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REGULARIZED TRACE FORMULA FOR HIGHER ORDER DIFFERENTIAL OPERATORS WITH UNBOUNDED COEFFICIENTS

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Abstract. In this work we obtain the regularized trace formula for an even-order differential operator with unbounded operator coefficient.

1. Introduction

The first work about the theory of regularized traces of differential operators belongs to Gelfand and Levitan [1]. They considered the Sturm-Liouville operator

$$-y'' + [q(x) - \lambda]y = 0,$$

with boundary conditions

$$y'(0) = y'(\pi) = 0,$$

where $q(x) \in C^1[0, \pi]$. Under the condition $\int_0^\pi q(x)dx = 0$ they obtained the formula

$$\sum_{n=0}^{\infty} (\mu_n - \lambda_n) = \frac{1}{4}(q(0) + q(\pi)).$$

Gul [2] obtained the formula

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4}[\operatorname{tr} Q(\pi) - \operatorname{tr} Q(0)]$$

for the regularized trace of the second order differential operator

$$l[y] = -y''(x) + Ay(x) + Q(x)y(x)$$

with unbounded operator coefficient and with the boundary conditions $y(0) = y'(\pi) = 0$. Here λ_k and μ_k are the eigen-elements of the operators

$$l_0[y] = -y''(x) + Ay(x)$$
$$l[y] = -y''(x) + Ay(x) + Q(x)y(x)$$

with the same boundary conditions $y(0) = y'(\pi) = 0$ respectively.

Adıgüzelov and Sezer [3] obtained a regularized trace formula for a self-adjoint differential operator of higher order with unbounded operator coefficient. Articles

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[4, 5, 6, 7, 8] are devoted to study of regularized trace formulas of differential operators with bounded operator coefficient. Bayramov et al [9] obtained the second regularized trace formula for the differential operator equation with the semi-periodic boundary conditions. Makin [10] established a formula for the first regularized trace of the Sturm-Liouville equation with a complex-valued potential and with irregular boundary conditions.

Let H be a separable Hilbert space and let $H_1 = L_2(H;[0,\pi])$ denote the set of all strongly measurable functions f with values in H and such that

$$\int_0^\pi \|f(t)\|_H^2 dt < \infty.$$

and the scalar function $(f(t), g)$ is Lebesgue measurable for every $g \in H$ in the interval $[0,\pi]$. Here $(\cdot,\cdot)$ denotes the inner product in H and $\|\cdot\|$ denotes the norm in H.

If the inner product of two arbitrary elements f and g of the space H_1 is defined by

$$(f,g)_{H_1} = \int_0^\pi (f(t), g(t))_H dt, \quad f, g \in H_1$$

then H_1 becomes a separable Hilbert space [11]. $\sigma_\infty(H)$ denotes the set of all compact operators from H into H. If $A \in \sigma_\infty(H)$, then AA^* is a nonnegative self-adjoint operator and $(A^*A)^{1/2} \in \sigma_\infty(H)$. Let the non-zero eigen-elements of the operator $(A^*A)^{1/2}$ be $s_1 \geq s_2 \geq \dots \geq s_q$ $(0 \leq q \leq \infty)$. Here, each eigen-element is counted according to its own multiplicity. The numbers $s_1, s_2, \dots, s_q$ are called s-numbers of the the operator A. $\sigma_1(H)$ is the set of all the operators $A \in \sigma_\infty(H)$ such that the s-numbers of which satisfy the condition $\sum_{q=1}^{\infty} s_q < \infty$. An operator is called a trace class operator if it belongs to $\sigma_1(H)$.

Let us consider the operators l_0 and l in H_1 defined by

$$l_0[u] = (-1)^m u^{(2m)}(t) + Au(t), \tag{1.1}$$

$$l[u] = (-1)^m u^{(2m)}(t) + Au(t) + Q(t)u(t) \tag{1.2}$$

with the same boundary conditions $y^{(2i-2)}(0) = y^{(2i-1)}(\pi) = 0$ $(i = 1, 2, \dots, m)$ respectively. Here $A : \Omega(A) \to H$ is a densely defined self-adjoint operator in H with $A = A^* \geq E$ where $E : H \to H$ is identity operator and $A^{-1} \in \sigma_\infty(H)$. We also should note that our problem's boundary conditions are different from the considered problem's boundary conditions in [3] which arise new difficulties.

Let $\eta_1 \leq \eta_2 \leq \dots \leq \eta_n \leq \dots$ be the eigen-elements of the operator A and $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be the orthonormal eigenvectors corresponding to these eigen-elements. Here, each eigenvalue is counted according to its own multiplicity number. Let $\Omega(L_0')$ denote the set of the functions $u(t)$ of the space H_1 satisfying the following conditions:

(a) $u(t)$ has continuous derivative of the $2m$ order with respect to the norm in the space H in the interval $[0,\pi]$;
(b) $u(t) \in \Omega(A)$ for every $t \in [0,\pi]$ and $Au(t)$ is continuous with respect to the norm in the space H.
(c) $y^{(2i-2)}(0) = y^{(2i-1)}(\pi) = 0$ $(i = 1, 2, \dots, m)$.

