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EXISTENCE OF SOLUTIONS FOR A NON-VARIATIONAL SYSTEM OF ELLIPTIC PDE'S VIA TOPOLOGICAL METHODS

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Abstract. In this article, we prove the existence of solutions for a non-variational system of elliptic PDE's. Also we study a system of bi-Laplacian equations with two nonlinearities and without variational assumptions. First, we prove a priori solution estimates, and then we use fixed point theory, to deduce the existence of solutions. Finally, to complement of the existence theorem, we establish a non-existence result.

1. Introduction

We consider the problem

$$\begin{gathered}\Delta^2 u = g(v), \quad v > 0 \quad \text{in } B,\\ \Delta^2 v = f(u), \quad u > 0 \quad \text{in } B,\\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, \quad v = 0, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B,\end{gathered} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$ $(N > 4)$, the functions f and g are continuous and positive on $(0, \infty)$ satisfying $f(0) = 0$ and $g(0) = 0$.

Existence results for elliptic nonlinear systems have earned a lot of interest in recent years, especially when the nonlinear term appears as a source in the equation, supplemented by the boundary conditions of Dirichlet or Neumann. There are two broad classes of systems, the first one with a variational structure, namely Hamiltonian or gradients systems; see [1, 12, 13]. The second one is the class of non-variational problems, which can be treated via topological arguments. For this type of results see [2, 3, 5, 6].

In this work, we address the existence problem for a given system without a variational assumption. More precisely we consider the non-variational case of f and g. Using a topological method (a fixed-point argument), some sufficient conditions for the study of this problem were established. It was shown that a priori estimate of positive solutions for elliptic equations provides a great deal of information about the existence and the structure set of positive solutions [4], [5], [14] and [15]. Our objective is to prove existence of results and a priori estimate of solutions.

This type of question was addressed in several works dealing with the Laplacian problem. One of the pioneering studies in this direction was [19]. In this work,

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we consider the bi-Laplacian operator. In recent years several authors studied polyharmonic problems and a lot of interesting and significant results were obtained see [7], [11], [18] the references therein, see also [20] for more general operator, namely nonhomogeneous differential operator with variable exponents.

The rest of the article is divided into two sections. In Section 2, we establish preliminary results which are helpful to study the bi-Laplacian system (1.1). Section 3 was devoted to present and prove the main results of this work.

2. Preliminary Results

Consider the problem (1.1) for radially symmetric solutions, let $r = |x|$, $u = u(r)$ and $v = v(r)$,

$$\begin{gathered}
u^{(4)} + \frac{2(N-1)}{r}u^{(3)} + \frac{(N-1)(N-3)}{r^2}u'' - \frac{(N-1)(N-3)}{r^3}u' = g(v), \\
v > 0 \quad \text{for } r \in (0,1), \\
v^{(4)} + \frac{2(N-1)}{r}v^{(3)} + \frac{(N-1)(N-3)}{r^2}v'' - \frac{(N-1)(N-3)}{r^3}v' = f(u), \\
u > 0 \quad \text{for } r \in (0,1), \\
u'(0) = 0 = v'(0), \quad u^{(3)}(0) = 0 = v^{(3)}(0), \\
u(1) = 0 = v(1), \quad u'(1) = 0 = v'(1).
\end{gathered} \tag{2.1}$$

We remark that any solution $(u(r), v(r)) \in \big(C^4(0,1)\big)^2$ of (2.1) is a radial symmetric solution of (1.1).

We recall the following result from [16, Lemma 2], which gives more information concerning the eigenvalue problem for the operator Δ^2.

Lemma 2.1. *There is a $\mu_1 > 0$ such that the problem*

$$\Delta^2 v = \mu_1 v \quad \text{in } B, \quad v = \frac{\partial v}{\partial \nu} \quad \text{on } \partial B$$

possesses a positive, radial symmetric solution $\varphi_1(x)$ which satisfies, for some positive constants C_1 and C_2,

$$C_1(1-|x|)^2 \le \varphi_1(x) \le C_2(1-|x|)^2, \quad x \in \overline{B}. \tag{2.2}$$

We recall from [16], see also [11], that the Green function $G(r,s)$ for the linear problem corresponding to (2.1) is defined, for $N > 4$, by

$$G(r,s) = \begin{cases} a_N(s) + r^2 b_N(s), & \text{for } 0 \le r \le s \le 1 \\ (\frac{s}{r})^{N-1}(a_N(r) + s^2 b_N(r)), & \text{for } 0 \le s \le r \le 1, \end{cases} \tag{2.3}$$

where

$$a_N(t) = \frac{t^3}{4(N-2)(N-4)}[2 + (N-4)t^{N-2} - (N-2)t^{N-4}]$$

and

$$b_N(t) = \frac{t}{4N(N-2)}[Nt^{N-2} - (N-2)t^N - 2].$$

The kernel $G(r,s)$ has the following properties (see [16]). There exists a positive constant C such that

$$0 \le G(r,s) \le Cs^{N-1}(1-s)^2\big(\max(r,s)\big)^{4-N}, \tag{2.4}$$

$$\frac{\partial}{\partial r}G(r,s)(r,s) \le 0, \tag{2.5}$$

$$\frac{\partial^2}{\partial r^2}G(r,s)\big|_{r=1} = \frac{1}{2}s^{N-1}(1-s^2). \tag{2.6}$$

Therefore, problem (2.1) is transformed into

$$\begin{aligned} u(r) &= \int_0^1 G(r,s)g(v(s))\mathrm{d}s, \\ v(r) &= \int_0^1 G(r,s)f(u(s))\mathrm{d}s. \end{aligned} \tag{2.7}$$

It is well known that problem (2.1) and problem (2.7) are equivalent.

