{p_0-1}(1+|DP|).
\end{aligned}
$$

In the above inequality, we used that $(\Phi^*_p)^{-1}(\theta t)\le\theta^{\frac{p-1}{p}}(\Phi^*_p)^{-1}(t)$, $\theta\in(0,1)$ and $t>0$, which can be easily derived from (2) of Proposition 2.1.

As for I_{6b}, using (2.5), Hölder's inequality, Jensen's inequality for the convex functions μ^{-1} and $\Psi(t):=\Phi_{p_0}(\Phi^{-1}_{p_0-1}(t^{\frac{2p_0-2}{2p_0-1}}))$, and (3.10), we have

$$
\begin{aligned}
I_{6b}&\le c\Big(\fint_{B_\rho}\mu(|u-P(x_0)|^2)\,dx\Big)^{\frac{1}{2p_0-1}}\Big(\fint_{B_\rho}[\Phi_{p_0-1}(1+|Du|)]^{\frac{2p_0-1}{2p_0-2}}\,dx\Big)^{\frac{2p_0-2}{2p_0-1}}\\
&\le c\Big[\mu\Big(\fint_{B_\rho}|u-P(x_0)|^2\,dx\Big)\Big]^{\frac{1}{2p_0-1}}\Big[\Psi^{-1}\Big(\fint_{B_\rho}\Phi_{p_0}(1+|Du|)\,dx\Big)\Big]^{\frac{2p_0-2}{2p_0-1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le c\Big[\mu\Big(\fint_{B_\rho}|u-P(x_0)|^2\,dx\Big)\Big]^{\frac{1}{2p_0-1}}\left[\Psi^{-1}\left(\Phi_{p_0}(1+|DP|)\right)\right]^{\frac{2p_0-2}{2p_0-1}}\\
&\le c\Big[\mu\Big(\fint_{B_\rho}|u-P(x_0)|^2\,dx\Big)\Big]^{\frac{1}{2p_0-1}}\Phi_{p_0-1}(1+|DP|).
\end{aligned}
$$

Therefore, from the previous two estimates and the definition of E, we have

$$
\frac{|I_6|}{\Phi_{p_0-1}(1+|DP|)}\le cE(x_0,\rho,P). \tag{3.18}
$$

Inserting (3.17) and (3.18) into (3.16) and using the definition of (3.3), we have (3.15) for every $\varphi\in C^\infty(B_\rho(x_0))$ with $\sup_{B_\rho}|D\varphi|=1$. Therefore, the standard normalization argument yields (3.15) for every $\varphi\in C^\infty(B_\rho(x_0))$. $\square$

We consider the affine function $P=(Du)_{x_0,\rho}(x-x_0)+(u)_{x_0,\rho}$, and set

$$
\begin{aligned}
C(x_0,\rho)&:=C(x_0,\rho,(Du)_{x_0,\rho}(x-x_0)+(u)_{x_0,\rho})\\
&=\fint_{B_\rho}\Big[\frac{|Du-(Du)_{x_0,\rho}|^2}{(1+|(Du)_{x_0,\rho}|)^2}+\frac{\Phi_{p_0}(|Du-(Du)_{x_0,\rho}|)}{\Phi_{p_0}(1+|(Du)_{x_0,\rho}|)}\Big]\,dx,
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
\tilde E(x_0,\rho)&:=E(x_0,\rho,(Du)_{x_0,\rho}(x-x_0)+(u)_{x_0,\rho})\\
&=C(x_0,\rho)+\Big[\mu\Big(\fint_{B_\rho}|u-(u)_{x_0,\rho}|^2\,dx\Big)+\omega(\rho)\log(\frac{1}{\rho})\Big]^{\frac{1}{2p_0-1}},
\end{aligned} \tag{3.20}
$$

$$
E(x_0,\rho):=C(x_0,\rho)+\Big[\mu\Big(\rho\fint_{B_\rho}|Du|^2\,dx\Big)+\omega(\rho)\log(\frac{1}{\rho})\Big]^{\frac{1}{2p_0-1}}. \tag{3.21}
$$

Then, from Poincaré's inequality along with the fact $\rho<1$, we see that

$$
\tilde E(x_0,\rho)\le cE(x_0,\rho). \tag{3.22}
$$

some $c=c(n,N)\ge 1$.

Lemma 3.3. *For $\theta\in(0,1/8)$, there exists $\epsilon_0=\epsilon_0(n,N,\gamma_1,\gamma_2,\nu,\Lambda,\theta)>0$ such that if $E(x_0,\rho)\le\epsilon_0$ and $\rho\le\epsilon_1:=\min\{\theta^n,\rho_0\}$, where ρ_0 is defined in (3.1), then*

$$
C(x_0,\theta\rho)\le c_*\theta^2E(x_0,\rho) \tag{3.23}
$$

for some $c_*=c_*(n,N,\gamma_1,\gamma_2,\nu,\Lambda)\ge 1$.

