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FRACTIONAL SCHRÖDINGER-POISSON EQUATIONS WITH GENERAL NONLINEARITIES

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Abstract. In this article we study the existence of positive solutions and ground state solutions for a class of fractional Schrödinger-Poisson equations in $\mathbb{R}^3$ with general nonlinearity.

1. Introduction

In this article we consider the Schrödinger-Poisson system

$$\begin{gathered}(-\Delta)^s u + V(x)u + \phi u = f(u), \quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t \phi = u^2, \quad \text{in } \mathbb{R}^3,\end{gathered} \tag{1.1}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian for $\alpha = s, t$. This article was motivated by [2]. There the authors show the existence of positive solutions for the system

$$\begin{gathered}-\Delta u + V(x)u + \phi u = f(u), \quad \text{in } \mathbb{R}^3,\\ -\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous periodic potential and positive. Our purpose is to show that when we consider this system with fractional Laplacian operator instead of the Laplacian, then we obtain a positive solution and a ground state solution for the system. We emphasize that we prove the existence of weak solution to the system and without using results of regularity, we show that the weak solution is positive almost everywhere in $\mathbb{R}^3$. To prove this, we present another version of the Logarithmic lemma and we deduce a weak comparison principle for the solution of the system (See Theorem 4.1).

We use the following assumptions for the potential function V and the function $f \in C(\mathbb{R},\mathbb{R})$:

- (A1) $V(x) \geq \alpha > 0$, $\forall x \in \mathbb{R}^3$, for some constant $\alpha > 0$,
- (A2) $V(x) = V(x + y)$, for all $x \in \mathbb{R}^3$, $y \in \mathbb{Z}^3$.
- (A3) $f(u)u > 0$, $u \neq 0$;
- (A4) $\lim_{u\to 0} f(u)/u = 0$;
- (A5) there exists $p \in (4, 2^*_s)$ and $C > 0$, such that
$$|f(u)| \leq C(|u| + |u|^{p-1}),$$
for all $u \in \mathbb{R}$, where $2^*_s = \frac{6}{3-2s}$.

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(A6) $\lim_{u\to+\infty} F(u)/u^4 = +\infty$, where $F(u) = \int_0^u f(z)dz$;
(A7) The function $u \longmapsto f(u)/|u|^3$ is increasing in $|u| \neq 0$.

We will denote $g(u) := f(u^+)$ and $G(t) = \int_0^t g(s)ds$.

System (1.1) was studied in [11], where the author consider the one dimensional system

$$\begin{gathered} -\Delta u + \phi u = a|u|^{p-1}u, \quad \text{in } \mathbb{R}, \\ (-\Delta)^t \phi = u^2, \quad \text{in } \mathbb{R}, \end{gathered}$$

for $p \in (1, 5)$ and $t \in (0, 1)$. In [18], the authors show the existence of positive solutions for the system

$$\begin{gathered} -\Delta u + u + \lambda\phi u = f(u), \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = \lambda u^2, \quad \text{in } \mathbb{R}, \end{gathered}$$

for $\lambda > 0$ and a general critical nonlinearity f. In [17], the authors proved the existence of radial ground state solutions of (1.1) when $V = 0$. In [16], the system was studied, although the sign of the solutions is not considered. In this article, we prove the existence of positive solutions for (1.1). Using a Nehari manifold, we ensure the existence of a ground state solution for the problem. Our main result reads as follows.

Theorem 1.1. *Suppose that $s \in (3/4, 1)$, $t \in (0, 1)$, and* (A1)–(A7) *are satisfied. Then* (1.1) *has a positive solution and a ground state solution.*

The hypothesis $s \in (3/4, 1)$ is required to ensure that the interval $(4, 2_s^*)$ is nondegenerate.

Remark 1.2. Condition (A7) implies that $H(u) = uf(u) - 4F(u)$ is a non-negative function.

In [10, Lemma 2.3], the authors proved the following version of the Lions lemma, which will be needed to prove our result.

Lemma 1.3. *If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^s(\mathbb{R}^3)$ such that for some $R > 0$ and $2 \leq q < 2_s^*$ we have*

$$\sup_{x\in\mathbb{R}^3} \int_{B_R(x)} |u_n|^q \to 0$$

when $n \to \infty$, then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2_s^*)$.

2. Preliminaries

For $s \in (0, 1)$, we denote by $\dot{H}^s(\mathbb{R}^3)$ the homogeneous fractional space. It is defined as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{\dot{H}^s} = \Big(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))^2}{|x - y|^{3+2s}} \, dx\, dy)^{1/2}$$

and we define

$$H^s(\mathbb{R}^3) := \big\{u \in L^2(\mathbb{R}^3); \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))^2}{|x - y|^{3+2s}} \, dx\, dy < \infty\big\}.$$

The space $H^s(\mathbb{R}^3)$ is a Hilbert space with the norm

$$\|u\|_{H^s} = \Big(\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))^2}{|x - y|^{3+2s}} \, dx\, dy\Big)^{1/2}$$

We define the fractional Laplacian operator $(-\Delta)^s : \dot{H}^s(\mathbb{R}^3) \to (\dot{H}^s(\mathbb{R}^3))'$ by $((-\Delta)^s u, v) = \frac{\zeta}{2}(u,v)_{\dot{H}^s}$, where $\zeta = \zeta(s) = (\int_{\mathbb{R}^3} \frac{1-cos(\xi_1)}{|\xi|^{3+2s}} d\xi)^{-1}$ and $(\cdot,\cdot)_{H^s}$ is an inner product of $H^s(\mathbb{R}^3)$. The constant ζ satisfies

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,dx\,dy = 2\zeta^{-1}\int_{\mathbb{R}^3} |\xi|^{2s}\mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}d\xi,$$

where $\mathcal{F}u$ is the Fourier transform of u (see [6, Proposition 3.4]). The fractional Laplacian operator is a bounded linear operator.

A pair (u,ϕ_u) is a solution of (1.1) if

$$\frac{\zeta(t)}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\phi_u(x)-\phi_u(y))(w(x)-w(y))}{|x-y|^{3+2t}}\,dx\,dy = \int_{\mathbb{R}^3} u^2 w\,dx.$$

for all $w \in \dot{H}^t(\mathbb{R}^3)$, and

$$((-\Delta)^s u, v) + \int_{\mathbb{R}^3} V(x)uv dx + \int_{\mathbb{R}^3} \phi_u uv dx = \int_{\mathbb{R}^3} f(u)v\,dx$$

for all $v \in H^s(\mathbb{R}^3)$.