Here $\overline{\Omega(L_0')} = H_1$. Let us consider the linear operator $L_0'u = l_0 u$ from $D(L_0)$ to H_1. L_0' is a symmetric operator. The eigen-elements of L_0' are $(\frac{1}{2} + k)^{2m} + \eta_j$

$(k = 0, 1, 2, \dots; j = 1, 2, \dots)$. and the orthonormal eigenvectors corresponding to these eigen-elements are

$$\sqrt{\frac{2}{\pi}}\sin\big((\frac{1}{2}+k)t\big)\varphi_j(k = 0, 1, 2, \dots; j = 1, 2, \dots).$$

2. Some relations about the eigen-elements and resolvents

Let the eigenvalues of the operators L_0 and L be $\mu_1 \le \mu_2 \le \dots \le \mu_n \le \dots$ and $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le \dots$ respectively. Let $N(\mu)$ be the number of eigen-elements of operator L_0 which is not greater than a positive number μ. If $\eta_j \sim aj^\alpha$ as $j \to \infty$ $(a > 0, \alpha > \frac{2m}{2m-1})$ that is, if

$$\lim_{j\to\infty}\frac{\eta_j}{aj^\alpha} = 1$$

then using the same method in [12] it can be found that $N(\mu) \sim d\mu^{\frac{2m+\alpha}{2m\alpha}}$, where

$$d = \frac{2}{\alpha a^{1/a}}\int_0^{\pi/2}(\sin\tau)^{\frac{2}{\alpha}-1}(\cos\tau)^{1+\frac{1}{m}}d\tau$$

and hence

$$\mu_n \sim d_0 n^{\frac{2m\alpha}{2m+\alpha}} \quad \text{as } j \to \infty \quad (d_0 = d^{\frac{2m\alpha}{2m+\alpha}}). \tag{2.1}$$

Let $Q(t)$ be an operator function satisfying the following conditions:

(1) $Q(t) : H \to H$ is a self-adjoint operator for every $t \in [0, \pi]$;
(2) $Q(t)$ is weakly measurable in the interval $[0, \pi]$;
(3) The norm function $\|Q(t)\|$ is bounded in the interval $[0, \pi]$;
(4) $Q(t)$ has weak derivative of the second order in the interval $[0, \pi]$;
(5) The function $(Q'(t)f, g)$ is continuous for every $f, g \in H$;
(6) $Q^{(i)}(t) : H \to H$ $(i = 0, 1, 2)$ are self-adjoint trace class operators and the functions $\|Q^{(i)}(t)\|_{\sigma_1(H)}$ $(i = 0, 1, 2)$ are bounded and measurable in the interval $[0, \pi]$.

Since Q is a self-adjoint operator from H_1 to H_1 for every $y \in H_1$ we have

$$|(Qy, y)_{H_1}| \le \|Qy\|_{H_1}\|y\|_{H_1} \le \|Q\|_{H_1}\|y\|^2_{H_1}$$

or

$$(-\|Q\|y, y)_{H_1} \le (Qy, y)_{H_1} \le (\|Q\|y, y)_{H_1}.$$

This means that

$$-\|Q\|_{H_1}E \le Q \le \|Q\|_{H_1}E.$$

And so

$$L_0 - \|Q\|_{H_1}E \le L = L_0 + Q \le L_0 + \|Q\|_{H_1}E.$$

In this situation, it is well-known that (Smirnov, [13])

$$\mu_n - \|Q\|_{H_1} \le \lambda_n \le \mu_n + \|Q\|_{H_1}.$$

According to this, we can write

$$1 - \frac{\|Q\|_{H_1}}{\mu_n} \le \frac{\lambda_n}{\mu_n} \le 1 + \frac{\|Q\|_{H_1}}{\mu_n}.$$

By applying limit to each side of this inequality and by considering the equality

$$\lim_{n\to\infty}\frac{\mu_n}{d_0 n^{\frac{2m\alpha}{2m+\alpha}}},$$

we get $\lim_{n\to\infty} \lambda_n/\mu_n = 1$. Thus we have

$$\lim_{n\to\infty} \frac{\lambda_n}{d_0 n^{\frac{2m\alpha}{2m+\alpha}}} = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \frac{\mu_n}{d_0 n^{\frac{2m\alpha}{2m+\alpha}}} = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \lim_{n\to\infty} \frac{\mu_n}{d_0 n^{\frac{2m\alpha}{2m+\alpha}}} = 1$$

or as $n \to \infty$,

$$\lambda_n \sim d_0 n^{\frac{2m\alpha}{2m+\alpha}}. \tag{2.2}$$

By using the formula (2.1), it is easily seen that the sequence $\{\mu_n\}$ has a subsequence $\mu_{n_1} < \mu_{n_2} < \dots < \mu_{n_p} < \dots$ such that

$$\mu_q - \mu_{n_p} > d_0\big(q^{\frac{2m\alpha}{2m+\alpha}} - n_p^{\frac{2m\alpha}{2m+\alpha}}\big) \quad \big(q = n_p + 1, n_p + 2, \dots; d_0 = d^{\frac{2m\alpha}{2m+\alpha}}\big). \tag{2.3}$$

