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EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS FOR SECOND-ORDER DISCRETE HAMILTONIAN SYSTEMS WITH PARTIALLY PERIODIC POTENTIALS

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ABSTRACT. In this article, we use critical point theory to obtain multiple periodic solutions for second-order discrete Hamiltonian systems, when the nonlinearity is partially periodic and its gradient is linearly and sublinearly bounded.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In this article, we are interested in the existence of multiple periodic solutions for the second-order discrete Hamiltonian system

$$\Delta^2 u(t-1) + \nabla F(t, u(t)) = 0, \quad \forall t \in \mathbb{Z}, \tag{1.1}$$

where $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $F(t,x) := \int_0^x \nabla F(t,s)ds$ is continuously differentiable in x for every $t \in \mathbb{Z}$, and satisfies the assumption

(A1) $F(t,x) = F(t+T,x)$ for all $t \in \mathbb{Z}[1,T]$ and $x \in \mathbb{R}^N$.

Here $\mathbb{R}$ denotes the set of real numbers, T denotes a positive integer, $\mathbb{Z}$ denotes the set of integers, and for $a, b \in \mathbb{Z}$ with $a \le b$, $\mathbb{Z}[a,b]$ denotes the set $\{a, a+1, \dots, b\}$.

It is worthwhile to explore discrete problems since they occur widely in numerous setting and form both in mathematics and in its applications to combinatorial analysis, quantum physics, chemical reactions, population dynamics, biology and other fields. Many authors have studied discrete problems and obtain important conclusions, see for example [1, 2, 9, 12, 13]. Especially, in 2003, Yu and Guo [4, 5, 6], using operator theory, established a variational structure and variational methods to study discrete Hamiltonian systems and obtained the solvability condition of periodic solution for discrete systems. Since then many authors have contributed to the study second-order discrete Hamiltonian systems. Using a powerful tool named the critical point theory, many interesting results have been obtained; see for example [3, 7, 10, 15, 16, 17, 18, 19]. Xue and Tang [16] constructed a variational setting unlike the one in [4] to study the second-order superquadratic discrete Hamiltonian system (1.1) and obtain the existence of periodic solutions. This result generalized the one in [3]. Xue and Tang [15] studied system (1.1) under condition (A1), the assumption

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(A2) There exist $M_1 > 0$, $M_2 > 0$, $\alpha \in [0,1)$ such that
$$|\nabla F(t,x)| \le M_1|x|^\alpha + M_2,$$
for all $(t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N$,

and the condition
$$|x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) \to +\infty \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^N,$$
for all $t \in \mathbb{Z}[1,T]$. Subsequently, Yan, Wu and Tang [17] extended the result in [15] and obtained multiple periodic solutions of system (1.1) under assumptions (A1), (A2),

(A3) $F(t,x)$ is T_i-periodic in x_i, $1 \le i \le r$, where the integer $r \in [0,N]$ and x_i is the ith component of $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$,

and the condition
$$|x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) \to +\infty \quad \text{or } -\infty \text{ as } |x| \to \infty,\ x \in \{0\} \times \mathbb{R}^{N-r}, \tag{1.2}$$
for all $t \in \mathbb{Z}[1,T]$.

We set
$$\begin{aligned} F(t,x) = \sin(\frac{2\pi t}{T})\Big(r+1+\sum_{j=1}^{r}\sin^2(x_j) + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2\Big)^{7/8} \\ - \frac{2^{3/4}\cdot 49}{8\lambda_1}\Big(r+1+\sum_{j=1}^{r}\sin^2(x_j) + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2\Big)^{3/4}, \end{aligned} \tag{1.3}$$
where the integer $r \in [0,N]$, $\lambda_1 = 2 - 2\cos(2\pi/T) > 0$ and $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$.