Let us recall some facts about the Schrödinger-Poisson equations (see [14, 3, 19, 9] for instance). We can transform (1.1) into a fractional Schrödinger problem with a nonlocal term. For all $u \in H^s(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \dot{H}^t(\mathbb{R}^3)$ such that

$$(-\Delta)^t\phi = u^2.$$

In fact, since $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{22_t^*}{2_t^*-1}}(\mathbb{R}^3)$ (continuously), a simple application of the Lax-Milgram theorem shows that ϕ_u is well defined and

$$\|\phi_u\|_{\dot{H}^t}^2 \le S^2\|u\|^4_{\frac{22_t^*}{2_t^*-1}},$$

where $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^3)$ norm and S is the best constant of the Sobolev immersion $H^s(\mathbb{R}^3) \to L^{2_t^*}(\mathbb{R}^3)$; that is

$$S = \inf_{u\in \dot{H}^t(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2_{\dot{H}^t}}{\|u\|^2_{2_t^*}}.$$

Lemma 2.1. *We have:*

(i) *there exists $C > 0$ such that $\|\phi_u\|_{\dot{H}^t} \le C\|u\|^2_{H^s}$ and*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\phi_u(x)-\phi_u(y))^2}{|x-y|^{3+2t}}\,dx\,dy \le C\|u\|^4_{H^s}$$

for all $u \in H^s(\mathbb{R}^3)$;

(ii) *$\phi_u \ge 0$, $\forall u \in H^s(\mathbb{R}^3)$;*

(iii) *$\phi_{tu} = t^2\phi_u$, $\forall t > 0, u \in H^s(\mathbb{R}^3)$.*

(iv) *If $\tilde{u}(x) := u(x+z)$ then $\phi_{\tilde{u}}(x) = \phi_u(x+z)$ and*

$$\int_{\mathbb{R}^3} \phi_u u^2 dx = \int_{\mathbb{R}^3} \phi_{\tilde{u}}\tilde{u}^2 dx.$$

for all $z \in \mathbb{R}^3$ and $u \in H^s(\mathbb{R}^3)$.

(v) *If $\{u_n\}$ converges weakly to u in $H^s(\mathbb{R}^3)$, then $\{\phi_{u_n}\}$ converges weakly to ϕ_u in $\dot{H}^t(\mathbb{R}^3)$.*

The proof of the above lemma is analogous to the case of Poisson equation in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ (See [2, 14, 19]).

At first, we are interested in showing the existence of a positive solution for (1.1). We will consider the Euler-Lagrange functional $I : H^s(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\begin{aligned} I(u) = \frac{\zeta(s)}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx \\ + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} G(u) dx, \end{aligned}$$

whose derivative is

$$\begin{aligned} I^{'}(u)(v) &= \frac{\zeta}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\, dx\, dy \\ &\quad + \int_{\mathbb{R}^3} V(x) uv dx + \int_{\mathbb{R}^3} \phi_u uv dx - \int_{\mathbb{R}^3} g(u) v dx \\ &= ((-\Delta)^s u, v) + \int_{\mathbb{R}^3} V(x) uv dx + \int_{\mathbb{R}^3} \phi_u uv dx - \int_{\mathbb{R}^3} g(u) v dx. \end{aligned}$$

We remark that critical points of I determine solutions for (1.1).

Lemma 2.2. *The function*

$$u \mapsto \|u\| := \Big(\frac{\zeta(s)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}\, dx\, dy + \int_{\mathbb{R}^3} V(x) u^2 dx \Big)^{1/2}$$

defines a norm in $H^s(\mathbb{R}^3)$ which is equivalent to the standard norm.

The proof of the previous lemma is trivial and therefore we will omit it in this paper.

3. Existence of the Solution

Theorem 3.1. *Suppose that $1 > s > 3/4$, $t \in (0,1)$, and* (A1)–(A6 *are satisfied. Then* (1.1) *has a nontrivial solution.*

Proof. By usual arguments, we prove that the functional I has the mountain pass geometry. By Montain Pass theorem, there is a Cerami's sequence for I at the mountain pass level c. That is, there is $\{u_n\}_{n\in\mathbb{N}} \subset H^s(\mathbb{R}^3)$ such that

$$\begin{gathered} I(u_n) \to c, \\ (1 + \|u_n\|) I^{'}(u_n) \to 0. \end{gathered}$$

where

$$\begin{gathered} c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)), \\ \Gamma = \{\gamma \in C([0,1], H^s(\mathbb{R}^3)); \gamma(0) = 0, \gamma(1) = e\}, \end{gathered}$$

where $e \in H^s(\mathbb{R}^3)$, and e satisfies $I(e) < 0$. By Remark 1.2

$$4I(u_n) - I'(u_n)u_n = \|u_n\|^2 + \int_{\mathbb{R}^3} [f(u_n)u_n - 4F(u_n)] dx \geq \|u_n\|^2$$

Therefore $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. So, there is $u \in H^s(\mathbb{R}^3)$ such that $\{u_n\}$ converges weakly to u. The Lemma 2.1, (A4), and (A5) imply that u is a critical point for I. If $u \neq 0$ then u is a nontrivial solution for (1.1). Suppose that $u = 0$.

We claim that $\{u_n\}$ does not converge to 0 in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2_s^*)$. Indeed, otherwise, by (A4), (A5) and the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} g(u_n)u_n dx \to 0;$$

By Lemma 2.1

$$\|u_n\|^2 \le \|u_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} g(u_n)u_n dx + I'(u_n)u_n.$$

The right-hand side of the above inequality converges to 0. In this case, $u_n \to 0$ in $H^s(\mathbb{R}^3)$. Consequently

$$c = \lim I(u_n) = 0.$$

This equality can not occur. Then, we can assume that there are $R > 0$ and $\delta > 0$ such that passing to a subsequence if necessary

$$\int_{B_R(y_n)} u_n^2 dx \ge \delta,$$

for some sequence $\{y_n\} \subset \mathbb{Z}^3$ (See Lemma 1.3). For each $n \in \mathbb{N}$, we define

$$w_n(x) := u_n(x + y_n).$$

Note that $w_n \in H^s(\mathbb{R}^3)$. Moreover, changing the variables in the integral below, we have

$$\begin{aligned}
I(w_n) &= \frac{\zeta}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x+y_n) - u_n(y+y_n))^2}{|(x+y_n) - (y+y_n)|^{3+2s}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u_n(x+y_n)^2 dx \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 dx - \int_{\mathbb{R}^3} G(u_n(x+y_n)) dx \\
&= \frac{\zeta}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(z) - u_n(w))^2}{|z-w|^{3+2s}} dz dw + \frac{1}{2} \int_{\mathbb{R}^3} V(z) u_n(z)^2 dz \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} G(u_n(z)) dz \\
&= I(u_n).
\end{aligned}$$