Let $R_\lambda^0 = (L_0 - \lambda E)^{-1}$, $R_\lambda = (L - \lambda E)^{-1}$ be the resolvents of the operators L_0 and L respectively. If $\alpha > \frac{2m}{2m-1}$ then, by the formulas (2.1) and (2.2), R_λ^0 and R_λ are trace class operators for $\lambda \neq \lambda_q, \mu_q$ $(q = 1, 2, \dots)$. In this situation

$$\operatorname{tr}(R_\lambda - R_\lambda^0) = trR_\lambda - trR_\lambda^0 = \sum_{q=1}^{\infty}(\frac{1}{\lambda_q - \lambda} - \frac{1}{\mu_q - \lambda}), \tag{2.4}$$

see Cohberg and Krein [14].

Let $|\lambda| = d_p = 2^{-1}(\mu_{n_p+1} + \mu_{n_p})$. It is easy to see that for large values of p the inequalities $\mu_{n_p} < d_p < \mu_{n_p+1}$ and $\lambda_{n_p} < d_p < \lambda_{n_p+1}$ are satisfied. The series $\sum_{q=1}^{\infty} \frac{\lambda}{\lambda_q - \lambda}$ and $\sum_{q=1}^{\infty} \frac{\lambda}{\mu_q - \lambda}$ are uniform convergent on the circle $|\lambda| = d_p$. Hence with the help of inequality (2.3), we obtain

$$\sum_{q=1}^{n_p}(\lambda_q - \mu_q) = -\frac{1}{2\pi i} \int_{|\lambda|=d_p} \lambda \operatorname{tr}(R_\lambda - R_\lambda^0) d\lambda, \tag{2.5}$$

where $i^2 = -1$.

Lemma 2.1. *If* $\eta_j \sim aj^\alpha$ *as* $j \to \infty$ $(a > 0, \alpha > \frac{2m}{2m-1})$ *then* $\|R_\lambda^0\|_{\sigma_1(H_1)} < 2d_0^{-1} \frac{(2\delta+1)}{\delta n_p^{\delta-1}}$ $(\delta = \frac{2m\alpha}{2m+\alpha} - 1)$ *on the circle* $|\lambda| = d_p$.

Proof. For $\lambda \notin \{\mu_q\}_{q=1}^{\infty}$, since R_λ^0 is a normal operator we have $\|R_\lambda^0\|_{\sigma_1(H_1)} = \sum_{q=1}^{\infty} \frac{1}{|\mu_q - \lambda|}$ [14]. On the circle $|\lambda| = d_p$ we have

$$\begin{aligned}
&\|R_\lambda^0\|_{\sigma_1(H_1)} \\
&\le \sum_{q=1}^{\infty} \frac{1}{||\lambda| - \mu_q|} = \sum_{q=1}^{n_p} \frac{2}{\mu_{n_p} + \mu_{n_p+1} - 2\mu_q} + \sum_{q=n_p+1}^{\infty} \frac{2}{2\mu_q - \mu_{n_p} - \mu_{n_p+1}} \\
&\le \sum_{q=1}^{n_p} \frac{2}{\mu_{n_p+1} - \mu_q} + \sum_{q=n_p+1}^{\infty} \frac{2}{2\mu_q - \mu_{n_p}} = \sum_{q=1}^{n_p} \frac{2}{\mu_{n_p+1} - \mu_q} + 2D_p,
\end{aligned} \tag{2.6}$$

where $D_p = \sum_{k=n_p+1}^{\infty} (\mu_k - \mu_{n_p})^{-1}$ $(p = 1, 2, \dots)$. By using the inequality (2.3) we obtain

$$\begin{aligned}
\sum_{q=1}^{n_p} \frac{2}{\mu_{n_p+1} - \mu_q} &< \frac{n_p}{\mu_{n_p+1} - \mu_{n_p}} < \frac{n_p}{d_0((n_p + 1)^{1+\delta} - n_p^{1+\delta})} \\
&< \frac{n_p}{d_0(n_p + 1)^\delta} < \frac{n_p^{1-\delta}}{d_0},
\end{aligned} \tag{2.7}$$

$$\begin{aligned} D_p &= \sum_{k=n_p+1}^{\infty}(\mu_k-\mu_{n_p})^{-1} < \frac{1}{d_0}\sum_{k=n_p+1}^{\infty}\frac{1}{k^{1+\delta}-n_p^{1+\delta}} \\ &= d_0^{-1}\Big[\frac{1}{((n_p+1)^{1+\delta}-n_p^{1+\delta})}+\sum_{k=n_p+2}^{\infty}\frac{1}{k^{1+\delta}-n_p^{1+\delta}}\Big]. \end{aligned} \tag{2.8}$$