Let $y = r+1+\sum_{j=1}^{r}\sin^2(x_j) + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2$. A simple computation yields
$$\nabla F(t,x) = \frac{7}{8}\sin(\frac{2\pi t}{T})y^{-1/8}z - \frac{2^{3/4}\cdot 49}{8\lambda_1}\cdot\frac{3}{4}y^{-1/4}z$$
where $z = (\sin 2x_1, \dots, \sin 2x_r, x_{r+1}, \dots x_N)$. Then one has
$$\begin{aligned} |\nabla F(t,x)| &\le \frac{7}{8}y^{-1/8}|z| + \frac{2^{3/4}\cdot 49}{8\lambda_1}\cdot\frac{3}{4}y^{-1/4}|z| \\ &\le \frac{7}{8}y^{-1/8}(2y)^{1/2} + \frac{2^{3/4}\cdot 49}{8\lambda_1}\cdot\frac{3}{4}y^{-1/4}(2y)^{1/2} \\ &\le \frac{7\sqrt{2}}{8}(r+1+|x|^2)^{3/8} + \frac{2^{3/4}\cdot 49}{8\lambda_1}\cdot\frac{3}{4}\sqrt{2}(r+1+|x|^2)^{1/4} \\ &\le (\frac{7\sqrt{2}}{8}+\varepsilon)|x|^{3/4} + A(\varepsilon) \end{aligned}$$
where $A(\varepsilon) > 0$ is a function for $\varepsilon \in (0,1)$.

Set $\alpha = 3/4$, $M_1 = 7\sqrt{2}/8 + \varepsilon$, $M_2 = A(\varepsilon)$. Thus $F(t,x)$ satisfies (A1)–(A3) with $T_i = \pi,\ i = 1, \dots, r$. However, noting $|x|^2 = \sum_{j=r+1}^{N} x_j^2$, for $x \in \{0\} \times \mathbb{R}^{N-r}$, one obtains
$$F(t,x) = \sin(\frac{2\pi t}{T})\Big(r+1+\frac{1}{2}|x|^2\Big)^{7/8} - \frac{2^{3/4}\cdot 49}{8\lambda_1}\Big(r+1+\frac{1}{2}|x|^2\Big)^{3/4}$$

and

$$\lim_{|x|\to\infty} \frac{\sum_{t=1}^{T} F(t,x)}{|x|^{2\cdot\frac{3}{4}}} = -\frac{49}{8\lambda_1}T$$

which means that such a function F does not satisfies (1.2). Hence it is valuable to further improve conditions (1.2).

Hence, it is natural to ask if existence of multiple solutions still holds for $\alpha = 1$. With $\alpha = 1$, (A2) changes to the linearly bounded gradient condition:

- (A2') There exist constants $M_1 \in (0, \lambda_1)$ and $M_2 > 0$ such that
$$|\nabla F(t,x)| \le M_1|x| + M_2$$
for all $(t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N$, where $\lambda_k = 2 - 2\cos(\frac{2k\pi}{T})$ are the eigenvalues of the problem
$$-\Delta^2 u(t-1) = \lambda_k u(t), \quad k \in \mathbb{Z}[0, [T/2]],$$
and
$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_{[T/2]} \le 4.$$

Motivated by references [14, 15, 17], we study the multiple periodic solutions for the second-order discrete Hamiltonian system (1.1) under the following assumptions:

- (A4) $\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) > \frac{2\lambda_{[T/2]}+4\lambda_1}{\lambda_1^2} M_1^2 T$, $x \in \{0\} \times \mathbb{R}^{N-r}$.
- (A5) $\limsup_{|x|\to+\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) < -\frac{M_1^2 T}{\lambda_1}$, $x \in \{0\} \times \mathbb{R}^{N-r}$.
- (A6) $\liminf_{|x|\to+\infty} |x|^{-2} \sum_{t=1}^{T} F(t,x) > \frac{\lambda_{[T/2]}+2\lambda_1-M_1}{2(\lambda_1-M_1)^2} M_1^2 T$, $x \in \{0\} \times \mathbb{R}^{N-r}$.
- (A7) $\limsup_{|x|\to+\infty} |x|^{-2} \sum_{t=1}^{T} F(t,x) < -\frac{M_1^2 T}{\lambda_1 - M_1}$, $x \in \{0\} \times \mathbb{R}^{N-r}$.

Our main results read as follows.

Theorem 1.1. *Under assumptions* (A1)–(A4)*, problem* (1.1) *possesses at least* $r+1$ *geometrically distinct periodic solutions.*

Theorem 1.2. *Under asumptions* (A1)–(A3), (A5)*, problem* (1.1) *possesses at least* $r+1$ *geometrically distinct periodic solutions.*

Theorem 1.3. *Under asumptions* (A1), (A2'), (A3), (A6)*, problem* (1.1) *possesses at least* $r+1$ *geometrically distinct periodic solutions.*

Theorem 1.4. *Under assumptions (A1), (A2'),(A3), (A7), problem* (1.1) *possesses at least* $r+1$ *geometrically distinct periodic solutions.*

Remark 1.5. Theorems 1.1 and 1.2 extend [17, Theorems 2.1 and 2.2] respectively. Theorems 1.3 and 1.4 are extensions of [17, Theorems 2.1 and 2.2] corresponding to $\alpha = 1$. There are functions F satisfying our Theorems but not satisfying the existing results. Detailed examples can be seen later.