Analogously, for every $\phi \in H^s(\mathbb{R}^3)$,

$$\begin{aligned}
I'(w_n)\phi &= \frac{\zeta}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(w_n(x) - w_n(y))(\phi(x) - \phi(y))}{|x-y|^{3+2s}}\, dx\, dy + \int_{\mathbb{R}^3} V(x) w_n \phi dx \\
&\quad + \int_{\mathbb{R}^3} \phi_{w_n} w_n \phi dx - \int_{\mathbb{R}^3} g(w_n)\phi dx \\
&= \frac{\zeta}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(x+y_n) - u_n(y+y_n))(\phi(x) - \phi(y))}{|(x+y_n) - (y+y_n)|^{3+2s}}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^3} V(x+y_n) u_n(x+y_n)\phi(x) dx + \int_{\mathbb{R}^3} \phi_{u_n}(x+y_n) u_n(x+y_n)\phi dx \\
&\quad - \int_{\mathbb{R}^3} g(u_n(x+y_n))\phi(x) dx \\
&= \frac{\zeta}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n(z) - u_n(w)(\phi(z-y_n) - \phi(w-y_n))}{|z-w|^{3+2s}} dz dw \\
&\quad + \int_{\mathbb{R}^3} V(z) u_n(z)\phi(z-y_n) dz + \int_{\mathbb{R}^3} \phi_{u_n}(z) u_n(z)\phi(z-y_n) dz
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^3} g(u_n(z))\phi(z-y_n)dz \\
&= I'(u_n)\overline{\phi}
\end{aligned}
$$

where $\overline{\phi}(x) = \phi(x - y_n)$. This implies that $\{w_n\}$ is a Cerami's sequence for I at the level c. Analogously, we can show that $\{w_n\}$ is bounded, $\{w_n\}$ converges weakly to some $w_0 \in H^s(\mathbb{R}^3)$ and that $I'(w_0) = 0$. Passing to a subsequence, if necessary, we can assume that $\{w_n\}$ converges on $L^2_{\rm loc}(\mathbb{R}^3)$ to w_0. Then

$$
\begin{aligned}
\int_{B_R(0)} w_0^2 dx &= \lim_{n\to\infty} \int_{B_R(0)} w_n^2 dx \\
&= \lim_{n\to\infty} \int_{B_R(0)} u_n(x+y_n)^2 dx \\
&= \lim_{n\to\infty} \int_{B_R(y_n)} u_n(z)^2 dz \geq \delta.
\end{aligned}
$$

Therefore, w_0 is a nontrivial solution for (1.1). Thus, if $u = 0$ we prove that there is a critical point for I, that is nontrivial. □

4. Positivity of solutions

In this section, we prove that the solution in Theorem 3.1 is positive. Initially, we prove a version of a logarithmic lemma. The logarithmic lemma was presented by Di Castro, Kuusi and Palatucci. [5, lemma 1.3]). In the Logarithmic lemma, the authors give an estimate for weak solutions of the equation

$$
\begin{gathered}
(-\Delta_p)^s u = 0 \quad \text{in } \Omega \\
u = g \quad \text{in } \mathbb{R}^n \setminus \Omega
\end{gathered}
$$

in $B_r(x_0) \subset B_{\frac{R}{2}}(x_0) \subset \Omega$, for $x_0 \in \Omega$ and $u \geq 0$ in $B_R(x_0)$. Following the ideas from Di Castro, Kuusi and Palatucci, we will show a similar estimate for a supersolution of the problem

$$
(-\Delta)^s u + a(x)u = 0 \quad \text{in } \mathbb{R}^n
$$

(See Lemma 4.1 bellow). Supersolutions are defined as follows

$$
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} a(x)u(x)v(x)dx \geq 0,
$$

for all $v \in H^s(\mathbb{R}^n)$ with $v \geq 0$ almost everywhere. Also, in this situation, we need not to assume that $u \geq 0$ in some subset of $\mathbb{R}^n$. With this estimate, we conclude that the supersolution satisfies $u > 0$ almost everywhere in $\mathbb{R}^3$ or $u = 0$ almost everywhere in $\mathbb{R}^3$.

Lemma 4.1. *Suppose that $a : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative function and $u \in H^s(\mathbb{R}^n)$. If*

$$
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} a(x)u(x)v(x)dx \geq 0.
$$

for all $v \in H^s(\mathbb{R}^n)$ with $v \geq 0$ almost everywhere, then $u \geq 0$ almost everywhere. In other words, if $(-\Delta)^s u + a(x)u \geq 0$ then $u \geq 0$ almost everywhere.

Proof. Define $v = u^- = \max\{0, -u\}$. By hypothesis

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} a(x)u(x)u^-(x)dx \geq 0.$$

However:

- if $u(x) > 0$ and $u(y) > 0$ then $(u(x)-u(y))(u^-(x)-u^-(y)) = 0$;
- if $u(x) < 0$ and $u(y) < 0$ then $(u(x)-u(y))(u^-(x)-u^-(y)) = -(u(x)-u(y))^2 \leq 0$;
- if $u(x) > 0$ and $u(y) < 0$ then $(u(x)-u(y))(u^-(x)-u^-(y)) = (u(x)-u(y))u(y) \leq 0$;
- if $u(x) < 0$ and $u(y) > 0$ then $(u(x)-u(y))(u^-(x)-u^-(y)) = (u(x)-u(y))(-u(x)) \leq 0$;
- if $u(x) < 0$, then $a(x)u(x)u^-(x) = -a(x)u^2(x) < 0$, and $a(x)u(x)u^-(x) = 0$ in the case $u(x) \geq 0$.

We conclude that each one of the integrals above is equal to zero. Therefore

$$\frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{n+2s}} = 0.$$

Also u^- is constant in $H^s(\mathbb{R}^n)$, that is, $u^- = 0$. □

Lemma 4.2. *Suppose that $\epsilon \in (0,1]$ and $a, b \in \mathbb{R}^n$. Then*

$$|a|^2 \leq |b|^2 + 2\epsilon|b|^2 + \frac{1+\epsilon}{\epsilon}|a-b|^2$$

Proof. Note taht

$$|a|^2 \leq (|b| + |a-b|)^2 = |b|^2 + 2|b||a-b| + |a-b|^2$$

By the Cauchy inequality with ϵ,

$$|b||a-b| \leq \epsilon|b|^2 + \frac{|a-b|^2}{4\epsilon} \leq \epsilon|b|^2 + \frac{|a-b|^2}{2\epsilon}$$

Replacing in the inequality above,

$$|a|^2 \leq |b|^2 + 2\epsilon|b|^2 + \frac{|a-b|^2}{\epsilon} + |a-b|^2 = |b|^2 + 2\epsilon|b|^2 + \frac{1+\epsilon}{\epsilon}|a-b|^2.$$

□

Lemma 4.3. *With the same assumptions as in Lemma 4.1 and $a \in L^1_{\rm loc}(\mathbb{R}^3)$, we have that for all $r, d > 0$ and $x_0 \in \mathbb{R}^n$,*

$$\int_{B_r}\int_{B_r} \Big|\log\Big(\frac{d+u(x)}{d+u(y)}\Big)\Big|^2 \frac{1}{|x-y|^{n+2s}}\,dx\,dy \leq Cr^{n-2s} + \int_{B_{2r}} a(x)dx, \tag{4.1}$$

where $B_r = B_r(x_0)$ and $C = C(n,s) > 0$ is a constant.

Proof. Consider $\phi \in C_0^\infty(B_{\frac{3r}{2}})$, $0 \leq \phi \leq 1$, $\phi = 1$ in B_r and $K > 0$ such that $\|D\phi\|_\infty \leq Kr^{-1}$. The function

$$\eta = \frac{\phi^2}{u+d}$$

is in $H^s(\mathbb{R}^n)$ and $\eta \geq 0$ (see [6, Lemma 5.3]). By hypothesis,

$$0 \leq \int_{R^n}\int_{R^n} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} a(x)u(x)\eta(x)dx$$

$$
\begin{aligned}
&= \int_{B_{2r}} \int_{B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{R^n - B_{2r}} \int_{B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{B_{2r}} \int_{R^n - B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{R^n - B_{2r}} \int_{R^n - B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^n} a(x)u(x)\eta(x)dx.
\end{aligned}
$$

We will prove some statements about the five integrals above.