Proof. **Step 1.** We first estimate the integrals

$$
\fint_{B_{2\theta\rho}}\frac{|u(x)-P_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx\quad\text{and}\quad\fint_{B_{2\theta\rho}}\Phi_{p_0}\Big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\Big)\,dx. \tag{3.24}
$$

where the affine function $P_{x_0,2\theta\rho}$ is defined in Section 2.4. Recall $\mathcal{A}$ and w denoted in (3.14) with $P=(Du)_\rho(x-x_0)+(u)_\rho$. Then

$$
w:=\frac{u-(Du)_\rho(x-x_0)-(u)_\rho}{(1+(Du)_\rho)\sqrt{\tilde E(x_0,\rho)}}\quad\text{and}\quad\fint_{B_\rho}|Dw|^2\,dx\le 1.
$$

Let $\epsilon\in(0,1)$ be a sufficiently small number determined later, for which we consider $\delta=\delta(n,N,\nu,\Lambda,\epsilon)>0$ determined in Lemma 2.6. Then by Lemma 3.2 with (3.20), (3.21) and (3.22), we have

$$
\Big|\fint_{B_\rho(x_0)}\mathcal{A}Dw:D\varphi\,dx\Big|\le\delta\sup_{B_\rho(x_0)}|D\varphi|,
$$

by taking sufficiently small $\epsilon_0 = \epsilon_0(n, N, \gamma_1, \gamma_2, \nu, \Lambda, \epsilon) \in (0, 1)$. We point out that when using Lemma 3.2, we need the assumption (3.4), which is clear since $C(x_0, \rho) \le E(x_0, \rho) \le \epsilon_0 \le 1$. Therefore, in view of Lemma 2.6, there exists an $\mathcal{A}$-harmonic map h such that

$$\fint_{B_\rho} |Dh|^2 \, dx \le 1 \quad \text{and} \quad \fint_{B_\rho} |w-h|^2 \, dx \le \epsilon\rho^2. \tag{3.25}$$

We notice by a basic regularity theory for $\mathcal{A}$-harmonic maps, see for instance [20, Chapter 10], that

$$\rho^{-2} \sup_{B_{\rho/2}} |Dh|^2 + \sup_{B_{\rho/2}} |D^2 h| \le c\rho^{-2} \fint_{B_\rho} |Dh|^2 \, dx \le c\rho^{-2}. \tag{3.26}$$

Moreover, the Taylor expansion of h implies that for $\theta \in (0, 1/4)$,

$$\sup_{x \in B_{2\theta\rho}} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^2 \le c\theta^4 \rho^2. \tag{3.27}$$

At this point we choose $\epsilon = \theta^{n+4}$. Then we have from the second inequality in (3.25) and (3.27) that

$$\fint_{B_{2\theta\rho}} \frac{|w - h(x_0) - Dh(x_0)(x - x_0)|^2}{(2\theta\rho)^2} \, dx \le c\theta^2,$$

hence, by the definitions of the affine function $P_{2\theta\rho} := P_{x_0, 2\theta\rho}$ denoted in Section 2.4 and w and (3.22), we obtain

$$\fint_{B_{2\theta\rho}} \frac{|u - P_{2\theta\rho}|^2}{(2\theta\rho)^2} \, dx \le c\theta^2 (1 + |(Du)_\rho|)^2 E(x_0, \rho). \tag{3.28}$$

Next we estimate the second integral in (3.24). Let $t_0 = t(x_0) \in (0, 1)$ be a number satisfying

$$\frac{1}{p_0} = (1 - t_0) + \frac{t_0}{p_0(1 + \sigma_s)}, \tag{3.29}$$

where σ_s is given in Lemma 2.2. Note that since $2 < \gamma_1 \le p_0 \le \gamma_2$, there exists $0 < t_m \le t_M < 1$ depending only on $\gamma_1, \gamma_2, \sigma_s$ such that $t_m \le t_0 \le t_M$. Then by Hölder's inequality, Jensen's inequality for the convex map $t \mapsto [(\Phi_{p_0})^{-1}(t^{p_0})]^2$, (3.28), (2.8) and (1) of Proposition 2.1, we have

$$\begin{aligned}
&\fint_{B_{2\theta\rho}} \Phi_{p_0}\Big(\frac{|u - P_{2\theta\rho}|}{2\theta\rho}\Big) \, dx \\
&\le \Big(\fint_{B_{2\theta\rho}} \Big[\Phi_{p_0}\Big(\frac{|u - P_{2\theta\rho}|}{2\theta\rho}\Big)\Big]^{1/p_0} \, dx \Big)^{(1-t_0)p_0} \\
&\quad \times \Big(\fint_{B_{2\theta\rho}} \Big[\Phi_{p_0}\Big(\frac{|u - P_{2\theta\rho}|}{2\theta\rho}\Big)\Big]^{1+\sigma_s} \, dx \Big)^{\frac{t_0}{1+\sigma_s}} \\
&\le \Big[\Phi_{p_0}\Big(\theta(1 + |(Du)_\rho|)\sqrt{E(x_0, \rho)}\Big)\Big]^{1-t_0} \Big(\fint_{B_{2\theta\rho}} \Phi_{p_0}(|Du - DP_{2\theta\rho}|) \, dx \Big)^{t_0} \\
&\le [\theta\sqrt{E(x_0, \rho)}]^{p_0(1-t_0)} \, [\Phi_{p_0}(1 + |(Du)_\rho|)]^{1-t_0} \Big(\fint_{B_{2\theta\rho}} \Phi_{p_0}(|Du - DP_{2\theta\rho}|) \, dx \Big)^{t_0}.
\end{aligned}$$