In the study of the problem (1.1), we need the following eigenvalue problem

$$\begin{gathered} \Delta^2\phi = \lambda_2\psi \quad \text{in } B, \\ \Delta^2\psi = \lambda_1\phi \quad \text{in } B, \\ \phi = 0 = \frac{\partial\phi}{\partial\nu}, \quad \psi = 0 = \frac{\partial\psi}{\partial\nu} \quad \text{on } \partial B, \end{gathered} \tag{2.8}$$

where $\lambda_1, \lambda_2 > 0$.

Let φ_1 be the corresponding eigenfunction of μ_1 which is the first eigenvalue of Δ^2 on the unit ball B, we have the following result.

Lemma 2.2. *Assume that $\lambda_1\lambda_2 = \mu_1^2$, then the problem (2.8) has a positive solution (ϕ,ψ) satisfying (modulo a constant) $\phi = \frac{1}{\sqrt{\lambda_1}}\varphi_1$ and $\psi = \frac{1}{\sqrt{\lambda_2}}\varphi_1$.*

Proof. According to the idea developed in [21] for a Laplacian eigenvalue problem, we define

$$w_1 = \sqrt{\lambda_1}\phi, \tag{2.9}$$

$$w_2 = \sqrt{\lambda_2}\psi. \tag{2.10}$$

We replace (2.9) and (2.10) in the problem (2.8), we obtain

$$\begin{gathered} \Delta^2 w_1 = \sqrt{\lambda_1\lambda_2}w_2 \quad \text{in } B, \\ \Delta^2 w_2 = \sqrt{\lambda_1\lambda_2}w_1 \quad \text{in } B, \\ w_1 = 0 = \frac{\partial w_1}{\partial\nu}, \quad w_2 = 0 = \frac{\partial w_2}{\partial\nu} \quad \text{on } \partial B. \end{gathered} \tag{2.11}$$

Adding the two first equations of (2.11), we write

$$\begin{gathered} \Delta^2(w_1+w_2) = \sqrt{\lambda_1\lambda_2}(w_1+w_2) \quad \text{in } B, \\ w_1+w_2 = 0, \ \frac{\partial(w_1+w_2)}{\partial\nu} = 0 \quad \text{on } \partial B. \end{gathered} \tag{2.12}$$

Subtracting the two first equations of (2.11), we write

$$\begin{gathered} \Delta^2(w_1-w_2) = \sqrt{\lambda_1\lambda_2}(w_2-w_1) \quad \text{in } B, \\ w_1-w_2 = 0, \quad \frac{\partial(w_1-w_2)}{\partial\nu} = 0 \quad \text{on } \partial B. \end{gathered} \tag{2.13}$$

We multiply (2.13) by $w_1 - w_2$ and we make a two integration by parts, we obtain

$$\int_B |\Delta(w_1-w_2)|^2\mathrm{d}x = -\sqrt{\lambda_1\lambda_2}\int_B |w_1-w_2|^2\mathrm{d}x,$$

which proves that $w_1 = w_2$ in $\overline{B}$. Since $\sqrt{\lambda_1\lambda_2} = \mu_1$ and looking at the properties of the eigenvalue problem for the bi-Laplacian, we have that the problem (2.12) has the first eigenfunction φ_1 as the only solution. Then, for a positive constant C, we have $w_1 = w_2 = C\varphi_1$ therefore $\phi = C\frac{1}{\sqrt{\lambda_1}}\varphi_1$ and $\psi = C\frac{1}{\sqrt{\lambda_2}}\varphi_1$. □

We end this section by giving the following identity which plays an important role in the study of our problem. Let F and G be, respectively, the primitives of f and g such that $F(0) = 0$ and $G(0) = 0$.

Lemma 2.3. *Let* (u, v) *a solution of the system* (1.1) *and* α, β *are some positive constants. We have the identity*

$$\begin{aligned}\int_{\partial B}(\Delta u, \Delta v)(x.\nu)\mathrm{d}\sigma_x = \int_B \left(NF(u) + NG(v) - \alpha uf(u) - \beta vg(v)\right)\mathrm{d}x \\ + (N - 4 - (\alpha + \beta))\int_B(\Delta u, \Delta v)\mathrm{d}x.\end{aligned} \tag{2.14}$$

Proof. According to [19, Proposition 4], [21, Theorem 2.1] and by easy computation, the following identity holds

$$\begin{aligned}&\frac{\partial}{\partial x_i}\Big[x_iL - \Big(x_k\frac{\partial u_l}{\partial x_k} + a_lu_l\Big)\Big(L - p_i - \frac{\partial}{\partial x_j}L_{r_{ij}}\Big) \\ &- \frac{\partial}{\partial x_j}\Big(x_k\frac{\partial u_l}{\partial x_k} + a_lu_l\Big)L_{r_{ij}}\Big] \\ &= NL + x_iL_{x_i} - a_lu_lL_{u_l} - (a_l + 1)\frac{\partial u_l}{\partial x_i}L_{p_i} - (a_l + 2)\frac{\partial^2 u_l}{\partial x_i\partial x_j}L_{r_{ij}},\end{aligned} \tag{2.15}$$

where $L = L(x, U, p, r)$ is a lagrangian with $U = (u_1, u_2)$, $p = (p_i^k)$, $p_i^k = \frac{\partial u_k}{\partial x_i}$, $r = (r_{ij})$, $i = 1, \dots, N$ and a_1, a_2 are some constants. We apply the last identity to the Lagrangian associate with problem (1.1); $L = L(x, U, \nabla U, \Delta U) = (\Delta u, \Delta v) + F(u) + G(v)$, $a_1 = \alpha$, $a_2 = \beta$. We integrate (2.15) over B and use the condition $u = 0 = v$, $\frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu}$ on ∂B, we obtain (2.14). □