Claim 1. There are constants $C_2, C_3 > 0$, such that, they depend only on n and s and

$$
\begin{aligned}
&\int_{B_{2r}} \int_{B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\le -C_2 \int_{B_{2r}} \int_{B_{2r}} \Big|\log\Big(\frac{d+u(x)}{d+u(y)}\Big)\Big|^2 \frac{1}{|x-y|^{n+2s}} \min\{\phi(y)^2, \phi(x)^2\}\,dx\,dy \\
&\quad + C_3 \int_{B_{2r}} \int_{B_{2r}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned}
$$

where $\min\{a,b\} = a$ if $a \le b$ and $\min\{a,b\} = b$ if $a \ge b$, for all $a, b \in \mathbb{R}$.

Fix $x, y \in B_{2r}$ and suppose that $u(x) > u(y)$. Define

$$
\epsilon = \delta \frac{u(x)-u(y)}{u(x)+d}
$$

where $\delta \in (0,1)$ is chosen small enough such that $\epsilon \in (0,1)$. Taking $a = \phi(x)$ and $b = \phi(y)$ in the Lemma 4.2, we obtain

$$
|\phi(x)|^2 \le |\phi(y)|^2 + 2\delta \frac{u(x)-u(y)}{u(x)+d} |\phi(y)|^2 + (\delta^{-1} \frac{u(x)+d}{u(x)-u(y)} + 1)|\phi(x)-\phi(y)|^2
$$

Then

$$
\begin{aligned}
&\frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} \\
&= (u(x)-u(y))\Big(\frac{\phi^2(x)}{u(x)+d} - \frac{\phi^2(y)}{u(y)+d}\Big)\frac{1}{|x-y|^{n+2s}} \\
&\le (u(x)-u(y))\Big(\frac{|\phi(y)|^2 + 2\delta \frac{u(x)-u(y)}{u(x)+d}|\phi(y)|^2 + (\delta^{-1}\frac{u(x)+d}{u(x)-u(y)} + 1)|\phi(x)-\phi(y)|^2}{u(x)+d} \\
&\quad - \frac{\phi^2(y)}{u(y)+d}\Big)\frac{1}{|x-y|^{n+2s}} \\
&= (u(x)-u(y))\frac{|\phi(y)|^2}{u(x)+d}\Big[1 + 2\delta \frac{u(x)-u(y)}{u(x)+d} \\
&\quad + (\delta^{-1}\frac{u(x)+d}{u(x)-u(y)} + 1)\frac{|\phi(x)-\phi(y)|^2}{|\phi(y)|^2} - \frac{u(x)+d}{u(y)+d}\Big]\frac{1}{|x-y|^{n+2s}}
\end{aligned}
$$

$$
\begin{aligned}
&= (u(x)-u(y))\frac{|\phi(y)|^2}{u(x)+d}\frac{1}{|x-y|^{n+2s}}(1+2\delta\frac{u(x)-u(y)}{u(x)+d}-\frac{u(x)+d}{u(y)+d})\\
&\quad + (\delta^{-1}+\frac{(u(x)-u(y))}{u(x)+d})|\phi(x)-\phi(y)|^2\frac{1}{|x-y|^{n+2s}}\\
&\le (u(x)-u(y))\frac{|\phi(y)|^2}{u(x)+d}\frac{1}{|x-y|^{n+2s}}(1+2\delta\frac{u(x)-u(y)}{u(x)+d}-\frac{u(x)+d}{u(y)+d})\\
&\quad + 2\delta^{-1}|\phi(x)-\phi(y)|^2\frac{1}{|x-y|^{n+2s}}.
\end{aligned}
$$

We rewrite the first part of the sum appearing on the right side of the above inequality as

$$
\begin{aligned}
&(u(x)-u(y))\frac{|\phi(y)|^2}{u(x)+d}\frac{1}{|x-y|^{n+2s}}\Big(1+2\delta\frac{u(x)-u(y)}{u(x)+d}-\frac{u(x)+d}{u(y)+d}\Big)\\
&= \Big(\frac{u(x)-u(y)}{u(x)+d}\Big)^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[\frac{u(x)+d}{u(x)-u(y)}+2\delta-\frac{u(x)+d}{u(y)+d}\cdot\frac{u(x)+d}{u(x)-u(y)}\Big]\\
&= (\frac{u(x)-u(y)}{u(x)+d})^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[\frac{1-\frac{u(x)+d}{u(y)+d}}{1-\frac{u(y)+d}{u(x)+d}}+2\delta\Big].
\end{aligned}
$$

Define the function $g:(0,1)\to\mathbb{R}$ defined by

$$
g(t)=\frac{1-t^{-1}}{1-t}
$$

satisfies $g(t)\le -\frac{1}{4}\frac{t^{-1}}{1-t}$ if $t\in(0,\frac{1}{2}]$ and $g(t)\le -1$ for all $t\in(0,1)$. We have two cases. If $\frac{u(y)+d}{u(x)+d}\le\frac{1}{2}$ then, we conclude that

$$
\begin{aligned}
&\Big(\frac{u(x)-u(y)}{u(x)+d}\Big)^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[\frac{1-\frac{u(x)+d}{u(y)+d}}{1-\frac{u(y)+d}{u(x)+d}}+2\delta\Big]\\
&\le (\frac{u(x)-u(y)}{u(x)+d})^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[-\frac{1}{4}\frac{\frac{u(x)+d}{u(y)+d}}{\frac{u(x)-u(y)}{u(x)+d}}+2\delta\Big]\\
&= \frac{u(x)-u(y)}{u(x)+d}\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[-\frac{1}{4}\frac{u(x)+d}{u(y)+d}+2\delta\frac{u(x)-u(y)}{u(x)+d}\Big]\\
&= \frac{u(x)-u(y)}{u(y)+d}\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[-\frac{1}{4}+2\delta\frac{(u(x)-u(y))(u(y)+d)}{(u(x)+d)^2}\Big]\\
&\le \frac{u(x)-u(y)}{u(y)+d}\phi(y)^2\frac{1}{|x-y|^{n+2s}}[-\frac{1}{4}+2\delta].
\end{aligned}
$$

In the last inequality, we used

$$
\frac{(u(x)-u(y))(u(y)+d)}{(u(x)+d)^2}\le 1.
$$

Choosing $\delta=1/16$ we have

$$
\Big(\frac{u(x)-u(y)}{u(x)+d}\Big)^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[\frac{1-\frac{u(x)+d}{u(y)+d}}{1-\frac{u(y)+d}{u(x)+d}}+2\delta\Big]
$$

$$
\begin{aligned}
&\le -\frac{1}{8}\frac{u(x)-u(y)}{u(y)+d}\phi(y)^2\frac{1}{|x-y|^{n+2s}}\\
&\le -\frac{1}{8}\Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}.
\end{aligned}
$$