It is easy to see that

$$\sum_{k=n_p+2}^{\infty}\frac{1}{k^{1+\delta}-n_p^{1+\delta}} \le \int_{n_p+1}^{\infty}\frac{dt}{t^{1+\delta}-n_p^{1+\delta}},$$

$$\int_{n_p+1}^{\infty}\frac{dt}{t^{1+\delta}-n_p^{1+\delta}} < \frac{1}{\delta[((n_p+1)^{1+\delta}-n_p^{1+\delta})]^{\frac{\delta}{1+\delta}}}.$$

Taking into account the inequality (2.8) we obtain

$$D_p < d_0^{-1}\frac{\delta+1}{\delta[((n_p+1)^{1+\delta}-n_p^{1+\delta})]^{\frac{\delta}{1+\delta}}} < d_0^{-1}\frac{\delta+1}{\delta n_p^{\frac{\delta^2}{1+\delta}}}. \tag{2.9}$$

With the help of (2.6), (2.7) and (2.9), it follows that on the circle $|\lambda| = d_p$,

$$\|R_\lambda^0\|_{\sigma_1(H_1)} < 2d_0^{-1}\frac{(2\delta+1)}{\delta n_p^{\delta-1}}.$$

□

Lemma 2.2. *If the operator function $Q(t)$ satisfies the conditions (1)–(3), and $\eta_j \sim a j^\alpha$ as $j \to \infty$ ($a > 0, \alpha > \frac{2m}{2m-1}$), then for $|\lambda| = d_p$ and large values of p,*

$$\|R_\lambda\|_{H_1} < 4d_0^{-1}n_p^{-\delta}.$$

Proof. Since the s-numbers of the trace class operator R_λ are $\{\frac{1}{\lambda_1-\lambda}, \frac{1}{\lambda_2-\lambda}, \dots, \frac{1}{\lambda_q-\lambda}, \dots\}$, it follows that

$$\|R_\lambda\|_{H_1} = \max\{\frac{1}{\lambda_1-\lambda}, \frac{1}{\lambda_2-\lambda}, \dots, \frac{1}{\lambda_q-\lambda}, \dots\}. \tag{2.10}$$

On the circle $|\lambda| = d_p$,

$$\big||\lambda_q|-|\lambda|\big| = \big||\lambda_q| - 2^{-1}(\mu_{n_p}+\mu_{n_p+1})\big| = 2^{-1}\big|\mu_{n_p}+\mu_{n_p+1}-2|\lambda_q|\big|. \tag{2.11}$$

Using the inequality $q \le n_p$ and for the large values of p, since $|\lambda_q| \le \lambda_{n_p}$, we have

$$\begin{aligned} &\mu_{n_p}+\mu_{n_p+1}-2|\lambda_q| \\ &\ge \mu_{n_p}+\mu_{n_p+1}-2\lambda_{n_p} = \mu_{n_p+1}-\mu_{n_p}+2(\mu_{n_p}-\lambda_{n_p}) \\ &\ge \mu_{n_p+1}-\mu_{n_p}-2|\mu_{n_p}-\lambda_{n_p}|. \end{aligned} \tag{2.12}$$

Considering $|\mu_q - \lambda_q| \le \|Q\|_{H_1}$ $(q = 1, 2, \dots)$ by (2.12) we obtain

$$\mu_{n_p}+\mu_{n_p+1}-2|\lambda_q| \ge \mu_{n_p+1}-\mu_{n_p}-2\|Q\|_{H_1} \quad (q \le n_p). \tag{2.13}$$

With the help of inequality $q \ge n_p + 1$ and for the large values of p, since $|\lambda_q| = \lambda_q \ge \lambda_{n_p+1}$ then

$$\begin{aligned} 2|\lambda_q|-\mu_{n_p}-\mu_{n_p+1} &\ge 2\lambda_{n_p+1}-\mu_{n_p}-\mu_{n_p+1} \\ &= 2(\lambda_{n_p+1}-\mu_{n_p+1})+\mu_{n_p+1}-\mu_{n_p} \\ &\ge \mu_{n_p+1}-\mu_{n_p}-2|\lambda_{n_p+1}-\mu_{n_p+1}|. \end{aligned}$$

Using the above inequality,

$$2|\lambda_q| - \mu_{n_p} - \mu_{n_p+1} \geq \mu_{n_p+1} - \mu_{n_p} - 2\|Q\|_{H_1} \quad (q \geq n_p + 1). \tag{2.14}$$