2. Proofs of main results

To apply critical point theory, we first define the Hilbert space

$$H_T = \{u : \mathbb{Z} \to \mathbb{R}^N \ |u(t+T) = u(t),\ t \in \mathbb{Z}\}$$

and equip it with the inner product

$$\langle u, v\rangle = \sum_{t=1}^{T} (u(t), v(t)), \quad \text{for } u, v \in H_T.$$

and the induced norm

$$\|u\| := \|u\|_2 = \Big(\sum_{t=1}^{T} |u(t)|^2\Big)^{1/2}, \quad \text{for } u \in H_T$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote the usual inner product and the usual norm in $\mathbb{R}^N$. Obviously, Hilbert space H_T is finite dimensional. We define a functional φ on H_T by

$$\varphi(u) = \frac{1}{2}\sum_{t=1}^{T} |\Delta u(t)|^2 - \sum_{t=1}^{T} F(t, u(t))$$

Then one has $\varphi \in C^1(H_T, \mathbb{R})$ and

$$\langle \varphi'(u), v\rangle = \sum_{t=1}^{T} (\Delta u(t), \Delta v(t)) - \sum_{t=1}^{T} (\nabla F(t, u(t)), v(t))$$

for all $u, v \in H_T$. From reference [16], we know that the problem of finding a T-periodic solution of (1.1) is equivalent to finding a critical point of the functional φ on H_T.

We can equip H_T with another norm $\|u\|_r$ for any positive number $r > 1$, where

$$\|u\|_r = \Big(\sum_{t=1}^{T} |u(t)|^r\Big)^{1/r}, \quad \text{for } u \in H_T.$$

From reference [16], one has

$$T^{-1}\|u\|_r \le \|u\| \le T\|u\|_r, \quad \forall u \in H_T. \tag{2.1}$$

For the reader' convenience, we give some useful lemmas presented in [16, 8, 11].

Lemma 2.1 ([16]). *As a subspace of H_T is defined as*

$$N_k := \{u \in H_T : -\Delta^2 u(t-1) = \lambda_k u(t)\},$$

where $\lambda_k = 2 - 2\cos(2k\pi/T)$, $k \in \mathbb{Z}[0, [T/2]]$. Then we claim that

- (i) $N_k \perp N_j$, $k \ne j$, $k, j \in \mathbb{Z}[0, [T/2]]$.
- (ii) $H_T = \oplus_{k=0}^{[T/2]} N_k$.

Lemma 2.2 ([16]). *Define $H_k := \oplus_{j=0}^{k} N_j$, $H_k^{\perp} := \oplus_{j=k+1}^{[T/2]} N_j$, $k \in \mathbb{Z}[0, [T/2]-1]$, then one has*

$$\sum_{t=1}^{T} |\Delta u(t)|^2 \le \lambda_k \|u\|^2, \quad \forall u \in H_k, \tag{2.2}$$

$$\sum_{t=1}^{T} |\Delta u(t)|^2 \ge \lambda_{k+1} \|u\|^2, \quad \forall u \in H_k^{\perp}. \tag{2.3}$$

For $u \in H_T$, put

$$\bar u = \frac{1}{T}\sum_{t=1}^{T} u(t), \quad \tilde u(t) = u(t) - \bar u, \quad \hat u(t) = P\bar u + Q\bar u + \tilde u(t),$$

where

$$P\bar u = \sum_{i=r+1}^{N} (\bar u, e_i)e_i, \quad Q\bar u = \sum_{i=1}^{r} \big((\bar u, e_i) - k_i T_i\big)e_i,$$

where $\{e_i | 1 \le i \le N\}$ *is the canonical basis of* $\mathbb{R}^N$ *and* k_i *is the unique integer such that*

$$0 \le (\bar{u}, e_i) - k_i T_i < T_i,$$

for $1 \le i \le r$. *Hence, there is a constant* $M > 0$ *satisfying*

$$|Q\bar{u}| < M. \tag{2.4}$$

By (A3), one obtains

$$\begin{gathered} F(t, u(t)) = F\Big(t, \hat{u}(t) + \sum_{i=1}^{r} k_i T_i e_i\Big) = F(t, \hat{u}(t)), \\ \nabla F(t, u(t)) = \nabla F\Big(t, \hat{u}(t) + \sum_{i=1}^{r} k_i T_i e_i\Big) = \nabla F(t, \hat{u}(t)). \end{gathered}$$