In addition, from (2.17), (1) of Proposition 2.1, (2.8), (2.18) and the definition of E, we have

$$\begin{aligned}
&\fint_{B_{2\theta\rho}} \Phi_{p_0}(|Du - DP_{2\theta\rho}|)\,dx \\
&\le c\theta^{-n}\fint_{B_\rho} \Phi_{p_0}(|Du - DP_\rho|)\,dx + c\Phi_{p_0}(|DP_\rho - DP_{2\theta\rho}|) \\
&\le c\theta^{-n}\fint_{B_\rho} \Phi_{p_0}(|Du - DP_\rho|)\,dx + c\theta^{-(p+1)}\fint_{B_{\theta\rho}} \Phi_p\big(\frac{|u-P_\rho|}{\rho}\big)\,dx \\
&\le c\theta^{-n-(p+1)}\fint_{B_\rho} \Phi_p(|Du - DP_\rho|)\,dx \\
&\le c\theta^{-n-(p+1)}\fint_{B_\rho} \Phi_{p_0}(|Du - (Du)_\rho|)\,dx \\
&\le c\theta^{-n-(p+1)}\Phi_{p_0}(1+|(Du)_\rho|)E(x_0,\rho).
\end{aligned}$$

Combining the two above estimates, we obtain

$$\begin{aligned}
&\fint_{B_{2\theta\rho}} \Phi_{p_0}\Big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\Big)\,dx \\
&\le c\theta^{p_0-(2p_0+n+1)t_0}\Phi_{p_0}(1+|(Du)_\rho|)E(x_0,\rho)^{(\frac{p_0}{2}-1)(1-t_0)+1}.
\end{aligned}$$

Therefore, taking $\epsilon_0 = \epsilon_0(n, N, \gamma_1, \gamma_2, \theta) > 0$ sufficiently small so that

$$\begin{aligned}
E(x_0,\rho)^{(p/2-1)(1-t_0)} &\le \epsilon_0^{(\frac{p_0}{2}-1)(1-t_0)} \le \epsilon_0^{(\frac{\gamma_1}{2}-1)(1-t_M)} \\
&\le \theta^{-\gamma_1+(2\gamma_2+n+1)t_M+2} \le \theta^{-p_0+(2p_0+n+1)t_0+2},
\end{aligned}$$

we obtain

$$\fint_{B_{2\theta\rho}} \Phi_{p_0}\Big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\Big)\,dx \le c\theta^2\Phi_{p_0}(1+|(Du)_\rho|)E(x_0,\rho). \tag{3.30}$$

Moreover, by assuming

$$\sqrt{E(x_0,\rho)} \le \sqrt{\epsilon_0} \le \theta^n/8,$$

we have

$$1+|(Du)_\rho| \le 2(1+|(Du)_{\theta\rho}|) \quad \text{and} \quad 1+|(Du)_{2\theta\rho}| \le 2(1+|(Du)_{\theta\rho}|), \tag{3.31}$$

see [18, page 483]. Therefore, inserting the first inequality in (3.31) into (3.28) and (3.30), we obtain

$$\fint_{B_{2\theta\rho}} \frac{|u-P_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx \le c\theta^2(1+|(Du)_{\theta\rho}|)^2E(x_0,\rho), \tag{3.32}$$

$$\fint_{B_{2\theta\rho}} \Phi_{p_0}\big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\big)\,dx \le c\theta^2\Phi_{p_0}(1+|(Du)_{\theta\rho}|)E(x_0,\rho). \tag{3.33}$$

Step 2. Now we prove (3.23). Suppose that

$$E(x_0,\rho) \le \epsilon_0 \le \theta^n. \tag{3.34}$$

Then we observe that

$$C(x_0, 8\theta\rho) \le (8\theta)^{-n}E(x_0,\rho) \le 1.$$

Hence, in view of Lemma 3.1 with ρ replaced by $\theta\rho$ and $P = P_{2\theta\rho}$, we have

$$\begin{aligned}
&\fint_{B_{\theta\rho}} \Phi_{p_0-2}(1+|DP_{2\theta\rho}|)|Du - DP_{2\theta\rho}|^2\,dx + \fint_{B_{\theta\rho}} \Phi_{p_0}(|Du - DP_{2\theta\rho}|)\,dx \\
&\le c\Phi_{p_0-2}(1+|DP_{2\theta\rho}|)\fint_{B_{2\theta\rho}} \frac{|u-P_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx + c\fint_{B_{2\theta\rho}} \Phi_{p_0}\Big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\Big)\,dx \\
&\quad + c\Phi_{p_0}(1+|DP_{2\theta\rho}|)\Big\{\mu\Big(\fint_{B_{2\theta\rho}} |u-(u)_{2\theta\rho}|^2\,dx\Big) + \omega(2\theta\rho)\log\big(\frac{1}{2\theta\rho}\big)\Big\}.
\end{aligned}$$