Above, we used that $(\log(t))^2 \le t-1$ for all $t\ge 2$, and that $\frac{u(x)+d}{u(y)+d}\ge 2$. But, if $\frac{u(y)+d}{u(x)+d} > 1/2$, then using $g(t)\le -1$ and $\delta = 1/16$,

$$
\begin{aligned}
&\Big(\frac{u(x)-u(y)}{u(x)+d}\Big)^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\Big[\frac{1-\frac{u(x)+d}{u(y)+d}}{1-\frac{u(y)+d}{u(x)+d}}+2\delta\Big]\\
&\le \Big(\frac{u(x)-u(y)}{u(x)+d}\Big)^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}[-1+2\delta]\\
&\le -\frac{7}{8}\Big(\frac{u(x)-u(y)}{u(x)+d}\Big)^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\\
&\le -\frac{7}{32}\Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}.
\end{aligned}
$$

Here, we used

$$
\Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2 = \Big[\log(1+\frac{u(x)-u(y)}{u(y)+d})\Big]^2 \le 4(\frac{u(x)-u(y)}{u(x)+d})^2.
$$

This is a consequence of the inequality $\log(1+t)\le t$ for all $t>0$, and

$$
t=\frac{u(x)-u(y)}{u(y)+d}=\frac{u(x)-u(y)}{u(x)+d}\cdot\frac{u(x)+d}{u(y)+d}\le 2\frac{u(x)-u(y)}{u(x)+d}.
$$

Therefore,

$$
\begin{aligned}
&(u(x)-u(y))\frac{|\phi(y)|^2}{u(x)+d}\frac{1}{|x-y|^{n+2s}}\Big(1+2\delta\frac{u(x)-u(y)}{u(x)+d}-\frac{u(x)+d}{u(y)+d}\Big)\\
&\le -\frac{1}{8}\Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}.
\end{aligned}
$$

We have proved that: if $u(x)>u(y)$, then

$$
\begin{aligned}
&\frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\\
&\le -\frac{1}{8}\Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}+32|\phi(x)-\phi(y)|^2\frac{1}{|x-y|^{n+2s}}.
\end{aligned}
$$

Integrating on B_{2r} the above inequality, we obtain

$$
\begin{aligned}
&\int_{B_{2r}}\int_{B_{2r}}\frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&=\int_{B_{2r}}\int_{\{x;u(x)>u(y)\}}\frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad+\int_{B_{2r}}\int_{\{x;u(x)<u(y)\}}\frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&\le -\frac{1}{8}\int_{B_{2r}}\int_{\{x;u(x)>u(y)\}}\Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2\phi(y)^2\frac{1}{|x-y|^{n+2s}}\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{8} \int_{B_{2r}} \int_{\{x;u(x)<u(y)\}} \Big[\log(\frac{u(y)+d}{u(x)+d}) \Big]^2 \phi(x)^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&+ 32 \int_{B_{2r}} \int_{B_{2r}} |\phi(x)-\phi(y)|^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy.
\end{aligned}
$$

Using that $|\log(x)| = |\log(\frac{1}{x})|$ for all $x \neq 0$, we obtain

$$
\Big[\log(\frac{u(y)+d}{u(x)+d}) \Big]^2 = \Big[\log(\frac{u(x)+d}{u(y)+d}) \Big]^2.
$$

Then

$$
\begin{aligned}
&\int_{B_{2r}} \int_{B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} \,dx\,dy \\
&\le -\frac{1}{8} \int_{B_{2r}} \int_{\{x;u(x)>u(y)\}} \Big[\log(\frac{u(x)+d}{u(y)+d}) \Big]^2 \phi(y)^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\quad - \frac{1}{8} \int_{B_{2r}} \int_{\{x;u(x)<u(y)\}} \Big[\log(\frac{u(x)+d}{u(y)+d}) \Big]^2 \phi(x)^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\quad + 32 \int_{B_{2r}} \int_{B_{2r}} |\phi(x)-\phi(y)|^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\le -\frac{1}{8} \int_{B_{2r}} \int_{\{x;u(x)>u(y)\}} \Big[\log(\frac{u(x)+d}{u(y)+d}) \Big]^2 \min\{\phi(y)^2, \phi(x)^2\} \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\quad - \frac{1}{8} \int_{B_{2r}} \int_{\{x;u(x)<u(y)\}} \Big[\log(\frac{u(x)+d}{u(y)+d}) \Big]^2 \min\{\phi(y)^2, \phi(x)^2\} \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\quad + 32 \int_{B_{2r}} \int_{B_{2r}} |\phi(x)-\phi(y)|^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&= -\frac{1}{8} \int_{B_{2r}} \int_{B_{2r}} \Big[\log(\frac{u(x)+d}{u(y)+d}) \Big]^2 \min\{\phi(y)^2, \phi(x)^2\} \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\quad + 32 \int_{B_{2r}} \int_{B_{2r}} |\phi(x)-\phi(y)|^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy,
\end{aligned}
$$

Thus, we have proved claim 1.

Claim 2. There exist $C_3 > 0$, depending only on s and n, such that

$$
\int_{R^n - B_{2r}} \int_{B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} \,dx\,dy \le C_3 r^{n-2s}.
$$

Indeed,

$$
\begin{aligned}
&\int_{R^n - B_{2r}} \int_{B_{2r}} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}} \,dx\,dy \\
&= \int_{\mathbb{R}^n - B_{2r}} \int_{\mathbb{R}^n} (u(x)-u(y)) \Big(\frac{\phi^2(x)}{u(x)+d} - \frac{\phi^2(y)}{u(y)+d} \Big) \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&= \int_{\mathbb{R}^n - B_{2r}} \int_{\mathbb{R}^n} |\phi(x)|^2 \frac{u(x)-u(y)}{u(x)+d} \frac{1}{|x-y|^{n+2s}} \,dx\,dy \\
&\le \int_{\mathbb{R}^n - B_{2r}} \int_{\mathbb{R}^n} |\phi(x)|^2 \frac{1}{|x-y|^{n+2s}} \,dx\,dy
\end{aligned}
$$

In the above equality, we used that $u(y) \geq 0$. Therefore

$$\frac{u(x)-u(y)}{u(x)+d} \leq 1.$$

A simple calculation shows that

$$\int_{\mathbb{R}^n - B_{2r}} \int_{\mathbb{R}^n} |\phi(x)|^2 \frac{1}{|x-y|^{n+2s}}\,dx\,dy \leq C_3 r^{n-2s}$$

and C_3 depends only on n and s. Therefore we obtain Claim 2.