Remark 2.4. Looking at (2.14), if we take $\alpha + \beta = N - 4$, we see that the critical conditions on f and g are $NF(u) - \alpha uf(u) = 0$ and $NG(v) - (N - 4 - \alpha)vg(v) = 0$, then

$$\frac{f(u)}{F(u)} = \frac{N/\alpha}{u} \quad \text{and} \quad \frac{g(v)}{G(v)} = \frac{N/(N - 4 - \alpha)}{v}.$$

An easy computation gives, for some positive constants c_1 and c_2,

$$f(u) = c_1u^{\frac{N}{\alpha} - 1} \quad \text{and} \quad g(v) = c_2v^{\frac{N}{N-4-\alpha} - 1}.$$

3. Main results and their proofs

Let F and G be, respectively, the primitives of f and g such that $F(0) = 0$ and $G(0) = 0$. We introduce the following critical exponents associate to the system (1.1) by

$$p^\star = \frac{N - \alpha}{\alpha}, \quad q^\star = \frac{4 + \alpha}{N - 4 - \alpha}, \quad \text{where } \alpha \in ((N - 4)/2,\ N/2). \tag{3.1}$$

Easily, we see that

$$\frac{1}{p^* + 1} + \frac{1}{q^* + 1} = \frac{N - 4}{N}.$$

Our main results are the following.

Theorem 3.1. *Suppose that f and g satisfy the following hypothesis*

(i) $\liminf_{s\to\infty} f(s)s^{-1} > \lambda_1$ *and* $\limsup_{s\to 0} f(s)s^{-1} < \lambda_1$,
$\liminf_{s\to\infty} g(s)s^{-1} > \lambda_2$ *and* $\limsup_{s\to 0} g(s)s^{-1} < \lambda_2$.

(ii) $NF(s) - \alpha sf(s) \geq \theta_1 sf(s)$, $s > 0$, *for some* $\theta_1 \geq 0$, $NG(s) - \beta sg(s) \geq \theta_2 sg(s)$, $s > 0$, *for some* $\theta_2 \geq 0$, *and* α *and* β *are positive reals satisfying* $\alpha + \beta = N - 4$.

We suppose that

(H1) *there exists a constant $C > 0$ such that for every positive solution (u, v) of problem* (1.1) *verifies $\|u\|_\infty \leq C$ and $\|v\|_\infty \leq C$.*

Then the system (1.1) *has a positive solution.*

Theorem 3.2. *Under conditions (i) and (ii) on f and g, the condition* (H1) *is satisfied, namely, every positive solution of system* (1.1) *is bounded in L^∞.*

The non-existence result is the following.

Theorem 3.3. *Suppose that f and g satisfy*
$$NF(t) - \alpha tf(t) \leq 0 \quad \text{and} \quad NG(t) - \beta t\, g(t) \leq 0 \quad \text{for } t > 0.$$
Then problem (1.1) *has no nontrivial solution $(u,\, v) \in \big(C^4(\bar{B})\big)^2$.*

Remark 3.4. Under conditions (i) and (ii) on f and g of Theorem 3.1 we have
$$\lim_{t\to\infty} \frac{f(t)}{t^{p^*}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{g(t)}{t^{q^*}} = 0.$$

Indeed, from condition (i), we see that there exists $t_0 > 0$ such that $f(t) > 0$ and $g(t) > 0$ for $t > t_0$. Hence, for $t > t_0$, from condition (ii) we write
$$NF(t) \geq -\theta_1 + \eta tf(t) \quad \text{and} \quad NG(t) \geq -\theta_2 + \mu tg(t), \tag{3.2}$$
where $\eta = \alpha + \theta_1$ and $\mu = \beta + \theta_2$. Therefore,
$$F'(t) - \frac{N}{\eta t} F(t) \leq \frac{\theta_1}{\eta t} \quad \text{and} \quad G'(t) - \frac{N}{\mu t} G(t) \leq \frac{\theta_3}{\mu t}.$$
Multiplying the two last inequalities by $t^{-\frac{N}{\eta}}$ and $t^{-\frac{N}{\mu}}$, respectively, we obtain
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(t^{-\frac{N}{\eta}} F(t)\Big) \leq \frac{\theta_1}{\eta} t^{-1-\frac{N}{\eta}} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(t^{-\frac{N}{\mu}} G(t)\Big) \leq \frac{\theta_1}{\eta} t^{-1-\frac{N}{\mu}}.$$
We deduce, for some positive constants C_1 and C_2, that
$$F(t) \leq C_1 t^{N/\eta} \quad \text{and} \quad G(t) \leq C_2 t^{N/\mu}.$$
We replace into (3.2), we obtain, for t large enough, that for some positive constant C and $\bar{C}$,
$$f(t) \leq C t^{\frac{N}{\eta}-1} \quad \text{and} \quad g(t) \leq \bar{C} t^{\frac{N}{\mu}-1}.$$
Or $\alpha + \beta = N - 4$, then, since $\eta = \alpha + \theta_1$, $\mu = \beta + \theta_2$ and $\theta_1,\ \theta_2 > 0$, we have $\eta + \mu > N - 4$.

Now, we return to the proofs of Theorem 3.1 and Theorem 3.2. The proof of Theorem 3.1 is based on a topological argument. More precisely, we apply the following fixed point theorem [9], see also [17].