We note that Young's inequalities such as (4) and (5) of Proposition 2.1 also hold for the Orlicz function $G(t) := t^{\frac{p}{2}}\log(e+\sqrt{t})$. Using this fact and Lemma 2.5 with (ρ,θ) replaced by $(\theta\rho, 1/2)$, we have

$$\begin{aligned}
&\fint_{B_{\theta\rho}} \Phi_{p_0-2}(1+|(Du)_{\theta\rho}|)|Du-(Du)_{\theta\rho}|^2\,dx \\
&\le c\fint_{B_{\theta\rho}} \Phi_{p_0-2}(1+|DP_{2\theta\rho}|)|Du-(Du)_{\theta\rho}|^2\,dx \\
&\quad + c\fint_{B_{\theta\rho}} \Phi_{p_0-2}(|(Du)_{\theta\rho} - DP_{\theta\rho}|)|Du-(Du)_{\theta\rho}|^2\,dx \\
&\quad + c\fint_{B_{\theta\rho}} \Phi_{p_0-2}(|DP_{\theta\rho} - DP_{2\theta\rho}|)|Du-(Du)_{\theta\rho}|^2\,dx \\
&\le c\fint_{B_{\theta\rho}} \Phi_{p_0-2}(1+|DP_{2\theta\rho}|)|Du-DP_{2\theta\rho}|^2\,dx + c\fint_{B_{\theta\rho}} \Phi_{p_0}(|Du-(Du)_{\theta\rho}|)\,dx \\
&\quad + c\,(\Phi_{p_0}(|(Du)_{\theta\rho} - DP_{\theta\rho}|) + \Phi_{p_0}(|DP_{\theta\rho} - DP_{2\theta\rho}|)) \\
&\le c\fint_{B_{\theta\rho}} \Phi_{p_0-2}(1+|DP_{2\theta\rho}|)|Du-DP_{2\theta\rho}|^2\,dx + c\fint_{B_{\theta\rho}} \Phi_{p_0}(|Du-(Du)_{\theta\rho}|)\,dx \\
&\quad + c\fint_{B_{2\theta\rho}} \Phi_{p_0}\big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\big)\,dx.
\end{aligned}$$

Combining the two estimates above with

$$\fint_{B_{\theta\rho}} \Phi_{p_0}(|Du-(Du)_{\theta\rho}|)\,dx \le c\fint_{B_{\theta\rho}} \Phi_{p_0}(|Du-DP_{2\theta\rho}|)\,dx,$$

we obtain

$$\begin{aligned}
&\fint_{B_{\theta\rho}} \Phi_{p_0-2}(1+|(Du)_{\theta\rho}|)|Du-(Du)_{\theta\rho}|^2\,dx + \fint_{B_{\theta\rho}} \Phi_{p_0}(|Du-(Du)_{\theta\rho}|)\,dx \\
&\le c\Phi_{p_0-2}(1+|DP_{2\theta\rho}|)\fint_{B_{2\theta\rho}} \frac{|u-P_{2\theta\rho}|^2}{(2\theta\rho)^2}\,dx + c\fint_{B_{2\theta\rho}} \Phi_{p_0}\Big(\frac{|u-P_{2\theta\rho}|}{2\theta\rho}\Big)\,dx \\
&\quad + c\Phi_{p_0}(1+|DP_{2\theta\rho}|)\Big\{\mu\Big(\fint_{B_{2\theta\rho}} |u-(u)_{2\theta\rho}|^2\,dx\Big) + \omega(2\theta\rho)\log\big(\frac{1}{2\theta\rho}\big)\Big\}.
\end{aligned} \tag{3.35}$$

We further estimate the right hand side on the above inequality. Applying (2.18), (3.34) and (3.31), we have

$$\begin{aligned}
\Phi_{p_0}(|DP_{2\theta\rho}|) &\le c\Phi_{p_0}(|DP_{2\theta\rho} - (Du)_{2\theta\rho}|) + c\Phi_{p_0}(|(Du)_{2\theta\rho}|) \\
&\le c\theta^{-n}\fint_{B_\rho} \Phi_{p_0}(|Du-(Du)_\rho|)\,dx + c\Phi_{p_0}(|(Du)_{2\theta\rho}|)
\end{aligned}$$

$$
\begin{aligned}
&\le c(\theta^{-n}E(x_0,\rho)+1)\Phi_{p_0}(1+|(Du)_\rho|)\\
&\le c\Phi_{p_0}(1+|(Du)_{\theta\rho}|)\\
&\le \Phi_{p_0}\left(c(1+|(Du)_{\theta\rho}|)\right),
\end{aligned}
$$

which implies

$$|DP_{2\theta\rho}|\le c(1+|(Du)_{\theta\rho}|).$$

Moreover, using Poincaré's inequality and the fact that $\rho\le\theta^n$, we have

$$⨍_{B_{2\theta\rho}}|u-(u)_{2\theta\rho}|^2\,dx\le c\theta^{-n}⨍_{B_\rho}|u-(u)_\rho|^2\,dx\le c\rho⨍_{B_\rho}|Du|^2\,dx.$$