Claim 3.

$$\int_{\mathbb{R}^n} a(x)u(x)\eta(x)dx \leq \int_{B_{2r}} a(x)dx$$

. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^n} a(x)u(x)\eta(x)dx &= \int_{\mathbb{R}^n} a(x)u(x)\frac{\phi^2(x)}{u(x)+d}dx \\
&= \int_{B_{2r}} a(x)u(x)\frac{\phi^2(x)}{u(x)+d}dx \\
&= \int_{B_{2r}} a(x)\frac{u(x)}{u(x)+d}\phi^2(x)dx \\
&\leq \int_{B_{2r}} a(x)dx
\end{aligned}$$

We used that $\operatorname{supp}(\eta) \subset B_{2r}$, that $\phi(x) \in (0,1)$ and that $\frac{u(x)}{u(x)+d} \leq 1$.

Claims 1,2 and 3 imply

$$\begin{aligned}
&\int_{B_{2r}} \int_{B_{2r}} \Big[\log(\frac{u(x)+d}{u(y)+d})\Big]^2 \min\{\phi(y)^2, \phi(x)^2\} \frac{1}{|x-y|^{n+2s}}\,dx\,dy \\
&\leq C_5 \int_{B_{2r}} \int_{B_{2r}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + C_6 r^{n-2s} + \int_{B_{2r}} a(x)dx.
\end{aligned}$$

for constants C_5, C_6. The constants C_5, C_6 depend only on n and s. Since $\phi = 1$ in B_r,

$$\begin{aligned}
&\int_{B_r} \int_{B_r} \Big|\log(\frac{d+u(x)}{d+u(y)})\Big|^2 \frac{1}{|x-y|^{n+2s}}\,dx\,dy \\
&\leq C_5 \int_{B_{2r}} \int_{B_{2r}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + C_6 r^{n-2s} + \int_{B_{2r}} a(x)dx
\end{aligned} \tag{4.2}$$

Finally, we show that

$$\int_{B_{2r}} \int_{B_{2r}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leq C_7 r^{n-2s}\,.$$

By assumption,

$$\begin{aligned}
\int_{B_{2r}} \int_{B_{2r}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\leq K r^{-2} \int_{B_{2r}} \int_{B_{2r}} \frac{|x-y|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= K r^{-2} \int_{B_{2r}} \int_{B_{2r}} \frac{1}{|x-y|^{n+2(s-1)}}\,dx\,dy \\
&\leq K r^{-2} \frac{r^{2(1-s)}}{2(1-s)} |B_{2r}| = C_7 r^{n-2s}
\end{aligned}$$

where C_7 depends only on n and s. Replacing the above estimate in (4.2), we obtain the Lemma 4.3. $\square$

Following the same ideas as in [4, Theorem A.1], we prove the theorem stated at the beginning of the section.

Theorem 4.4. *Suppose that $u \in H^s(\mathbb{R}^n)$ and $a \geq 0$ with $a \in L^1_{\rm loc}(\mathbb{R}^n)$. We assume that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} a(x)u(x)v(x)dx \geq 0,$$

for all $v \in H^s(\mathbb{R}^n)$ with $v \geq 0$ almost everywhere. Then $u > 0$ almost everywhere in $\mathbb{R}^n$ or $u = 0$ almost everywhere in $\mathbb{R}^n$.

Proof. By Lemma 4.1, $u \geq 0$. Suppose that $x_0 \in \mathbb{R}^n$ and $r > 0$. Define

$$Z := \{x \in B_r(x_0); u(x) = 0\}$$

If $|Z| > 0$, then we define $F_\delta : B_r(x_0) \to \mathbb{R}$ as

$$F_\delta(x) = \log(1 + \frac{u(x)}{\delta})$$

for all $\delta > 0$. We have $F_\delta(y) = 0$ for all $y \in Z$. Therefore, if $x \in B_r(x_0)$ and $y \in Z$,

$$|F_\delta(x)|^2 = \frac{|F_\delta(x) - F_\delta(y)|^2}{|x-y|^{n+2s}}|x-y|^{n+2s}$$

Integrating with respect to $y \in Z$ we obtain

$$|Z||F_\delta(x)|^2 = \int_Z \frac{|F_\delta(x) - F_\delta(y)|^2}{|x-y|^{n+2s}}|x-y|^{n+2s}dy \leq 2r^{n+2s}\int_Z \frac{|F_\delta(x) - F_\delta(y)|^2}{|x-y|^{n+2s}}dy$$

Now, integrating with respect to $x \in B_r$ we obtain

$$\begin{aligned}
\int_{B_r(x_0)} |F_\delta(x)|^2dx &\leq \frac{1}{|Z|}2r^{n+2s}\int_{B_r(x_0)}\int_Z \frac{|F_\delta(x) - F_\delta(y)|^2}{|x-y|^{n+2s}}\,dy\,dx \\
&\leq \frac{1}{|Z|}2r^{n+2s}\int_{B_r(x_0)}\int_{B_r(x_0)} \frac{|F_\delta(x) - F_\delta(y)|^2}{|x-y|^{n+2s}}\,dy\,dx \\
&= \frac{1}{|Z|}2r^{n+2s}\int_{B_r(x_0)}\int_{B_r(x_0)} \Big|\log(\frac{\delta+u(x)}{\delta+u(y)})\Big|^2\frac{1}{|x-y|^{n+2s}}\,dx\,dy \\
&\leq \frac{1}{|Z|}2r^{n+2s}\Big(Cr^{n-2s} + \int_{B_{2r}} a(x)dx\Big) \\
&= \frac{1}{|Z|}2Cr^{2n} + \frac{1}{|Z|}2r^{n+2s}\int_{B_{2r}} a(x)dx := L.
\end{aligned}$$

The number L does not depend on δ. In summary, we have proved that

$$\int_{B_r(x_0)} \Big|\log(1+\frac{u(x)}{\delta})\Big|^2dx \leq C$$

for some constant $C > 0$ which does not depend on δ. If $u(x) \neq 0$ then $F_\delta(x) \to \infty$ when $\delta \to 0$. By Fatou's lemma, if $|B_r \cap Z^c| > 0$,

$$+\infty \leq \liminf_{\delta\to 0}\int_{B_r\cap Z^c} |F_\delta(x)|^2 \leq C.$$

Therefore $|Z| = |B_r|$ and $u = 0$ almost everywhere in $B_r(x_0)$. Now, we define

$$\begin{gathered} A = \{B_r(x); r > 0, x \in \mathbb{R}^n, u > 0 \ in\ B_r(x)\}, \\ B = \{B_r(x); r > 0, x \in \mathbb{R}^n, u = 0 \ in\ B_r(x)\}, \\ S = \cup_{V \in A} V, \quad W = \cup_{V \in B} V\,. \end{gathered}$$

Note that S and W are open sets. Consider $x \in \mathbb{R}^n$ and $r > 0$. We have two possibilities, either $u \neq 0$ in $B_r(x)$ or $u = 0$ in $B_r(x)$. If $u \neq 0$ in B_r then $u > 0$ in B_r. In this case, $x \in S$. If $u = 0$ in $B_r(x)$, then $x \in W$. Consequently

$$\mathbb{R}^n = S \cup W.$$

By connectedness, we should have $S = \emptyset$ or $W = \emptyset$. If $\mathbb{R}^n = S$ then $u > 0$ almost everywhere in $\mathbb{R}^n$. If $\mathbb{R}^n = W$ then $u = 0$ almost everywhere in $\mathbb{R}^n$. □