Taking into account that $\lim_{p\to\infty}(\mu_{n_p+1} - \mu_{n_p}) = \infty$ and by (2.11), (2.13) and (2.14), on the circle $|\lambda| = d_p$ we have

$$\big||\lambda_q| - |\lambda|\big| > 4^{-1}(\mu_{n_p+1} - \mu_{n_p}). \tag{2.15}$$

By (2.3) and (2.15) we obtain

$$\big||\lambda_q| - |\lambda|\big| > 4^{-1}d_0((n_p+1)^{1+\delta} - n_p^{1+\delta}) > 4^{-1}d_0(n_p+1)^{\delta}.$$

From the above inequality and $|\lambda| = d_p$ for the sufficiently large values of p, we have

$$|\lambda_q - \lambda| > 4^{-1}d_0 n_p^{\delta}.$$

From (2.10) and the above inequality we have $4d_0^{-1}n_p^{-\delta}$. □

3. Regularized trace formula

We know from operator theory that for the resolvents of the operators L_0 and L the following formula holds:

$$R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0 \quad (\lambda \in \rho(L_0) \cap \rho(L)).$$

Using the above formula and (2.5), it can be easily shown that

$$\sum_{q=1}^{n_p}(\lambda_q - \mu_q) = \sum_{j=1}^{s} U_{pj} + U_p^{(s)}, \tag{3.1}$$

where

$$U_{pj} = \frac{(-1)^j}{2\pi i j}\int_{|\lambda|=d_p} tr[(QR_\lambda^0)^j]d\lambda \;\; (i^2 = -1; j = 1, 2, \dots), \tag{3.2}$$

$$U_p^{(s)} = \frac{(-1)^s}{2\pi i}\int_{|\lambda|=d_p} \lambda tr[R_\lambda(QR_\lambda^0)^{s+1}]d\lambda \;\; (i^2 = -1). \tag{3.3}$$

Theorem 3.1. *If the operator function $Q(t)$ satisfies the the conditions* (1)–(3) *and $\eta_j \sim aj^\alpha$ as $j \to \infty$ ($a > 0, \alpha > \frac{2m(1+\sqrt{2})}{2\sqrt{2}m - \sqrt{2} - 1}$) then*

$$\lim_{p\to\infty} U_{pj} = 0 \quad (j = 2, 3, 4, \dots).$$

Proof. According to (3.2) for U_{p2} we have the equality

$$U_{p2} = \frac{1}{2\pi i}\sum_{j=1}^{n_p}\sum_{k=n_p+1}^{\infty}\Big[\int_{|\lambda|=d_p}\frac{d\lambda}{(\lambda - \mu_j)(\lambda - \mu_k)}\Big](Q\psi_j, \psi_k)_{H_1}(Q\psi_k, \psi_j)_{H_1}. \tag{3.4}$$

Therefore,

$$|U_{p2}| \leq \|Q\|_{H_1}^2 D_p. \tag{3.5}$$

By (2.9) and (3.5) we obtain

$$\lim_{p\to\infty} U_{p2} = 0 \quad (\alpha > \frac{2m}{2m-1}). \tag{3.6}$$

Let us show that

$$\lim_{p\to\infty} U_{p3} = 0. \tag{3.7}$$

By using (3.2) it follows that

$$\begin{aligned} U_{p3} = &\sum_{J=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}[G(j,k,s)+G(s,k,j)+G(j,s,k] \\ &+\sum_{J=1}^{n_p}\sum_{k=n_p+1}^{\infty}\sum_{s=n_p+1}^{\infty}[G(j,k,s)+G(s,k,j)+G(k,j,s], \end{aligned} \tag{3.8}$$

where

$$\begin{gathered} G(j,k,s)=g(j,k,s)(Q\psi_j,\psi_k)_{H_1}(Q\psi_k,\psi_s)_{H_1}(Q\psi_s,\psi_j)_{H_1}, \\ g(j,k,s)=\frac{1}{6\pi i}\int_{|\lambda|=d_p}\frac{d\lambda}{(\lambda-\mu_j)(\lambda-\mu_k)(\lambda-\mu_s)}. \end{gathered}$$

Taking into account $g(j,k,s)=\overline{g(j,k,s)}$ and $Q=Q^*$ we obtain

$$G(s,k,j)=\overline{G(j,k,s)},\quad G(k,j,s)=\overline{G(j,k,s)},\quad G(j,s,k)=\overline{G(j,k,s)}. \tag{3.9}$$