Theorem 3.5 ([9])**.** *Let C be a cone in a Banach space X and $\Phi : C \to C$ a compact map such that $\Phi(0) = 0$. Assume that there exist numbers $0 < r < R$ such that*

(a) $x \neq \lambda\Phi(x)$ *for* $0 \leq \lambda \leq 1$ *and* $\|x\| = r$,
(b) *there exists a compact map* $F : \overline{B_R} \times [0, \infty) \to C$ *such that*

$$F(x,0) = \Phi(x) \quad \textit{if } \|x\| = R,$$
$$F(x,\mu) \neq x \quad \textit{if } \|x\| = R \textit{ and } 0 \leq \mu < \infty,$$
$$F(x,\mu) \neq x \quad \textit{if } x \in \overline{B_R} \textit{ and } \mu \geq \mu_0.$$

Then if $U = \{x \in C : r < \|x\| < R\}$ *and* $B_\rho = \{x \in C : \|x\| < \rho\}$, *we have*

$$i_C(\Phi, B_R) = 0, \quad i_C(\Phi, B_r) = 1, \quad i_C(\Phi, U) = -1,$$

where $i_C(\Phi,\Omega)$ *denotes the index of* Φ *with respect to* Ω. *In particular,* Φ *has a fixed point in* U.

Proof of Theorem 3.1. We apply Theorem 3.5, then consider the Banach space $X = \big(C^*(0,1)\big)^2$, where $C^*(0,1)$ denote the space of continuous bounded functions defined on $(0,1)$, endowed with the norm $\|u\| := \sup_{t\in(0,1)} |u(t)|$. We define the cone C by

$$C := \big\{w \in X : w(t) \geq 0, \text{ for all } t \in (0,1)\big\},$$

where $w = (y,z) \geq 0$ means that $y \geq 0$ and $z \geq 0$. We define the compact map $\Phi : X \to X$ by

$$\Phi(w)(r) = \int_0^1 G(r,s)h(w(s))\,\mathrm{d}s, \quad h(w) = (g(v), f(u)).$$

We observe that a fixed point of Φ is a solution of (2.7) and then a solution of (1.1).

Now we shall verify the conditions of Theorem 3.5.

Verification of condition (a): From condition (i) of Theorem 3.1 there exists positive constants $q_1 < 1$ and $q_2 < 1$ such that $f(u(x)) \leq q_1\lambda_1 u(x)$ and $g(v(x)) \leq q_2\lambda_2 v(x)$. Then we have

$$\begin{aligned}\lambda_2 \int v\psi \mathrm{d}x &= \int v\Delta^2\phi \mathrm{d}x = \int \Delta^2 v\phi \mathrm{d}x\\ &= \int f(u)\phi \mathrm{d}x \leq q_1\lambda_1 \int u\phi \mathrm{d}x.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda_1 \int u\phi \mathrm{d}x &= \int u\Delta^2\psi \mathrm{d}x = \int \Delta^2 u\psi \mathrm{d}x\\ &= \int g(v)\psi \mathrm{d}x \leq q_2\lambda_2 \int v\psi \mathrm{d}x.\end{aligned}$$

Combining these two inequalities, we write

$$\lambda_2 \int v\psi \mathrm{d}x \leq q_1\lambda_1 \int u\phi \mathrm{d}x \leq q_1q_2\lambda_2 \int v\psi \mathrm{d}x, \tag{3.3}$$
$$\lambda_1 \int u\phi \mathrm{d}x \leq q_2\lambda_2 \int v\psi \mathrm{d}x \leq q_2q_1\lambda_1 \int u\phi \mathrm{d}x. \tag{3.4}$$

Or $q_1q_2 < 1$ then (3.3) and (3.4) give a contraction since the integrals are nonzero. Moreover, if u and v are replaced respectively by $\lambda\, u$ and $\lambda\, v$ in, respectively, (3.3) and (3.4), for $\lambda \in [0,1]$, then a contradiction also follows and therefore

$$w(t) \neq \lambda\Phi(w(t)) \quad \text{with} \quad \lambda \in [0,1], \quad \|w\| = r, \quad w \in C.$$

Verification of condition (b): Define the compact mapping $F : C \times [0, \infty) \to C$ by

$$F(w, \mu)(r) = \Phi(w + \mu)(r) \tag{3.5}$$

Easily, we see that $F(w, 0) = \Phi(w)$. The condition (i) of Theorem 3.1 gives the existence of constants $k_1 > \lambda_1$, $k_2 > \lambda_2$ and $\mu_0 > 0$ such that $f(y + \mu) \geq k_1 y$ and $g(z + \mu) \geq k_2 z$ if $\mu \geq \mu_0$ for all $(y, z) \geq (0, 0)$. We have

$$\begin{aligned} \lambda_2 \int v\psi \mathrm{d}x &= \int v\Delta^2\phi \mathrm{d}x = \int \Delta^2 v\phi \mathrm{d}x \\ &= \int f(u)\phi \mathrm{d}x \geq k_1 \int u\phi \mathrm{d}x \geq \lambda_1 \int u\phi \mathrm{d}x. \end{aligned}$$

Or

$$\begin{aligned} \lambda_1 \int u\phi \mathrm{d}x &= \int u\Delta^2\psi \mathrm{d}x = \int \Delta^2 u\psi \mathrm{d}x \\ &= \int g(v)\psi \mathrm{d}x \geq k_2 \int v\psi \mathrm{d}x, \end{aligned}$$

then

$$\lambda_2 \int v\psi \mathrm{d}x \geq k_2 \int v\psi \mathrm{d}x. \tag{3.6}$$

In the same way, we have

$$\begin{aligned} \lambda_1 \int u\phi \mathrm{d}x &= \int u\Delta^2\psi \mathrm{d}x = \int \Delta^2 u\psi \mathrm{d}x \\ &= \int g(v)\psi \mathrm{d}x \geq k_2 \int v\psi \mathrm{d}x \geq \lambda_2 \int v\psi \mathrm{d}x. \end{aligned}$$