Thus, $\varphi(u) = \varphi(\hat{u})$ *and* $\varphi'(u) = \varphi'(\hat{u})$. *Set*

$$\begin{gathered} G = \big\{ \sum_{i=1}^{r} k_i T_i e_i : k_i \in Z, 1 \le i \le r \big\}, \\ Y = \operatorname{span}\{e_{r+1}, \dots, e_N\}, \quad V = \operatorname{span}\{e_1, \dots, e_r\}/G, \\ X = Y + W, \quad W = \big\{ u \in H_T | \bar{u} = \frac{1}{T} \sum_{t=1}^{T} u(t) = 0 \big\}. \end{gathered}$$

It is obvious that $H_T/G = X \times V$ *and* V *is isomorphic to the torus* T^r, *Define* $f : X \times V \to R$ *by* $f(\pi(u)) = \varphi(u)$, *where* $\pi : H_T \to H_T/G$ *is the canonical surjection.*

Lemma 2.3 (generalized saddle point theorem [8]). *Let* X *be a Banach space with a decomposition* $X = Y + W$, *where* Y *and* W *are two subspaces of* X *with* $\dim W < +\infty$. *Let* V *be a finite-dimensional, compact* C^2*-manifold without boundary. Let* $f : X \times V \longrightarrow R$ *be a* C^1*-function and satisfy the (PS) condition. Suppose that there exist constants* $\rho > 0$ *and* $\gamma < \beta$ *such that*

(a) $\inf_{x \in Y \times V} f(x) \ge \beta$,
(b) $\sup_{x \in S \times V} f(x) \le \gamma$,

where $S = \partial D, D = \{z \in W ||z| \le \rho\}$. *Then the functional* φ *has at least cuplength*$(V) + 1$ *critical points.*

Lemma 2.4 ([11, Theorem 4.12]). *Let* $\varphi \in C^1(H_T, R)$ *be a* G*-invariant functional satisfying the (PS) condition. If* φ *is bounded from below and if the dimension* r *of the space generated by* G *is finite, then* φ *has at least* $r + 1$ *critical orbits.*

Proof of Theorem 1.1. This proof relies on Lemma 2.3. Firstly, we prove that φ satisfies the (PS) condition. Suppose that $\{\pi(u_k)\} \subset H_T$ is a (PS) sequence, that is $f(\pi(u_k))$ is bounded and $f'(\pi(u_k)) \to 0$ as $k \to \infty$. Then $\varphi(u_k)$ is bounded, and $\varphi'(u_k) \to 0$ as $k \to \infty$. Then for sufficiently large k, one has

$$-\|u_k\| \le \langle \varphi'(u_k), u_k \rangle \le \|u_k\|.$$

By (A4), one chooses a constant $a_1 > 1/\lambda_1^2$ such that

$$\liminf_{|x| \to +\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t, x) > \big(2\lambda_{[T/2]} a_1 + 4\sqrt{a_1}\big) M_1^2 T, \quad x \in \{0\} \times \mathbb{R}^{N-r}. \tag{2.5}$$

By (A2), (2.1), (2.4), Hölder inequality and Young inequality, one has

$$
\begin{aligned}
&\big|\sum_{t=1}^{T}(F(t,\hat{u}(t))-F(t,P\bar{u}))\big| \\
&\le \sum_{t=1}^{T}\int_0^1 |(\nabla F(t,P\bar{u}+s(Q\bar{u}+\tilde{u}(t))),Q\bar{u}+\tilde{u}(t))|ds \\
&\le \sum_{t=1}^{T}\int_0^1 (M_1|P\bar{u}+s(Q\bar{u}+\tilde{u}(t))|^{\alpha}+M_2)\cdot|Q\bar{u}+\tilde{u}(t)|ds \\
&\le \sum_{t=1}^{T}2^{\alpha}M_1|P\bar{u}|^{\alpha}|Q\bar{u}|+\sum_{t=1}^{T}2^{\alpha}M_1|P\bar{u}|^{\alpha}|\tilde{u}(t)| \\
&\quad +\sum_{t=1}^{T}2^{\alpha}M_1(|Q\bar{u}|+|\tilde{u}(t)|)^{\alpha+1}+\sum_{t=1}^{T}M_2(|Q\bar{u}|+|\tilde{u}(t)|) \\
&\le 2M_1MT|P\bar{u}|^{\alpha}+\sum_{t=1}^{T}2M_1|P\bar{u}|^{\alpha}|\tilde{u}(t)| \\
&\quad +\sum_{t=1}^{T}2M_1(|Q\bar{u}|+|\tilde{u}(t)|)^{\alpha+1}+M_2MT+M_2\sqrt{T}\|\tilde{u}\| \\
&\le 2M_1MT|P\bar{u}|^{\alpha}+2\sqrt{a_1}M_1^2T|P\bar{u}|^{2\alpha}+\frac{1}{2\sqrt{a_1}}\|\tilde{u}\|^2 \\
&\quad +8M_1M^{\alpha+1}T+8M_1\sqrt{T^{\alpha+1}}\|\tilde{u}\|^{\alpha+1}+M_2MT+M_2\sqrt{T}\|\tilde{u}\|.
\end{aligned}
\tag{2.6}
$$