Corollary 4.5. *The solution u found in Theorem 3.1 is positive almost everywhere in $\mathbb{R}^3$.*

Proof. For some $v \in H^s(\mathbb{R}^3)$, with $v \geq 0$ almost everywhere, we have

$$\begin{aligned} &\frac{(\zeta)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}}\, dx\, dy + \int_{\mathbb{R}^3} V(x) uv dx + \int_{\mathbb{R}^3} \phi_u uv\, dx \\ &= \int_{\mathbb{R}^3} g(u) v\, dx \geq 0. \end{aligned}$$

If we define $a(x) = \frac{2}{\zeta}(V(x) + \phi_u(x))$, we have that $a \in L^1_{\rm loc}(\mathbb{R}^3)$, because $L^{2^*_t}(\mathbb{R}^3) \subset L^1_{\rm loc}(\mathbb{R}^3)$ and V is continuous. By (A1) and Lemma 2.1 we have $a(x) > 0$ in $\mathbb{R}^3$. Therefore

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}}\, dx\, dy + \int_{\mathbb{R}^3} a(x) uv\, dx \geq 0.$$

for all $v \in H^s(\mathbb{R}^3)$ with $v \geq 0$. But $u \neq 0$. Then, Theorem 4.4 implies that $u > 0$ almost everywhere in $\mathbb{R}^3$. □

Remark 4.6. Define $\mathcal{N} = \{u \in H^s(\mathbb{R}^3) \setminus \{0\}; I'(u)u = 0\}$, where

$$\begin{aligned} I(u) = \frac{\zeta(s)}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))^2}{|x - y|^{3+2s}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \\ - \int_{\mathbb{R}^3} F(u) dx. \end{aligned}$$

If f satisfies (A3)–(A7), then

$$I_\infty = \inf_{u \in \mathcal{N}} I(u)$$

coincides with the mountain pass level associated with I.

Theorem 4.7. *If* (A1)–(A7) *are satisfied, then* (1.1) *has a ground state solution.*

Proof. Defining the Euler-Lagrange functional $I : H^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$\begin{aligned} I(u) = \frac{\zeta(s)}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))^2}{|x - y|^{3+2s}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \\ - \int_{\mathbb{R}^3} F(u) dx, \end{aligned}$$

and following with the ideas in Theorem 3.1, we prove that there is a nonzero solution u to the system (1.1). Also, we prove that there is a Cerami's sequence

$\{w_n\}$ in the mountain pass level associated with I converging to u. By Remark 1.2 and Fatou's lemma

$$\begin{aligned}4c &= \liminf_{n\to\infty}(4I(w_n) - I'(w_n)w_n)\\ &= \liminf_{n\to\infty}(\|w_n\|^2 + \int_{\mathbb{R}^3} H(w_n)dx)\\ &\geq \liminf_{n\to\infty} \|w_n\|^2 + \liminf_{n\to\infty} \int_{\mathbb{R}^3} H(w_n)dx\\ &\geq \|u\|^2 + \int_{\mathbb{R}^3} H(u)dx\\ &= 4I(u) - I'(u)u\\ &= 4I(u).\end{aligned}$$

where $H(u) = uf(u) - 4F(u)$. By definition $u \in \mathcal{N}$. Then $I(u) \leq \inf_{u\in\mathcal{N}} I(u)$. By Remark 4.6

$$I(u) = \inf_{u\in\mathcal{N}} I(u).$$

□

5. Asymptotically periodic potential

In this section, we study problem (1.1), when V satisfies the Assumption (A1) and

(A8) There is a function V_p satisfying (V_1) such that
$$\lim_{|x|\to\infty} |V(x) - V_p(x)| = 0;$$

(A9) $V(x) \leq V_p(x)$ and there is a open set $\Omega \subset \mathbb{R}^3$ with $|\Omega| > 0$ and $V(x) < V_p(x)$ in Ω.

Here V_p is a periodic continuous potential. This case follows the ideas already studied in Schrödinger-Poisson system with asymptotically periodic potential in [2]. We are writing this case to make a complete work for the reader.

Theorem 5.1. *Suppose that* (A1), (A3)–(A9) *are satisfied. Then* (1.1) *has a ground state solution.*

Proof. In $H^s(\mathbb{R}^3)$ we define the norm

$$\|u\|_p = \Big(\frac{\zeta}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} V_p(x)u^2dx\Big)^{1/2}.$$

Consider the functional

$$I_p(u) = \frac{1}{2}\|u\|_p^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2dx - \int_{\mathbb{R}^3} F(u)dx.$$

We claim that there is $w_p \in H^s(\mathbb{R}^3)$ such that $I_p'(w_p) = 0$ and $I_p(w_p) = c_p$, where c_p is the mountain pass level associated with I_p. We consider another norm in $H^s(\mathbb{R}^3)$:

$$\|u\| = \Big(\frac{\zeta}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} V(x)u^2dx\Big)^{1/2}.$$

Then, we define

$$I(u)=\frac{1}{2}\|u\|^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2dx-\int_{\mathbb{R}^3}F(u)dx.$$

The functional I has a mountain pass geometry. If c is the mountain pass level associated with I then $c<c_p$. Indeed, there is a t_* such that $t_*w_p\in\mathcal{N}$ (see remark 4.6) and it is the unique with this property. Then

$$\begin{aligned}
c&\le I(t^*w_p)\\
&<I_p(t^*w_p)\\
&\le\max_{t\ge0}I_p(tw_p)\\
&=I_p(w_p)=c_p
\end{aligned}$$

Consider $\{u_n\}_{n\in\mathbb{N}}$ a Cerami's sequence at the mountain pass level c associated with I. Similarly to the periodic case, we prove that the sequence $\{u_n\}$ is bounded and therefore, converges weakly to $u\in H^s(\mathbb{R}^3)$. Additionally $I'(u)=0$. Now we prove that $u\neq0$. Suppose that $u=0$. Regarding the sequence $\{u_n\}$, the following equalities are true

(1) $\lim_{n\to\infty}\int_{\mathbb{R}^3}|V(x)-V_p(x)|u_n^2dx=0$
(2) $\lim_{n\to\infty}|\|u_n\|-\|u_n\|_p|=0$.
(3) $\lim_{n\to\infty}|I_p(u_n)-I(u_n)|=0$
(4) $\lim_{n\to\infty}|I_p'(u_n)u_n-I'(u_n)u_n|=0$

We will prove (1). The limits (2), (3) and (4) are immediate consequences of (1). Consider $\epsilon>0$ and $A>0$ such that $\|u_n\|_2^2<A$ for all $n\in\mathbb{N}$. By (A8), there is $R>0$ such that, for all $|x|>R$ we have

$$|V(x)-V_p(x)|<\frac{\epsilon}{2A}.$$

But $\{u_n\}$ converges weakly to $u=0$. Then $u_n\to0$ in $L^2(B_R(0))$. This convergence implies that there is $n_0\in\mathbb{N}$ such that

$$\int_{B_R(0)}|V(x)-V_p(x)|u_n^2dx<\frac{\epsilon}{2}$$

for all $n\ge n_0$. Then, if $n\ge n_0$

$$\begin{aligned}
&\int_{\mathbb{R}^3}|V(x)-V_p(x)|u_n^2dx\\
&=\int_{B_R(0)}|V(x)-V_p(x)|u_n^2dx+\int_{(B_R(0))^c}|V(x)-V_p(x)|u_n^2dx\\
&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