Or

$$\begin{aligned} \lambda_2 \int v\psi \mathrm{d}x &= \int v\Delta^2\phi \mathrm{d}x = \int \Delta^2 v\phi \mathrm{d}x \\ &= \int f(u)\phi dx \geq k_1 \int u\phi \mathrm{d}x, \end{aligned}$$

then

$$\lambda_1 \int u\phi \mathrm{d}x \geq k_1 \int u\phi \mathrm{d}x. \tag{3.7}$$

Since the integrals $\int u\phi dx$ and $\int v\psi dx$ are nonzero and $k_1 > \lambda_1$, $k_2 > \lambda_2$, the inequalities (3.6) and (3.7) give a contradiction. Therefore, there exists a constant $\mu_0 > 0$ such that

$$w(t) \neq F(w, \mu)(t), \quad \text{for all } w \in C \text{ and } \mu \geq \mu_0. \tag{3.8}$$

This prove the third condition of (b). Now, to prove the second condition of (b), we choose the family of nonlinearities $(f(y + \mu), g(z + \mu))$ for $\mu \in [0, \mu_0]$, using the a priori estimates (H1) which does not depend on μ. Thus, choosing $R > r$ large enough, we have

$$w(r) \neq F(w, \mu)(r), \quad \text{for all } \mu \in [0, \mu_0], \quad w \in C, \quad \|w\| = R. \tag{3.9}$$

The relations (3.8) and (3.9) prove the second condition of (b).

Now, since all conditions of Theorem 3.5 are satisfied, we apply Theorem 3.5 and we conclude the existence of a nontrivial positive solution of problem (2.7) and so the existence of positive solution of problem (1.1). $\square$

Proof of Theorem 3.2. We give this proof in four steps.

Step 1. We claim that there exist positive constants $C_1,\dots,C_4$ such that

$$\int_B f(u)\phi \mathrm{d}x \le C_1, \quad \int_B g(v)\psi \mathrm{d}x \le C_2, \tag{3.10}$$

$$\int_B u\phi \mathrm{d}x \le C_3, \quad \int_B v\psi \mathrm{d}x \le C_4. \tag{3.11}$$

Indeed, using the first and second equations of (1.1), we write

$$\int_B f(u)\phi \mathrm{d}x = \int_B \Delta^2 v\phi \mathrm{d}x = \int_B v\Delta^2\phi \mathrm{d}x = \lambda_2 \int_B v\psi \mathrm{d}x.$$

From condition (i) of Theorem 3.1, there exist $k_2 > \lambda_2$ and $A > 0$ such that $g(v) \ge k_2 v - A$. Thus, for a positive constant C, we have

$$\int_B f(u)\phi \mathrm{d}x = \lambda_2 \int_B v\psi \mathrm{d}x \le C + \frac{\lambda_2}{k_2}\int_B g(v)\psi \mathrm{d}x. \tag{3.12}$$

In the same way, we have

$$\int_B g(v)\psi \mathrm{d}x = \int_B \Delta^2 u\psi \mathrm{d}x = \int_B u\Delta^2\psi \mathrm{d}x = \lambda_1 \int_B v\phi \mathrm{d}x.$$

Again, from condition (i) of Theorem 3.1, there exist $k_1 > \lambda_1$ and $D > 0$ such that $f(u) \ge k_1 u - D$. Thus, for a positive constant $\bar{C}$, we have

$$\int_B g(v)\psi \mathrm{d}x = \lambda_1 \int_B u\phi \mathrm{d}x \le \bar{C} + \frac{\lambda_1}{k_1}\int_B f(u)\phi \mathrm{d}x. \tag{3.13}$$

Combining (3.12) and (3.13), for positive constants M_1 and M_2, we obtain

$$\begin{gathered}
\int_B f(u)\phi \mathrm{d}x \le M_1 + \frac{\lambda_1\lambda_2}{k_1k_2}\int_B f(u)\phi \mathrm{d}x,\\
\int_B g(v)\psi \mathrm{d}x \le M_2 + \frac{\lambda_1\lambda_2}{k_1k_2}\int_B g(v)\psi \mathrm{d}x.
\end{gathered}$$

Since $\frac{\lambda_1\lambda_2}{k_1k_2} < 1$, we deduce (3.10). Using, again, condition (i) of Theorem 3.1 and (3.10) we easily deduce (3.11).

Step 2. We claim that there exist positive constants C_1,..., C_4 such that

$$u(r) \le C_1, \quad v(r) \le C_2, \quad \text{for } \frac{2}{3} \le r \le 1, \tag{3.14}$$

$$u''(1) \le C_3, \quad v''(1) \le C_4. \tag{3.15}$$

Indeed, we see that

$$\begin{gathered}
u(r) = \int_0^1 G(r,s)g(v(s))\mathrm{d}s,\\
v(r) = \int_0^1 G(r,s)f(u(s))\mathrm{d}s.
\end{gathered}$$

Using the fact that $r \to G(r,s)$ is decreasing (see (2.5) and (2.4)), we deduce that $u(r)$ and $v(r)$ are decreasing in r and, for arbitrary $\frac{2}{3} \le r \le 1$,

$$u(r) \le u(\frac{2}{3}) \le 3\int_{1/3}^{2/3} u(s)\mathrm{d}s$$

$$\leq C\int_0^1 s^{N-1}(1-s)^2u(s)\mathrm{d}s \leq C+\int_0^1 s^{N-1}(1-s)^2u(s)\mathrm{d}s.$$

Looking at (2.2) and Lemma 2.2, we write

$$u(r)\leq C\Big(1+\int_0^1 s^{N-1}(1-s)^2u(s)\mathrm{d}s\Big)\leq C\Big(1+\int_B \phi u\mathrm{d}x\Big).$$

We conclude by (3.11) that $u(r)\leq C_1$, for $\frac{2}{3}\leq r\leq 1$ and by the same way that $v(r)\leq C_2$ for $\frac{2}{3}\leq r\leq 1$.