By using the same method, we obtain

$$
\begin{aligned}
&\big|\sum_{t=1}^{T}(\nabla F(t,\hat{u}(t)),\tilde{u}(t))\big| \\
&=\sum_{t=1}^{T}|(\nabla F(t,Q\bar{u}+\tilde{u}(t)+P\bar{u}),\tilde{u}(t))| \\
&\le 2a_1\lambda_1M_1^2T|P\bar{u}|^{2\alpha}+\frac{1}{2a_1\lambda_1}\|\tilde{u}\|^2+2M_1\sqrt{T^{\alpha+1}}\|\tilde{u}\|^{\alpha+1} \\
&\quad +(2M_1M^{\alpha}+M_2)\sqrt{T}\|\tilde{u}\|.
\end{aligned}
\tag{2.7}
$$

It follows from (2.7) and (2.3), respectively, that

$$
\begin{aligned}
\sum_{t=1}^{T}(\Delta u_k(t),\Delta\tilde{u}_k(t)) &= -\langle\varphi'(u_k),\tilde{u}_k\rangle+\sum_{t=1}^{T}(\nabla F(t,u_k(t)),\tilde{u}_k(t)) \\
&\le \|\tilde{u}_k\|+2a_1\lambda_1M_1^2T|P\bar{u}_k|^{2\alpha}+\frac{1}{2a_1\lambda_1}\|\tilde{u}_k\|^2 \\
&\quad +2M_1\sqrt{T^{\alpha+1}}\|\tilde{u}_k\|^{\alpha+1}+(2M_1M^{\alpha}+M_2)\sqrt{T}\|\tilde{u}_k\|
\end{aligned}
$$

and

$$
\sum_{t=1}^{T}(\Delta u_k(t),\Delta\tilde{u}_k(t))=\sum_{t=1}^{T}(\Delta\tilde{u}_k(t),\Delta\tilde{u}_k(t))\ge\lambda_1\|\tilde{u}_k\|^2. \tag{2.8}
$$

Thus from the two inequalities above, one has

$$\begin{aligned}
&2a_1\lambda_1 M_1^2 T|P\bar{u}_k|^{2\alpha}\\
&\geq (\lambda_1 - \frac{1}{2a_1\lambda_1})\|\tilde{u}_k\|^2 - 2M_1T^{\frac{\alpha+1}{2}}\|\tilde{u}_k\|^{\alpha+1} - (1 + 2M_1M^\alpha T^{1/2} + M_2T^{1/2})\|\tilde{u}_k\|\\
&\geq \frac{\lambda_1}{2}\|\tilde{u}_k\|^2 + C_1,
\end{aligned}$$

where

$$C_1 = \min_{s\in[0,+\infty)}\big\{(\frac{\lambda_1}{2} - \frac{1}{2a_1\lambda_1})s^2 - 2M_1T^{\frac{\alpha+1}{2}}s^{\alpha+1} - (1+2M^\alpha M_1T^{1/2} + M_2T^{1/2})s\big\} < 0.$$