Therefore, inserting the two above inequalities, (3.32) and (3.33) into (3.35), we obtain

$$C(x_0,\theta\rho)\le c\theta^2E(x_0,\rho)+c[E(x_0,\rho)]^{2p_0-1}$$

Finally, assuming

$$[E(x_0,\rho)]^{2(p_0-1)}\le[E(x_0,\rho)]^{2(\gamma_1-1)}\le\epsilon_0^{2(\gamma_1-1)}\le\theta^2,$$

we obtain (3.23). □

Now, we are ready to prove Theorem 1.1

Proof of Theorem 1.1. Fix $\alpha\in(0,1)$. Let us determine several constants such that

$$\lambda:=n-2(1-\alpha)\in(n-2,n), \tag{3.36}$$

$$\theta:=\min\left\{\frac18,\frac{1}{\sqrt{c_*}},\frac{1}{3^{1/(n-\lambda)}}\right\} \tag{3.37}$$

$$\epsilon_2:=\min\left\{\frac{\theta^n}{16},\frac{\epsilon_0}{2}\right\}, \tag{3.38}$$

where c_* and ϵ_0 are determined in Lemma 3.3. We note from the continuity of $\mu(\cdot)$ and (1.2) that one can find $\delta_1=\delta_1(\mu,\omega,\epsilon_2)>0$ such that

$$\mu(r)+\omega(r)\log\left(\frac1r\right)\le\epsilon_2\quad\text{for every } r\in(0,\delta_1]. \tag{3.39}$$

Then we denote

$$\rho_m:=\min\{\delta_1,\epsilon_1,\rho_0\}. \tag{3.40}$$

Step 1. In this step, we fix $B_\rho=B_\rho(x_0)\subset\Omega$ with $x_0\in\Omega$ and $\rho\in(0,\rho_m]$, and suppose that

$$C(x_0,\rho)\le\epsilon_2\quad\text{and}\quad M(x_0,\rho):=\rho⨍_{B_\rho}|Du|^2\,dx\le\delta_1. \tag{3.41}$$

Then we claim that for any $k=0,1,2,\dots$,

$$C(x_0,\theta^k\rho)\le\epsilon_2\quad\text{and}\quad M(x_0,\theta^k\rho):=\theta^k\rho⨍_{B_{\theta^k\rho}}|Du|^2\,dx\le\delta_1. \tag{3.42}$$

We prove the claim by induction. For the sake of convenience, for $k=0,1,2,\dots$, we write $(3.42)_{k,1}$ (resp. $(3.42)_{k,2}$) with the first (resp. second) inequality in (3.42).

Now we suppose that the inequalities in (3.42) hold for k, and then prove that (3.42) holds for k replaced by $k+1$. We first observe from $(3.42)_{k,1}$ and Hölder's inequality that

$$\begin{aligned} \fint_{B_{\theta^k\rho}} |Du-(Du)_{\theta^k\rho}|^2\,dx &\le (1+|(Du)_{\theta^k\rho}|)^2 C(x_0,\theta^k\rho) \\ &\le 2\epsilon_2\Big(1+\fint_{B_{\theta^k\rho}}|Du|^2\,dx\Big). \end{aligned} \tag{3.43}$$

and so, by $(3.42)_{k,2}$,

$$\theta^k\rho\fint_{B_{\theta^k\rho}}|Du-(Du)_{\theta^k\rho}|^2\,dx \le 2\epsilon_2\theta^k\rho+2\epsilon_2\delta_1.$$

This together with (3.37), (3.38) and (3.40) imply

$$\begin{aligned} M(x_0,\theta^{k+1}\rho) &\le 2\theta^{k+1}\rho\fint_{B_{\theta^{k+1}\rho}}|Du-(Du)_{\theta^k\rho}|^2\,dx+2\theta^{k+1}\rho|(Du)_{\theta^k\rho}|^2 \\ &\le 2\theta^{1-n}\theta^k\rho\fint_{B_{\theta^k\rho}}|Du-(Du)_{\theta^k\rho}|^2\,dx+2\theta M(x_0,\theta^k\rho) \\ &\le 4\theta^{k+1-n}\epsilon_2\rho+4\theta^{1-n}\epsilon_2\delta_1+2\theta\delta_1 \\ &\le 4\theta^{-n}\epsilon_2\rho+4\theta^{-n}\epsilon_2\delta_1+2\theta\delta_1\le\delta_1, \end{aligned}$$

which shows $(3.42)_{k+1,2}$. It remains to prove $(3.42)_{k+1,1}$. We notice from (3.21), (3.38), (3.39), $(3.42)_{k,1}$, $(3.42)_{k,2}$ and the fact that $\theta^k\rho\le\rho_m\le\delta_1$ that

$$E(x_0,\theta^k\rho):=\epsilon_2+\big[\mu(\delta_1)+\omega(\theta^k\rho)\log\big(\frac{1}{\theta^k\rho}\big)\big]^{\frac{1}{2p_0-1}}\le 2\epsilon_2<\epsilon_0.$$

Therefore, applying Lemma 3.3, we have

$$C(x_0,\theta^{k+1}\rho)\le c_*\theta^2\epsilon_2\le\epsilon_2.$$

This shows $(3.42)_{k+1,1}$. Hence, we prove that (3.42) holds for every $k=0,1,2,\dots$. Here, we note that (3.43) also holds for every $k=0,1,2,\dots$.