Now, we prove (3.15). We have

$$u(r)=\int_0^1 G(r,s)g(v(s))\mathrm{d}s,\quad v(r)=\int_0^1 G(r,s)f(u(s))\mathrm{d}s.$$

We differentiate the two previous relations two times with respect to r, we obtain

$$u''(r)=\int_0^1 \frac{\partial^2G(r,s)}{\partial r^2}g(v(s))\mathrm{d}s,\quad v''(r)=\int_0^1 \frac{\partial^2G(r,s)}{\partial r^2}f(u(s))\mathrm{d}s.$$

Since the integrals converge, we take the limit when r approaches 1, we write

$$u''(1)=\int_0^1 \frac{\partial^2G(r,s)}{\partial r^2}\Big|_{r=1}g(v(s))\mathrm{d}s\quad v''(1)=\int_0^1 \frac{\partial^2G(r,s)}{\partial r^2}\Big|_{r=1}f(u(s))\mathrm{d}s.$$

Using (2.6), we obtain

$$u''(1)=\frac{1}{2}\int_0^1 s^{N-1}(1-s^2)g(v(s))\mathrm{d}s,\quad v''(1)=\frac{1}{2}\int_0^1 s^{N-1}(1-s^2)f(u(s))\mathrm{d}s.$$

From (2.2) and Lemma 2.2, we see, for some positive constant C, that

$$u''(1)\leq C\int_B \psi g(v)\mathrm{d}x,\quad v''(1)\leq C\int_B \phi f(u)\mathrm{d}x.$$

Finally, we obtain (3.15) by (3.10).

Step 3. Under conditions (i) and (ii) of Theorem 3.1, we claim that, for a small number $0<l<1$, there exist positive constants $C_1,\dots,C_4$ such that

$$\int_0^l s^{N-1}f(u(s))\mathrm{d}s\leq C_1,\quad \int_0^l s^{N-1}g(v(s))\mathrm{d}s\leq C_2, \tag{3.16}$$

$$\int_B uf(u)\mathrm{d}x\leq C_3,\quad \int_B vg(v)\mathrm{d}x\leq C_4. \tag{3.17}$$

Indeed, for the proof of (3.16), looking at proof of Step 1, namely at (3.10) and (3.11), and using Lemma 2.1 and Lemma 2.2 we obtain, for small $0<l<1$,

$$\begin{aligned}
\int_0^l s^{N-1}f(u(s))\mathrm{d}s &\leq \int_0^l s^{N-1}\frac{(1-s)^2}{(1-l)^2}f(u(s))\mathrm{d}s\\
&\leq \frac{1}{(1-l)^2}\int_0^l s^{N-1}(1-s)^2f(u(s))\mathrm{d}s\\
&\leq C\int_0^1 s^{N-1}\phi(s)f(u(s))\mathrm{d}s\\
&= C\int_B f(u)\phi\mathrm{d}x\leq M_1,
\end{aligned}$$

and

$$\begin{aligned}\int_0^l s^{N-1} g(v(s))\mathrm{d}s &\le \int_0^l s^{N-1}\frac{(1-s)^2}{(1-l)^2} g(v(s))\mathrm{d}s\\ &\le \frac{1}{(1-l)^2}\int_0^l s^{N-1}(1-s)^2 g(v(s))\mathrm{d}s\\ &\le \bar{C}\int_0^1 s^{N-1}\psi(s)g(v(s))\mathrm{d}s\\ &= \bar{C}\int_B g(v)\psi\mathrm{d}x \le M_2,\end{aligned}$$

where C, $\bar{C}$, M_1 and M_2 are some constants. This gives (3.16).

For the proof of (3.17), we rewrite the identity (2.14) of Lemma 2.3, considering the fact that $\alpha+\beta = N-4$, as

$$\int_B NF(u) - \alpha u f(u)\mathrm{d}x + \int_B NG(v) - \beta v g(v)\mathrm{d}x = \int_{\partial B}(\Delta u, \Delta v)(x.\nu)\mathrm{d}\sigma_x.$$

Using condition (ii) of Theorem 3.1 for the left hand side of the last equality and after easy computation on the right hand side, we obtain

$$\theta_1\int_B uf(u)dx + \theta_2\int_B vg(v)dx \le Cu''(1)v''(1),$$

where C is a generic constant and θ_1, θ_2 are the constants given by hypothesis of Theorem 3.1. Therefore

$$\theta_1\int_B u\,f(u)dx + \theta_2\int_B v\,g(v)dx \le C.$$

Since the two both left hand sides are positive we obtain directly (3.17).

Step 4. Under conditions (i) and (ii) of Theorem 3.1, we claim that there exist positive constants C_1 and C_2 such that, for any solution (u,v) of problem (1.1),

$$\|u\|_\infty \le C_1, \quad \|v\|_\infty \le C_2. \tag{3.18}$$

Indeed, for u, we have

$$\begin{aligned}\|u\|_\infty \le u(0) &\le \int_0^1 G(0,s)\,g(v(s))\mathrm{d}s\\ &\le C\int_0^1 s^3(1-s)^2\,g(v)\mathrm{d}s \le C\int_0^1 s^3\,g(v)\mathrm{d}s\\ &\le C\int_0^t s^3\,g(v(s))\mathrm{d}s + C\int_t^1 s^3\,g(v)\mathrm{d}s,\end{aligned}$$

where $t\in(0,1)$ is arbitrary and C denotes a positive constant whose value may vary from line to line.