With the help of (3.8) and (3.9) we obtain

$$U_{p3}=E_1+E_2$$

with

$$\begin{gathered} E_1=\sum_{J=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}(G(j,k,s)+2\overline{G(j,k,s)}), \\ E_2=\sum_{J=1}^{n_p}\sum_{k=n_p+1}^{\infty}\sum_{s=n_p+1}^{\infty}(G(j,k,s)+2\overline{G(j,k,s)}) \end{gathered}$$

and

$$E_1=E_{11}+\overline{E_{11}},\quad E_2=E_{21}+\overline{E_{21}}, \tag{3.10}$$

with

$$\begin{gathered} E_{11}=\sum_{J=1}^{n_p}\sum_{k=1}^{n_p}\sum_{s=n_p+1}^{\infty}G(j,k,s), \\ E_{21}=\sum_{J=1}^{n_p}\sum_{k=n_p+1}^{\infty}\sum_{s=n_p+1}^{\infty}G(j,k,s). \end{gathered}$$

It is not hard to see that the following inequalities hold:

$$|E_{11}|\le\frac{1+\delta}{d_0^2\delta}\|Q\|_{H_1}^3 n_p^{\frac{1-2\delta^2}{1+\delta}}, \tag{3.11}$$

$$|E_{21}|\le(\frac{1+\delta}{d_0\delta})^2\|Q\|_{H_1}^3 n_p^{\frac{-2\delta^2}{1+\delta}}. \tag{3.12}$$

It follows that

$$\lim_{p\to\infty}U_{p3}=0.$$

Now, let us show that the equality $\lim_{p\to\infty} U_{pj} = 0$ $(j = 4, 5, \dots)$ holds. According to (3.2),

$$\begin{aligned} |U_{pj}| &\le \frac{1}{2\pi j}\int_{|\lambda|=d_p} |\operatorname{tr}(QR_\lambda^0)^j||d\lambda| \\ &\le \int_{|\lambda|=d_p} \|(QR_\lambda^0)^j\|_{\sigma_1(H_1)}|d\lambda| \\ &\le \int_{|\lambda|=d_p} \|(QR_\lambda^0)\|_{\sigma_1(H_1)}\|(QR_\lambda^0)^{j-1}\|_{H_1}|d\lambda| \\ &\le \int_{|\lambda|=d_p} \|Q\|_{H_1}\|R_\lambda^0\|_{\sigma_1(H_1)}\|(QR_\lambda^0)^{j-1}\|_{H_1}|d\lambda| \\ &\le \|Q\|_{H_1}\int_{|\lambda|=d_p} \|R_\lambda^0\|_{\sigma_1(H_1)}\|(QR_\lambda^0)\|_{H_1}^{j-1}|d\lambda| \\ &\le const.\int_{|\lambda|=d_p} \|R_\lambda^0\|_{\sigma_1(H_1)}\|R_\lambda^0\|_{H_1}^{j-1}|d\lambda|. \end{aligned} \tag{3.13}$$

Since $R_\lambda = R_\lambda^0$ for $Q(t) \equiv 0$ according to Lemma 2.2, on the circle $|\lambda| = d_p$,

$$\|R_\lambda^0\|_{H_1} < 4d_0^{-1}n_p^{-\delta} \quad (\delta = \frac{2m\alpha}{2m+\alpha} - 1). \tag{3.14}$$

Using Lemma 2.1 and the inequalities (3.13) and (3.14) one obtains

$$|U_{pj}| < const.n_p^{1-\delta j}\int_{|\lambda|=d_p} |d\lambda| < \text{const}.n_p^{1-\delta j}d_p.$$

For the sufficiently large values of p, since $d_p = 2^{-1}(\mu_{n_p} + \mu_{n_p+1}) \le \text{const}.n_p^{1+\delta}$, then we obtain

$$|U_{pj}| < \text{const}.n_p^{2-\delta(j-1)}.$$

It is easy to see that if $\delta > \frac{2}{3}$ or $\alpha > \frac{10m}{6m-5}$, then

$$\lim_{p\to\infty} U_{pj} = 0 \quad (j = 4, 5, \dots). \tag{3.15}$$

However, if

$$\frac{2m(1+\sqrt{2})}{2\sqrt{2}m - \sqrt{2} - 1} > \frac{10m}{6m-5},$$

considering (3.6) and (3.7) as $\alpha > \frac{2m(1+\sqrt{2})}{2\sqrt{2}m-\sqrt{2}-1}$ one obtains $\lim_{p\to\infty} U_{pj} = 0$ $(j = 2, 3, \dots)$. □

Since the eigen-elements of the operator L_0 are

$$\big(k + \frac{1}{2}\big)^{2m} + \eta_j \quad (k = 0, 1, 2, \dots; j = 1, 2, \dots),$$

then for $q = 1, 2, \dots$,

$$\mu_q = \big(k_q + \frac{1}{2}\big)^{2m} + \eta_{j_q}. \tag{3.16}$$

Theorem 3.2. *If the operator function $Q(t)$ satisfies the conditions* (4)–(6) *and $\eta_j \sim aj^\alpha$ as $j \to \infty$ $(a > 0, \alpha > \frac{2m(1+\sqrt{2})}{2\sqrt{2}m-\sqrt{2}-1})$ then*

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\Big[\lambda_q - \mu_q - \pi^{-1}\int_0^\pi (Q(t)\varphi_{j_q}, \varphi_{j_q})dt\Big] = 4^{-1}[\operatorname{tr} Q(\pi) - \operatorname{tr} Q(0)]$$

where $\{j_q\}_{q=1}^{\infty}$ *is a set of natural numbers satisfying* (3.16).