Next we observe from (3.38) and (3.43) that

$$\begin{aligned} \fint_{B_{\theta^{k+1}\rho}(x_0)}|Du|^2\,dx &\le 2\fint_{B_{\theta^{k+1}\rho}}|Du-(Du)_{\theta^k\rho}|^2\,dx+2|(Du)_{\theta^k\rho}|^2 \\ &\le 2\theta^{-n}\fint_{B_{\theta^k\rho}}|Du-(Du)_{\theta^k\rho}|^2\,dx+2\fint_{B_{\theta^k\rho}}|Du|^2\,dx \\ &\le 4\theta^{-n}\epsilon_2+(4\theta^{-n}\epsilon_2+2)\fint_{B_{\theta^k\rho}}|Du|^2\,dx \\ &\le 4\theta^{-n}\epsilon_2+3\fint_{B_{\theta^k\rho}}|Du|^2\,dx, \end{aligned}$$

and so, by (3.37),

$$\int_{B_{\theta^{k+1}\rho}}|Du|^2\,dx\le\theta^\lambda\int_{B_{\theta^k\rho}}|Du|^2\,dx+4|B_1|(\theta^k\rho)^n.$$

Then applying Lemma 2.3 with $\phi(r)=\int_{B_r}|Du|^2\,dx$, for every $r\in(0,\rho)$, we have

$$\begin{aligned}\int_{B_r}|Du|^2\,dx &\le c\big\{\big(\frac{r}{\rho}\big)^\lambda\int_{B_\rho}|Du|^2\,dx+r^\lambda\big\}\\ &\le \frac{c}{\rho^\lambda}\Big(\int_\Omega|Du|^2\,dx+1\Big)r^\lambda.\end{aligned}\tag{3.44}$$

Step 2. To complete the proof, we define

$$\Omega_u:=\big\{x_0\in\Omega: u\in C^{\tilde\alpha}(U_{x_0},\mathbb{R}^N)\text{ for every }\tilde\alpha\in(0,1)\text{ and for some }U_{x_0}\subset\Omega\big\}.$$

where U_{x_0} is an open neighborhood of x_0. Suppose that $x_0\in\Omega$ satisfies

$$\begin{gathered}\liminf_{\rho\downarrow0}\fint_{B_\rho(x_0)}|Du-(Du)_\rho|\,dx=0,\\ M_{x_0}:=\limsup_{\rho\downarrow0}\fint_{B_\rho(x_0)}|Du|^2\,dx<\infty.\end{gathered}\tag{3.45}$$

For $p\in[\gamma_1,\gamma_2]$, set $t_p\in(0,1)$ such that

$$\frac{1}{p}=2t_p+\frac{(1-t_p)}{p(1+\frac{\sigma_h}{2})},\tag{3.46}$$

where σ_h is determined in Theorem 2.4. Note that Since $p\in[\gamma_1,\gamma_2]\mapsto t_p$ is continuous there exists $t_m=t_m(\gamma_1,\gamma_2,\sigma_h)\in(0,1)$ such that $t_m\le t_p$ for every $p\in[\gamma_1,\gamma_2]$. We define

$$s:=\min\Big\{\Big[\frac{(\epsilon_2/2)^{\gamma_2/2}}{c_1(M_{x_0}+2)}\Big]^{\frac{1}{\gamma_1t_m}},\delta_1\Big\}<1,\tag{3.47}$$

where $c_1>0$ will be determined later. Then in view of (3.45), one can find

$$\rho<\min\{\rho_m,(2^n(M_{x_0}+1)+1)^{-1}\delta_1\}.\tag{3.48}$$

with $B_{3\rho}(x_0)\subset\Omega$ such that

$$\fint_{B_\rho(x_0)}|Du-(Du)_\rho|dx<s\quad\text{and}\quad\fint_{B_{2\rho}(x_0)}\Phi(x,|Du|)\,dx<M_{x_0}+1.\tag{3.49}$$

Then by Hölder's inequality with (3.46),

$$\begin{aligned}\fint_{B_\rho(x_0)}\Phi_{p_0}(|Du-(Du)_\rho|)\,dx&\le\Big(\fint_{B_\rho(x_0)}[\Phi_{p_0}(|Du-(Du)_\rho|)]^{\frac{1}{2p_0}}\,dx\Big)^{2p_0t_0}\\ &\quad\times\Big(\fint_{B_\rho(x_0)}[\Phi_{p_0}(|Du-(Du)_\rho|)]^{1+\frac{\sigma_h}{2}}\,dx\Big)^{\frac{1-t_0}{1+\frac{\sigma_h}{2}}},\end{aligned}$$

where $t_0:=t_{p_0}$. From Jensen's inequality for the convex function $G(t):=\Phi_{p_0}^{-1}(t^{2p_0})$, we have

$$\begin{aligned}\fint_{B_\rho(x_0)}[\Phi_{p_0}(|Du-(Du)_\rho|)]^{\frac{1}{2p_0}}\,dx&\le G^{-1}\Big(\fint_{B_\rho(x_0)}|Du-(Du)_\rho|\,dx\Big)\\ &<G^{-1}(s)=[\Phi_{p_0}(s)]^{\frac{1}{2p_0}}\\ &\le(2s^{p_0})^{\frac{1}{2p_0}}.\end{aligned}$$