Hence one gets

$$\|\tilde{u}_k\|^2 \leq 4a_1M_1^2T|P\bar{u}_k|^{2\alpha} - \frac{2C_1}{\lambda_1} \tag{2.9}$$

and so

$$\|\tilde{u}_k\| \leq 2M_1\sqrt{a_1T}|P\bar{u}_k|^\alpha + C_2, \tag{2.10}$$

holds for all large k, where $C_2 > 0$. From the boundedness of $\varphi(u_k)$, (2.6), (2.9) and (2.10) it follows that, for all large k,

$$\begin{aligned}
C_3 &\geq \varphi(u_k) = \varphi(\hat{u}_k)\\
&= -\frac{1}{2}\sum_{t=1}^{T}|\Delta\hat{u}_k(t)|^2 + \sum_{t=1}^{T}[F(t,\hat{u}_k(t)) - F(t,P\bar{u}_k)] + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&\geq -\frac{1}{2}\lambda_{[T/2]}\|\tilde{u}_k\|^2 - 2M_1MT|P\bar{u}_k|^\alpha - 2\sqrt{a_1}M_1^2T|P\bar{u}_k|^{2\alpha}\\
&\quad - \frac{1}{2\sqrt{a_1}}\|\tilde{u}_k\|^2 - 8M_1M^{\alpha+1}T - 8M_1\sqrt{T^{\alpha+1}}\|\tilde{u}_k\|^{\alpha+1}\\
&\quad - M_2MT - M_2\sqrt{T}\|\tilde{u}_k\| + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&\geq -\big(\frac{\lambda_{[T/2]}}{2} + \frac{1}{2\sqrt{a_1}}\big)\big(4a_1M_1^2T|P\bar{u}_k|^{2\alpha} - \frac{2C_1}{\lambda_1}\big) - 2M_1MT|P\bar{u}_k|^\alpha\\
&\quad - 2\sqrt{a_1}M_1^2T|P\bar{u}_k|^{2\alpha} - C_4 - 8M_1\sqrt{T^{\alpha+1}}\big(2M_1\sqrt{a_1T}|P\bar{u}_k|^\alpha\\
&\quad + C_2\big)^{\alpha+1} - M_2\sqrt{T}(2M_1\sqrt{a_1T}|P\bar{u}_k|^\alpha + C_2) + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&\geq -\big(2\lambda_{[T/2]}a_1 + 4\sqrt{a_1}\big)M_1^2T|P\bar{u}_k|^{2\alpha} - C_5 - C_6|P\bar{u}_k|^{\alpha(\alpha+1)}\\
&\quad - C_7|P\bar{u}_k|^\alpha + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&= |P\bar{u}_k|^{2\alpha}\Big[\frac{\sum_{t=1}^{T}F(t,P\bar{u}_k)}{|P\bar{u}_k|^{2\alpha}} - \big(2\lambda_{[T/2]}a_1 + 4\sqrt{a_1}\big)M_1^2T\\
&\quad - C_6|P\bar{u}_k|^{\alpha(\alpha-1)} - C_7|P\bar{u}_k|^{-\alpha}\Big] - C_5,
\end{aligned} \tag{2.11}$$

where $C_i > 0$, $i = 3,4,5,6,7$. With it and (2.5), $\{P\bar{u}_k\}$ is bounded. Then it follows from (2.10) that $\{\tilde{u}_k\}$ is bounded. Hence $\{\hat{u}_k\}$ is bounded. Since H_T is a finite dimensional space and $\pi(u_k) = \pi(\tilde{u}_k)$, $\{u_k\}$ contains a convergent subsequence in H_T. Thus φ satisfies (PS) condition.

Secondly, we need to verify the linking conditions (a) and (b) of Lemma 2.3. For $\pi(u) \in Y \times V$, $u = Q\bar{u} + P\bar{u}$, we have

$$f(\pi(u)) = \varphi(u) = \sum_{t=1}^{T} F(t, Q\bar{u} + P\bar{u}).$$

It follows from (A4) that

$$\sum_{t=1}^{T} F(t, Q\bar{u} + P\bar{u}) \to +\infty$$

uniformly for $\pi(Q\bar{u}) \in V$ as $|P\bar{u}| \to \infty$. Thus part (a) of Lemma 2.3 is verified.