Let $g(m) := \max_{s\in[0,m]} g(s)$ for $m\in(0,\infty)$, applying Hölder's inequality, we obtain

$$\begin{aligned}\|u\|_\infty &\le Ct^4 g(\|v\|_\infty) + C\Big(\int_t^1 s^{\gamma_1(q^*+1)}\mathrm{d}s\Big)^{\frac{1}{q^*+1}}\Big(\int_t^1 s^{N-1}(g(v(s)))^{\frac{q^*+1}{q^*}}\mathrm{d}s\Big)^{\frac{q^*}{q^*+1}}\\ &\le Ct^4\,g(\|v\|_\infty) + C\Big(\int_t^1 s^{\gamma_1(q^*+1)}\mathrm{d}s\Big)^{\frac{1}{q^*+1}}\end{aligned}$$

$$\times \Big(\int_t^1 s^{N-1} g(v(s))(g(v(s)))^{\frac{1}{q^*}} \mathrm{d}s \Big)^{\frac{q^*}{q^*+1}},$$

where $\gamma_1 = 3 - (N-1)\frac{q^*}{q^*+1}$. Or from Remark 3.4, there exists a positive constant M such that

$$\begin{aligned} g(s) &< M(1+s)^{q^*}, \quad \text{for all } s \geq 0, \\ f(s) &< M(1+s)^{p^*}, \quad \text{for all } s \geq 0. \end{aligned} \tag{3.19}$$

Then

$$\begin{aligned} \|u\|_\infty &\leq Ct^4 g(\|v\|_\infty) + M^{\frac{1}{q^*}} C \Big(\int_t^1 s^{\gamma_1(q^*+1)} \mathrm{d}s \Big)^{\frac{1}{q^*+1}} \\ &\quad \times \Big(\int_t^1 s^{N-1} g(v(s))\,(1+v(s)) \mathrm{d}s \Big)^{\frac{q^*}{q^*+1}} \\ &\leq Ct^4 g(\|v\|_\infty) + M^{\frac{1}{q^*}} C \Big(\int_t^1 s^{\gamma_1(q^*+1)} \mathrm{d}s \Big)^{\frac{1}{q^*+1}} \\ &\quad \times \Big(\int_B g(v) \mathrm{d}x + \int_B g(v)\, v(x) \mathrm{d}x \Big)^{\frac{q^*}{q^*+1}}. \end{aligned}$$

Using (3.16) and (3.17), we obtain

$$\|u\|_\infty \leq Ct^4\, g(\|v\|_\infty) + C \Big(\int_t^1 s^{\gamma_1(q^*+1)} \mathrm{d}s \Big)^{\frac{1}{q^*+1}}.$$

In a similar way, for v, we have

$$\|v\|_\infty \leq Ct^4 f(\|u\|_\infty) + C \Big(\int_t^1 s^{\gamma_2(p^*+1)} \mathrm{d}s \Big)^{\frac{1}{p^*+1}},$$

where $\gamma_2 = 3 - (N-1)\frac{p^*}{p^*+1}$.

In all the next inequalities C will always represent a positive constant, not necessarily the same in each occurrence. After some computations, we obtain

$$\|u\|_\infty \leq Ct^4 g(\|v\|_\infty) + Ct^{\frac{4+(4-N)q^*}{q^*+1}}, \tag{3.20}$$

$$\|v\|_\infty \leq Ct^4 f(\|u\|_\infty) + Ct^{\frac{4+(4-N)p^*}{p^*+1}}. \tag{3.21}$$

Note that if g and f are bounded then (3.18) comes immediately. However, if g is not bounded then, by Remark 3.4, there exists a positive K, see (3.19), such that $g(r) \leq Kr^{q^*}$, for $r \geq 1$. Therefore, we can write (3.20) as

$$\|u\|_\infty \leq Ct^4 (\|v\|_\infty)^{q^*} + Ct^{\frac{4+(4-N)q^*}{q^*+1}}. \tag{3.22}$$

Now, inserting (3.21) into (3.22), and using the inequality $(a+b)^m \leq C_m(a^m + b^m)$ for $a, b, m \geq 0$ where C_m is a positive constant depending of m, we obtain

$$\|u\|_\infty \leq Ct^{4(q^*+1)} \left(f(\|u\|_\infty)\right)^{q^*} + Ct^{\frac{[4+(4-N)p^*]q^*}{p^*+1}+4} + Ct^{\frac{4+(4-N)q^*}{q^*+1}}. \tag{3.23}$$

Easy computations show that

$$\frac{[4+(4-N)p^*]q^*}{p^*+1} + 4 = \frac{4+(4-N)q^*}{q^*+1} = -\frac{N}{p^*+1} = 4 - \frac{Nq^*}{q^*+1}.$$

Therefore,

$$\|u\|_\infty \leq Ct^{4(q^*+1)} (f(\|u\|_\infty))^{q^*} + Ct^{4-\frac{Nq^*}{q^*+1}}. \tag{3.24}$$

Let $r = 4q^*\big[1 + \frac{1}{(p^*+1)(q^*+1)}\big]$. Since $t \in [0,\,1]$, we write (3.24) as

$$\|u\|_\infty \le Ct^r\left(f(\|u\|_\infty)\right)^{q^*} + Ct^{4-\frac{Nq^*}{q^*+1}}.$$

Let $h(t) = t^r(f(\|u\|_\infty))^{q^*} + t^{4-\frac{Nq^*}{q^*+1}}$. The function h attains its infimum at

$$t_0 = C(f(\|u\|_\infty))^{-\frac{q^*(q^*+1)}{(r-4)(q^*+1)+Nq^*}}$$

and we have

$$\|u\|_\infty \le C\left(f(\|u\|_\infty)\right)^{\frac{-q^*(q^*+1)r}{(r-4)(q^*+1)+Nq^*}+q^*} + C(f(\|u\|_\infty))^{\frac{-q^*(q^*+1)}{(r-4)(q^*+1)+Nq^*}(4-\frac{Nq^*}{q^*+1})}.$$