Proof. From (3.2),

$$U_{p1} = -\frac{1}{2\pi i}\int_{|\lambda|=d_p} \operatorname{tr}(QR_\lambda^0)d\lambda. \tag{3.17}$$

Since QR_λ^0 is a trace class operator for each $\lambda \in \rho(L_0)$ and $\{\Psi_1(t), \Psi_2(t), \dots\}$ is an orthonormal basis of the space H_1, then

$$\operatorname{tr}(QR_\lambda^0) = \sum_{q=1}^{\infty}(QR_\lambda^0\Psi_q, \Psi_q)_{H_1}.$$

By putting $\operatorname{tr}(QR_\lambda^0)$ into (3.17) and considering

$$R_\lambda^0\Psi_q = (L_0 - \lambda E)^{-1}\Psi_q = (\mu_q - \lambda)^{-1}\Psi_q,$$

one obtains

$$\begin{aligned} U_{p1} &= -\frac{1}{2\pi i}\int_{|\lambda|=d_p}[\sum_{q=1}^{\infty}(QR_\lambda^0\Psi_q, \Psi_q)_{H_1}]d\lambda \\ &= -\frac{1}{2\pi i}\int_{|\lambda|=d_p}[\sum_{q=1}^{\infty}(\mu_q - \lambda)^{-1}(Q\Psi_q, \Psi_q)_{H_1}]d\lambda \\ &= [\sum_{q=1}^{\infty}(Q\Psi_q, \Psi_q)_{H_1}]\frac{1}{2\pi i}\int_{|\lambda|=d_p}(\lambda - \mu_q)^{-1}d\lambda. \end{aligned} \tag{3.18}$$

Since the orthonormal eigenvectors according to the eigen-elements $(k+\frac{1}{2})^{2m} + \eta_j$ $(k = 0, 1, 2, \dots; j = 1, 2, \dots)$ of the operator L_0 are $\sqrt{\frac{2}{\pi}}\sin((k+\frac{1}{2})t)\varphi_j$ $(k = 0, 1, 2, \dots; j = 1, 2, \dots)$, it follows that

$$\Psi_q(t) = \sqrt{\frac{2}{\pi}}\sin\big((k+\frac{1}{2})t\big)\varphi_{j_q} \quad q = 1, 2, \dots. \tag{3.19}$$

Further,

$$\frac{1}{2\pi i}\int_{|\lambda|=d_p}(\lambda - \mu_q)^{-1}d\lambda = \begin{cases} 1, & q \le n_p, \\ 0, & q > n_p \end{cases} \tag{3.20}$$

and by using (3.18)–(3.20) we find that

$$\begin{aligned} U_{p1} &= \sum_{q=1}^{n_p}(Q\Psi_q, \Psi_q)_{H_1} = \sum_{q=1}^{n_p}\int_0^{\pi}(Q(t)\Psi_q(t), \Psi_q(t)) \\ &= \sum_{q=1}^{n_p}\int_0^{\pi}\Big(Q(t)\sqrt{\frac{2}{\pi}}\sin\big((k_q+\frac{1}{2})t\big)\varphi_{j_q}, \sqrt{\frac{2}{\pi}}\sin\big((k_q+\frac{1}{2})t\big)\varphi_{j_q}\Big)dt \\ &= \frac{2}{\pi}\sum_{q=1}^{n_p}\int_0^{\pi}\sin^2\big((k_q+\frac{1}{2})t\big)(Q(t)\varphi_{j_q}, \varphi_{j_q})dt \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^{\pi}\big(1 - \cos\big((2k_q+1)t\big)\big)\big(Q(t)\varphi_{j_q}, \varphi_{j_q}\big)dt \\ &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^{\pi}(Q(t)\varphi_{j_q}, \varphi_{j_q})dt - \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^{\pi}\cos((2k_q+1)t)(Q(t)\varphi_{j_q}, \varphi_{j_q})dt. \end{aligned}$$

By subtracting and adding the expression $(Q(t)\varphi_{j_q},\varphi_{j_q})\cos(2k_q t)$ into the second integral on the right side of last equality one obtains

$$\begin{aligned} U_{p1} &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi (Q(t)\varphi_{j_q},\varphi_{j_q})dt + \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi \cos(2k_q t)(Q(t)\varphi_{j_q},\varphi_{j_q})dt \\ &\quad - \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi [\cos((2k_q+1)t)+\cos(2k_q t)](Q(t)\varphi_{j_q},\varphi_{j_q})dt \end{aligned}$$