By (A2), there is a constant $C_8 > 0$, such that for all $t \in \mathbb{Z}[1, T]$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} |F(t,x)| &\le |\int_0^1 (\nabla F(t, sx), x) ds| + |F(t, 0)| \\ &\le \int_0^1 |\nabla F(t, sx)||x| ds + C_8 \\ &\le \frac{M_1}{\alpha + 1}|x|^{\alpha+1} + M_2|x| + C_8. \end{aligned}$$

By (2.1), (2.4) and the above inequality, for any $\pi(u) \in W \times V$, $u = Q\bar{u} + \tilde{u}$, one obtains

$$\begin{aligned} f(\pi(u)) &= f(\pi(Q\bar{u} + \tilde{u})) = \varphi(Q\bar{u} + \tilde{u}) \\ &= -\frac{1}{2}\sum_{t=1}^{T} |\Delta\tilde{u}(t)|^2 + \sum_{t=1}^{T} F(t, Q\bar{u} + \tilde{u}(t)) \\ &\le -\frac{1}{2}\lambda_1\|\tilde{u}\|^2 + \sum_{t=1}^{T} \Big[\frac{M_1}{\alpha+1}|Q\bar{u} + \tilde{u}(t)|^{\alpha+1} + M_2|Q\bar{u} + \tilde{u}(t)| + C_8\Big] \\ &\le -\frac{1}{2}\lambda_1\|\tilde{u}\|^2 + C_9\|\tilde{u}\|^{\alpha+1} + C_{10}\|\tilde{u}\| + C_{11}, \end{aligned}$$

where $C_i > 0$, $i = 9, 10, 11$. Noting $0 \le \alpha < 1$, we choose $\|\tilde{u}\|$ so large enough that

$$\sup_{\pi(u) \in W \times V} f(\pi(u)) \le \gamma < \beta.$$

Then part (b) of Lemma 2.3 holds and f has at least $r + 1$ critical points. Thus the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. We use Lemma 2.4 in this proof. By (A5), one takes a constant $a_2 > \frac{1}{\lambda_1}$ satisfying

$$\limsup_{|x| \to +\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t, x) < -2a_2 M_1^2 T, \quad x \in \{0\} \times \mathbb{R}^{N-r}. \tag{2.12}$$

In a way similar to (2.6), one obtains

$$\begin{aligned}
&\Big|\sum_{t=1}^{T}(F(t,\hat{u}(t))-F(t,P\bar{u}))\Big| \\
&\le 2M_1MT|P\bar{u}|^{\alpha}+2a_2M_1^2T|P\bar{u}|^{2\alpha}+\frac{1}{2a_2}\|\tilde{u}\|^2 \\
&\quad +8M_1\sqrt{T^{\alpha+1}}\|\tilde{u}\|^{\alpha+1}+M_2\sqrt{T}\|\tilde{u}\|+C_{12},
\end{aligned} \tag{2.13}$$

where $C_{12}>0$. We se $\psi(u)=-\varphi(u),\quad u\in H_T$. It is easy to see that $\psi(u)$ is a G-invariant functional. Hence, from (2.13), for all $u\in H_T$, we obtain

$$\begin{aligned}
\psi(u)&=\psi(\hat{u}) \\
&=\frac12\sum_{t=1}^{T}|\Delta\tilde{u}(t)|^2-\sum_{t=1}^{T}[F(t,\hat{u})-F(t,P\bar{u})]-\sum_{t=1}^{T}F(t,P\bar{u}) \\
&\ge\frac{\lambda_1}{2}\|\tilde{u}\|^2-(2M_1MT|P\bar{u}|^{\alpha}+2a_2M_1^2T|P\bar{u}|^{2\alpha}+\frac{1}{2a_2}\|\tilde{u}\|^2 \\
&\quad +8M_1\sqrt{T^{\alpha+1}}\|\tilde{u}\|^{\alpha+1}+M_2\sqrt{T}\|\tilde{u}\|+C_{12})-\sum_{t=1}^{T}F(t,P\bar{u}) \\
&\ge\Big(\frac{\lambda_1}{2}-\frac{1}{2a_2}\Big)\|\tilde{u}\|^2-8M_1\sqrt{T^{\alpha+1}}\|\tilde{u}\|^{\alpha+1}-M_2\sqrt{T}\|\tilde{u}\|-C_{12} \\
&\quad -|P\bar{u}|^{2\alpha}\Big(\frac{\sum_{t=1}^{T}F(t,P\bar{u})}{|P\bar{u}|^{2\alpha}}+2a_2M_1^2T+2M_1MT|P\bar{u}|^{-\alpha}\Big).
\end{aligned} \tag{2.14}$$

Thus (2.11) and (2.14) imply that ψ is bounded from below.