On the other hand, using Jensen's inequality for the convex map $t \mapsto [\Phi(t)]^{1+\frac{\sigma_h}{2}}$, (2.13) and (3.49), we have

$$\begin{aligned}
\fint_{B_\rho(x_0)} [\Phi_{p_0}(|Du-(Du)_\rho|)]^{1+\frac{\sigma_h}{2}}\,dx &\le c \fint_{B_\rho(x_0)} [\Phi_{p_0}(|Du|)]^{1+\frac{\sigma_h}{2}}\,dx \\
&\le c\Big\{\Big(\fint_{B_{2\rho}(x_0)} \Phi(x,|Du|)\,dx\Big)^{1+\frac{\sigma_h}{2}} + 1\Big\} \\
&< c(M_{x_0}+2)^{1+\frac{\sigma_h}{2}}.
\end{aligned}$$

Therefore, we have

$$\fint_{B_\rho(x_0)} \Phi_{p_0}(|Du-(Du)_\rho|)\,dx < cs^{p_0t_0}(M_{x_0}+2)^{1-t_0} \le c_1(M_{x_0}+2)s^{\gamma_1 t_m},$$

where $c_1>0$ depends only on $n, N, \gamma_1, \gamma_2, \nu, \Lambda$, and so by (3.47),

$$\fint_{B_\rho(x_0)} \Phi_{p_0}(|Du-(Du)_\rho|)\,dx < \big(\frac{\epsilon_2}{2}\big)^{\gamma_2/2}.$$

On the other hand, by (3.48) and (3.49), we have

$$\rho \fint_{B_\rho(x_0)} |Du|^2\,dx \le \rho\Big(2^n \fint_{B_{2\rho}} \Phi(x,|Du|)\,dx + 1\Big) < \delta_1.$$

In addition, by the continuity of integrals, there exists $\varrho>0$ such that (3.49) with x_0 replaced by y holds for every $y \in B_\varrho(x_0)$. Without loss of generality, we can assume that $\varrho \le \rho$. We note that using Jensen's inequality for the convex function $\Psi_{p(y)}(t) := \Phi_{p(y)}(\sqrt{t})$, for $y \in B_\varrho(x_0)$ we have

$$\begin{aligned}
C(y,\rho) &\le \Psi_{p(y)}^{-1}\Big(\fint_{B_\rho(y)} \Phi_{p_0}(|Du-(Du)_\rho|)\,dx\Big) + \fint_{B_\rho(y)} \Phi_{p(y)}(|Du-(Du)_\rho|)\,dx \\
&\le \Psi_{p(y)}^{-1}\big(\big(\frac{\epsilon_2}{2}\big)^{\gamma_2/2}\big) + \big(\frac{\epsilon_2}{2}\big)^{\gamma_2/2} \\
&\le \big(\frac{\epsilon_2}{2}\big)^{\frac{\gamma_2}{2}\frac{2}{p(y)}} + \big(\frac{\epsilon_2}{2}\big)^{\gamma_2/2} \\
&\le \frac{\epsilon_2}{2} + \frac{\epsilon_2}{2} \le \epsilon_2,
\end{aligned}$$

where $\Psi_p(t) = t^{\frac{p}{2}}\log(e+t)$ hence $\Psi_{p(y)}^{-1}(t) \le t^{\frac{2}{\gamma_1}}$. Therefore, in view of Step 1, we see that (3.44) with x_0 replaced by y hold for every $y \in B_\varrho(x_0)$ and $t \le \rho$. Therefore, by Morrey-Campanato's embedding theorem, we have u is $C^\alpha(B_\varrho(x_0),\mathbb{R}^N)$, that is, $x_0 \in \Omega_u$. □

References

[1] E. Acerbi and G. Mingione; *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584** (2005), 117-148.

[2] P. Baroni, M. Colombo, G. Mingione; *Harnack inequalities for double phase functionals*, Nonlinear Anal. TMA, **121** (2015), 206-222.

[3] P. Baroni, M. Colombo, G. Mingione; *Non-autonomous functionals, borderline cases and related function classes*, St. Petersburg Math. J. **27** (3) (2016), 347-379.

[4] V. Bögelein, M. Foss, G. Mingione; *Regularity in parabolic systems with continuous coefficients*, Math. Z. **270** (3-4) (2012), 903-938.

[5] V. Bögelein, F. Duzaar, J. Habermann, C. Scheven; *Partial Hlder continuity for discontinuous elliptic problems with VMO-coefficients*, Proc. Lond. Math. Soc. (3) **103** (3) (2011), 371-404.

[6] S. Campanato; *Partial Hölder continuity of solutions of quasilinear parabolic systems of second order with linear growth*, Rend. Sem. Math. Univ. Padova **64** (1981), 59-75.