Some computation show that

$$\frac{-q^*(q^*+1)r}{(r-4)(q^*+1)+Nq^*} + q^* = \frac{-q^*(q^*+1)}{(r-4)(q^*+1)+Nq^*}(4-\frac{Nq^*}{q^*+1}) = \frac{1}{p^*}.$$

Therefore

$$\|u\|_\infty \le C(f(\|u\|_\infty))^{1/p^*}.$$

Or Remark 3.4 implies that $f(x) = o(x^{p^*})$ for $x \to \infty$ then the last inequality becomes

$$\|u\|_\infty \le C(1 + o(\|u\|_\infty)),$$

which proves that $\|u\|_\infty$ is bounded and then implies, by (3.21), that $\|v\|_\infty$ is bounded. This completes Step 4. $\square$

Proof of Theorem 3.3. In the identity (2.14), we take $\alpha + \beta = N - 4$. Since $u = 0 = \frac{\partial u}{\partial \nu}$ and $v = 0 = \frac{\partial v}{\partial \nu}$, we have $(\Delta u, \Delta v) = \frac{\partial^2 u}{\partial \nu^2}\frac{\partial^2 v}{\partial \nu^2}$. If (u,v) is a non trivial solution of (1.1), since B is star-shaped domain about 0, then $x.\nu \ge 0$ on ∂B and the identity (2.14) gives a contradiction when

$$NF(s) - \alpha sf(s) \le 0 \quad \text{and} \quad NG(s) - \beta sg(s) \le 0.$$

The proof is complete. $\square$

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References

[1] C. O. Alves, D. C. de Morais Filho, M .A. S. Souto; On systems of elliptic equations involving subcritical or critical Sobolev exponents, *Nonl. Anal.*, **42**(5) (2000), 771–787.

[2] H. Aydi, K. Nashine, B. Samet, H. Yazidi; Coincidence and common fixed point results of Choudhury-Metiya type in partially ordered cone metric, *Nonl. Anal.: T.M.A.*, **74** (2011), 6814–6825.

[3] L. Boccardo, D. G. de Figueiredo; Some remarks on a system of quasilinear elliptic equations, *Nonl. Diff. Eq. Appl.*, **9** (2002), 309–323.

[4] H. Brezis, R. E. L. Turner; On a class of superlinear elliptic problems, *Comm. in P.D.E.*, **2** (1977), 601–614.

[5] Ph. Clement, D. G. de Figueiredo, E. Mitidieri; Positive solutions of semilinear elliptic systems, *Comm. Part. Diff. Eq.*, **17** (1992), 923–940.

[6] C. Cosner; Positive solutions for superlinear elliptic systems without variational structure, *Nonlinear. Anal. T.M.A.*, **8** (1984), 1427–1436.

[7] R. Demarque, O. Miyagaki; Radial solutions of inhomogeneous fourth order elliptic equations and weighted Sobolev embeddings, *Adv. Nonlinear Anal.*, **4** (2015), no. 2, 135–151.

[8] D. E. Edmunds, D. Fortunato, E. Janelli; Critical exponents, critical dimensions and the biharmonic operator, *Arch. Rat. Mech. Anal.*, **112** (1990), 269–289.

[9] D. G. de Figueiredo, P. L. Lions, R. D. Nussbaum; A Priori estimates and existence of positive solutions of semilinear elliptic equations, *J. Math. P. Appl.*, **61** (1982), 41–63.

[10] D. G. de Figueiredo, C. A. Magalhaes; On nonquadratic Hamiltonian elliptic systems, *Adv. Diff. Eq.*, **1**(5) (1996), 881–898.
[11] F. Gazzola, H. C. Grunau, G. Sweers; Polyharmonic boundary value problems, *Lecture notes in Mathematics*, **1991**, Springer, Berlin (2010).
[12] F. Gazzola, G. Sweers; On positivity for the biharmonic operator under Steklov boundary conditions, *Arch. Rat. Mech. Anal.*, **188**, (2008), 339–427.
[13] R. Hadiji, H. Yazidi; A nonlinear general Neumann problem involving two critical exponents, *Asymptotic Analysis*, **89** (2014), 123–150.
[14] D. D. Joseph, T. S. Lundgren; Quasilinear Dirichlet problems driven by positive sources, *Arch. Rat. Mech. Anal.*, **49** (1972), 241–269.
[15] R. D. Nussbaum; Positive solutions of nonlinear elliptic boundary value problems, *J. Math. Anal. Appl.*, **51** (1975), 461–482.
[16] P. Oswald; On a priori estimates for positive solutions of a semilinear biharmonic equation in a Ball, *Comm. Math. Univ. Carol.*, **26**(3) (1985) 565–577.
[17] L. A. Peletier, R. C. A. M. Van Der Vost; Existence and non-existence of positive solutions of non-linear elliptic systems and The Biharmonic equation, *Diff. Int. Eq.*, **5**(4) (1982), 441–467.
[18] P. Pucci, V. Radulescu; Remarks on a polyharmonic eigenvalue problem, *C. R. Math. Acad. Sci. Paris*, **348** (2010), no. 3-4, 161–164.
[19] P. Pucci, J. Serrin; A general variational identity, *Indiana Univ. Math., J.*, **35** (1986) 681–703.
[20] V. Radulescu, D. Repovs; Partial differential equations with variable exponents. Variational methods and qualitative analysis, *Monographs and Research Notes in Mathematics*, CRC Press, Boca Raton, FL, 2015.
[21] R. C. A. M. Van Der Vost; Variational identities and applications to differential systems, *Arch. Rat. Mech. Anal.*, **116** (1990), 375–398.

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