We can write the expresssion

$$-\frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi \cos(r_q t)(Q(t)\varphi_{j_q},\varphi_{j_q})dt,$$

instead of first term in the right-hand side of the above equality. Thus, we have

$$\begin{aligned} U_{p1} &= \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi (Q(t)\varphi_{j_q},\varphi_{j_q})dt + \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi \cos(2k_q t)(Q(t)\varphi_{j_q},\varphi_{j_q})dt \\ &\quad - \frac{1}{\pi}\sum_{q=1}^{n_p}\int_0^\pi \cos(r_q t)(Q(t)\varphi_{j_q},\varphi_{j_q})dt. \end{aligned}$$

We can write this equation in the form

$$\begin{aligned} \lim_{p\to\infty} U_{p1} &= \frac{1}{\pi}\sum_{j=1}^{\infty}\int_0^\pi (Q(t)\varphi_j,\varphi_j)dt - \frac{1}{\pi}\sum_{j=1}^{\infty}\sum_{r=1}^{\infty}\int_0^\pi \cos(rt)(Q(t)\varphi_j,\varphi_j)dt \\ &\quad + \frac{1}{2\pi}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big[\int_0^\pi \cos(kt)(Q(t)\varphi_j,\varphi_j)dt \\ &\quad + (-1)^k\int_0^\pi \cos(kt)(Q(t)\varphi_j,\varphi_j)dt\Big] \end{aligned}$$

and so we have

$$\begin{aligned} \lim_{p\to\infty} U_{p1} &= \frac{1}{\pi}\sum_{j=1}^{\infty}\int_0^\pi (Q(t)\varphi_j,\varphi_j)dt \\ &\quad - \frac{1}{2}\sum_{j=1}^{\infty}\Big\{\sum_{r=1}^{\infty}\Big[\frac{2}{\pi}\int_0^\pi (Q(t)\varphi_j,\varphi_j)\cos rt\,dt\Big]\cos r0\Big\} \\ &\quad + \frac{1}{4}\sum_{j=1}^{\infty}\Big\{\sum_{k=1}^{\infty}\Big[\frac{2}{\pi}\int_0^\pi (Q(t)\varphi_j,\varphi_j)\cos ktdt\Big]\cos k0 \\ &\quad + \sum_{k=1}^{\infty}\Big[\frac{2}{\pi}\int_0^\pi (Q(t)\varphi_j,\varphi_j)\cos kt\,dt\Big]\cos k\pi\Big\}. \end{aligned} \tag{3.21}$$

The sum with respect to r in the first term on the right side of this expression in the value at 0 of Fourier series according to functions $\{\cos rx\}_{r=0}^{\infty}$ in the interval $[0,\pi]$ of the function $\int_0^\pi (Q(t)\varphi_j,\varphi_j)_H$ having the derivative of second order. Analogically, the sums in the second term with respect to k are the values at the points 0 and π respectively of Fourier series with respect to the functions $\{\cos kx\}_{k=0}^{\infty}$ in the same interval of that function.

Also

$$\Big|\sum_{j=1}^{p}(Q(t)\varphi_j,\varphi_j)\Big| \le \sum_{j=1}^{p}|(Q(t)\varphi_j,\varphi_j)| \le \|Q(t)\|_{\sigma_1(H)}, \quad (p=1,2,\dots). \tag{3.22}$$

Since the norm function $\|Q(t)\|_{\sigma_1(H)}$ is bounded and measurable in the interval $[0,\pi]$, we have

$$\int_0^\pi \|Q(t)\|_{\sigma_1(H)}dt < \infty. \tag{3.23}$$

By using (3.22), (3.23) and Lebesgue theorem we obtain

$$\sum_{j=1}^{\infty}\int_0^\pi (Q(t)\varphi_j,\varphi_j)dt = \int_0^\pi \Big[\sum_{j=1}^{\infty}(Q(t)\varphi_j,\varphi_j)\Big]dt = \int_0^\pi \operatorname{tr} Q(t)dt. \tag{3.24}$$

Furthermore, as in the proof of (3.15) by Lemma 2.1 and Lemma 2.2, we can show that

$$\lim_{p\to\infty} U_p^{(s)} = 0 \quad (s > \frac{3}{\delta}). \tag{3.25}$$

Therefore by (3.1), (3.21), (3.24) and (3.25), we obtain

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\Big[\lambda_q - \mu_q - \pi^{-1}\int_0^\pi (Q(t)\varphi_{j_q},\varphi_{j_q})dt\Big] = 4^{-1}[\operatorname{tr} Q(\pi) - \operatorname{tr} Q(0)]. \tag{3.26}$$

□

The limit on the left hand side of the equality (44) is said to be regularized trace of the operator L (see [1]).

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