[7] S. Campanato; *On the nonlinear parabolic systems in divergence form Hölder continuity and partial Hölder continuity of the solutions*, Ann. Mat. Pura Appl. **137** (4) (1984), 83-122.

[8] A. Cianchi; *Boundedness of solutions to variational problems under general growth conditions*, Comm. Part. Diff. Equ., **22** (1997), 1629-1646.

[9] A. Cianchi; *Local boundedness of minimizers of anisotropic functionals.* Ann. Inst. H. Poincaré Anal. Non Linèaire, **17** (2000), 147-168.

[10] M. Colombo, G. Mingione; *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2) (2015), 443-496.

[11] M. Colombo, G. Mingione; *Bounded minimizers of double phase variational integrals*, Arch. Ration. Mech. Anal.**218** (1) (2015), 219-273.

[12] M. Colombo, G. Mingione; *Calderón-Zygmund estimates and non-uniformly elliptic operators*, J. Funct. Anal. **270** (4) (2016), 1416-1478.

[13] L. Diening, F. Ettwein; *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math., **20** (3) (2008), 523-556.

[14] L. Diening, P. Harjulehto, P. Hästö, M. Ružička; *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math., vol. 2017, Springer-Verlag, Berlin, 2011.

[15] F. Duzaar and K. Steffen; *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals*, J. Reine Angew. Math. **546** (2002), 73-138.

[16] M. Eleuteri, P. Marcellini, E. Mascolo; *Lipschitz continuity for energy integrals with variable exponents*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **27** (2016), 61–87.

[17] M. Eleuteri, P. Marcellini, E. Mascolo; *Lipschitz estimates for systems with ellipticity conditions at infinity*, Ann. Mat. Pura Appl., to appear.

[18] M. Foss, G. Mingione; *Partial continuity for elliptic problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (3) (2008), 471-503.

[19] F. Giannetti, A. Passarelli di Napoli; *Regularity results for a new class of functionals with non-standard growth conditions*, J. Differential Equations, **254** (3) (2013), 1280-1305.

[20] E. Giusti; *Direct methods in the calculus of variations*, World Scientific, River Edge, NJ, 2003.

[21] J. Habermann; *Partial regularity for nonlinear elliptic systems with continuous growth exponent*, Ann. Mat. Pura Appl. (4) **192** (3) (2013), 475-527.

[22] P. Harjulehto, P. Hästö, R. Klén; *Generalized Orlicz spaces and related PDE*, Nonlinear Anal. TMA, **143** (2016), 155-173.

[23] P. Harjulehto, P. Hästö, O. Toivanen; *Harnack's inequality for quasiminimizers with generalized Orlicz growth conditions*, preprint.

[24] T. Iwaniec, A Verde; *On the operator* $L(f) = f \log |f|$, J. Funct. Anal., **169** (2) (1999), 391-420.

[25] M. Kronz; *Partial regularity results for minimizers of quasiconvex functionals of higher order*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2) (2002), 81-112.

[26] T. Kuusi, G. Mingione; *Nonlinear potential theory of elliptic systems*, Nonlinear Anal. TMA, **138** (2016), 277-299.

[27] T. Kuusi, G. Mingione; *Vectorial nonlinear potential theory*, J. Eur. Math. Soc., to appear.

[28] F. Leonetti; *Higher integrability for minimizers of integral functionals with non-standard growth,* J. Differential Equations **112** (1994), 308–324.

[29] F. Leonetti, F. Siepe; *Integrability for solutions to some anisotropic elliptic equations.* Nonlinear Anal. TMA, **75** (2012), 2867-2873.

[30] G. M. Lieberman; *The natural generalization of the natural conditions of Ladyzenskaja and Uralâtzeva for elliptic equations*, Comm. Partial Differential Equations, **16** (2-3) (1991), 311-361.

[31] P. Marcellini; *Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions*, Arch. Ration. Mech. Anal., **105** (1989), 267-284.

[32] P. Marcellini; *Regularity and existence of solutions of elliptic equations with* p, q*-growth conditions*, J. Differential Equations **90** (1991), 1-30.

[33] P. Marcellini; *Regularity for elliptic equations with general growth conditions*, J. Differential Equations **105** (1993), 296-333.

[34] P. Marcellini; *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV), **23** (1996), 1-25.

[35] G. Mingione; *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math., **51** (4) (2006), 355-426.
[36] J. Ok; *Gradient estimates for elliptic equations with $L^{p(\cdot)}\log L$ growth*, Calc. Var. Partial Differential Equations, **55** (2) (2016), 55:26.
[37] J. Ok; *Harnack inequality for a class of functionals with non-standard growth via De Giorgi's method*, Adv. Nonlinear Anal. DOI 10.1515/anona-2016-0083.
[38] J. Ok; *Regularity results for a class of obstacle problems with non-standard growth*, J. Math. Anal. Appl. **444** (2) (2016), 957-979.
[39] J. Ok; *Calderón-Zygmund estimates for a class of obstacle problems with non-standard growth*, NoDEA Nonlinear Differential Equations Appl. **23** (4) (2016), 1-21

Jihoon Ok
School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Korea
E-mail address: jihoonok@kias.re.kr