Consider $s_n>0$ such that $s_nu_n\in\mathcal{N}_p$ for every $n\in\mathbb{N}$. Where $N_p=\{u\in H^s(\mathbb{R}^3)\setminus\{0\};I_p'(u)u=0\}$. We claim that $\limsup_{n\to\infty}s_n\le1$. Otherwise, there is $\delta>0$ such that, passing to a subsequence if necessary, we can assume that $s_n\ge1+\delta$ for all $n\in\mathbb{N}$. By (4) we have $I_p'(u_n)u_n\to0$; that is,

$$\|u_n\|_p^2+\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx=\int_{\mathbb{R}^3}f(u_n)u_ndx+o_n(1)$$

From $s_nu_n \in \mathcal{N}_p$ we have $I_p'(s_nu_n)u_n = 0$. Equivalently

$$s_n\|u_n\|_p^2 + s_n^3 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} f(s_nu_n)u_n dx$$

Therefore,

$$\int_{\mathbb{R}^3} \Big[\frac{f(s_nu_n)}{(s_nu_n)^3} - \frac{f(u_n)}{(u_n)^3}\Big] u_n^4 dx = \big(\frac{1}{s_n^2} - 1\big)\|u_n\|_p^2 + o_n(1) \le o_n(1). \tag{5.1}$$

If $\{u_n\}_{n\in\mathbb{N}}$ converges to 0 in $L^q(\mathbb{R}^3)$ for all $q \in (2, 2_s^*)$, then by Lemma 2.1,

$$\|u_n\|^2 \le \|u_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} f(u_n)u_n + I'(u_n)u_n$$

consequently $\{u_n\}$ would have limit 0 in $H^s(\mathbb{R}^3)$ and this would contradict the fact that $c > 0$. Therefore, there is a sequence $\{y_n\} \subset \mathbb{Z}^n$, $R > 0$ and $\beta > 0$ such that

$$\int_{B_R(y_n)} u_n^2 dx \ge \beta > 0$$

Taking $v_n(x) := u_n(x+y_n)$ we have $\|v_n\| = \|u_n\|$ and therefore we can assume that $\{v_n\}_{n\in\mathbb{N}}$ converges weakly to some $v \in H^s(\mathbb{R}^3)$. Note that

$$\int_{B_R(0)} v^2 dx \ge \beta > 0\,.$$

The inequality (5.1), Remark 1.2 and Fatou's lemma imply that

$$\begin{aligned}
0 &< \int_{\mathbb{R}^3} \Big[\frac{f((1+\delta)v)}{((1+\delta)v)^3} - \frac{f(v)}{(v)^3}\Big] v^4 dx \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^3} \Big[\frac{f((1+\delta)v_n)}{((1+\delta)v_n)^3} - \frac{f(v_n)}{(v_n)^3}\Big] v_n^4 dx \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^3} \Big[\frac{f(s_nv_n)}{(s_nv_n)^3} - \frac{f(v_n)}{(v_n)^3}\Big] v_n^4 dx \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^3} \Big[\frac{f(s_nu_n)}{(s_nu_n)^3} - \frac{f(u_n)}{(u_n)^3}\Big] u_n^4 dx \\
&\le \liminf_{n\to\infty} o_n(1) = 0.
\end{aligned}$$

The above inequality is a contradiction. Therefore $\limsup_{n\to\infty} s_n \le 1$. Now, we will prove that for n large enough, $s_n > 1$. Suppose that the statement is false. In this case, passing to a subsequence if necessary, we can assume that $s_n \le 1$ for all $n \in \mathbb{N}$. Note that by (f_5), the function $H(u) := uf(u) - 4F(u)$ is increasing in $|u| \ne 0$. Then

$$\begin{aligned}
4c_p &= 4 \inf_{u\in N_p} I_p(u) \\
&\le 4I_p(s_nu_n) \\
&= 4I_p(s_nu_n) - I_p'(s_nu_n)(s_nu_n) \\
&= s_n^2\|u_n\|_p^2 + \int_{\mathbb{R}^3} f(s_nu_n)(s_nu_n) - 4F(s_nu_n)dx \\
&\le \|u_n\|_p^2 + \int_{\mathbb{R}^3} f(u_n)(u_n) - 4F(u_n)dx \\
&\le 4I(u_n) - I'(u_n)u_n + \int_{\mathbb{R}^3} |V(x) - V_p(x)|u_n^2 dx.
\end{aligned}$$

This implies $4c_p \le 4c$. But, this last inequality is false, because we have proved that $c < c_p$. Therefore, we have that $s_n > 1$ for n large enough. Then we have proved that

$$1 \le \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n \le 1.$$

and therefore

$$\lim_{n\to\infty} s_n = 1. \tag{5.2}$$

The Fundamental Theorem of Calculus implies

$$\int_{\mathbb{R}^3} F(s_n u_n)dx - \int_{\mathbb{R}^3} F(u_n)dx = \int_1^{s_n} \Big[\int_{\mathbb{R}^3} f(\tau u_n)u_n dx\Big] d\tau. \tag{5.3}$$

Also, by (A5) we obtain $C > 0$ such that

$$\int_{\mathbb{R}^3} f(\tau u_n)u_n dx \le C(s_n\|u_n\|^2 + s_n^{p-1}\|u_n\|^p). \tag{5.4}$$

for all $\tau \in (1, s_n)$. We have that the sequence $\{u_n\}$ is bounded. Then, by (5.2), (5.3) and (5.4),

$$\int_{\mathbb{R}^3} F(s_n u_n)dx - \int_{\mathbb{R}^3} F(u_n)dx = o_n(1).$$

Then

$$\begin{aligned}
&I_p(s_n u_n) - I_p(u_n)\\
&= \frac{(s_n^2-1)}{2}\|u_n\|^2 + \frac{(s_n^4-1)}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} F(s_n u_n)dx + \int_{\mathbb{R}^3} F(u_n)dx\\
&= o_n(1)
\end{aligned}$$

because $\{u_n\}$ is bounded and $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \|\phi_{u_n}\|^2_{\dot{H}^t(\mathbb{R}^3)} \le C\|u_n\|^4$. By (3),

$$c_p \le I_p(s_n u_n) = I_p(u_n) + o_n(1) = I(u_n) + o_n(1)$$

In the limit as $n \to \infty$ we obtain $c_p \le c$ But, this last inequality is false, because we have proved that $c < c_p$. This contradiction was generated because we assumed that $u = 0$. It follows that u is nontrivial. In Particular,

$$I(u) \ge \inf_{u\in\mathcal{N}} I(u).$$

As in the periodic case

$$I(u) \le c = \inf_{u\in\mathcal{N}} I(u).$$

Therefore u is a ground state solution for system (1.1). □

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