Moreover, we draw a conclusion that the functional ψ satisfies the (PS) condition. In fact, the boundedness of $\psi(u_k)$, (A5) and (2.14) imply that $|P\bar{u}_k|$ and $\|\tilde{u}_k\|$ are bounded and then $\hat{u}_k=P\bar{u}_k+Q\bar{u}_k+\tilde{u}_k$ is bounded by (2.4). In finite-dimensional space H_T, $\{\hat{u}_k\}$ contains a convergent subsequence. Hence, $\pi(u_k)$ contains a convergent subsequence by $\pi(u_k)=\pi(\hat{u}_k)$. Then the proof is complete by using Lemma 2.4. □

Proof of Theorem 1.3. By (A6), one chooses a positive constant $a_3>\frac{1}{\lambda_1-M_1}$ such that

$$\liminf_{|x|\to+\infty}|x|^{-2}\sum_{t=1}^{T}F(t,x)>\Big[\Big(\frac{\lambda_{[T/2]}+M_1}{2}+\frac{1}{2a_3}\Big)\frac{a_3}{\lambda_1-M_1}+\frac{a_3}{2}\Big]M_1^2T, \tag{2.15}$$

for all $x\in\{0\}\times\mathbb{R}^{N-r}$. In a way similar to (2.6), one has

$$\begin{aligned}
&\Big|\sum_{t=1}^{T}(F(t,\hat{u}(t))-F(t,P\bar{u}))\Big| \\
&\le\sum_{t=1}^{T}\int_0^1|(\nabla F(t,P\bar{u}+s(Q\bar{u}+\tilde{u}(t))),Q\bar{u}+\tilde{u}(t))|ds \\
&\le\sum_{t=1}^{T}(M_1|P\bar{u}|+\frac12M_1|Q\bar{u}+\tilde{u}(t)|+M_2)\cdot|Q\bar{u}+\tilde{u}(t)|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{t=1}^{T} M_1|P\bar{u}||Q\bar{u}| + \sum_{t=1}^{T} M_1|P\bar{u}||\tilde{u}(t)| \\
&\quad + \frac{1}{2}\sum_{t=1}^{T} M_1(|Q\bar{u}|^2 + |\tilde{u}(t)|^2 + 2|Q\bar{u}||\tilde{u}(t)|) + \sum_{t=1}^{T} M_2(|Q\bar{u}| + |\tilde{u}(t)|) \\
&\le M_1MT|P\bar{u}| + \frac{a_3M_1^2T}{2}|P\bar{u}|^2 + \frac{1}{2a_3}\|\tilde{u}\|^2 \\
&\quad + \frac{1}{2}M_1\|\tilde{u}\|^2 + (M_1M + M_2)\sqrt{T}\|\tilde{u}\| + C_{13}
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
&\big|\sum_{t=1}^{T}(\nabla F(t,\hat{u}(t)),\tilde{u}(t))\big| \\
&= \sum_{t=1}^{T}|(\nabla F(t, Q\bar{u} + \tilde{u}(t) + P\bar{u}), \tilde{u}(t))| \\
&\le \frac{a_3}{2}M_1^2T|P\bar{u}|^2 + \frac{1}{2a_3}\|\tilde{u}\|^2 + M_1\|\tilde{u}\|^2 + (MM_1 + M_2)\sqrt{T}\|\tilde{u}\|,
\end{aligned} \tag{2.17}
$$

where $C_{13} > 0$. By (2.17), one obtains

$$
\begin{aligned}
&\sum_{t=1}^{T}(\Delta u_k(t), \Delta\tilde{u}_k(t)) \\
&= -\langle\varphi'(u_k), \tilde{u}_k\rangle + \sum_{t=1}^{T}(\nabla F(t, u_k(t)), \tilde{u}_k(t)) \\
&\le \|\tilde{u}_k\| + \frac{a_3}{2}M_1^2T|P\bar{u}|^2 + \frac{1}{2a_3}\|\tilde{u}\|^2 + M_1\|\tilde{u}_k\|^2 + (MM_1 + M_2)\sqrt{T}\|\tilde{u}_k\|.
\end{aligned}
$$

With this and (2.8),

$$
\begin{aligned}
\frac{a_3}{2}M_1^2T|P\bar{u}_k|^2 &\ge (\lambda_1 - M_1 - \frac{1}{2a_3})\|\tilde{u}_k\|^2 - (1 + MM_1T + \sqrt{T}M_2)\|\tilde{u}_k\| \\
&\ge \frac{\lambda_1 - M_1}{2}\|\tilde{u}_k\|^2 + C_{14}
\end{aligned}
$$

is implied, where

$$
C_{14} = \min_{s\in[0,+\infty)}\Big\{(\frac{\lambda_1 - M_1}{2} - \frac{1}{2a_3})s^2 - (1 + MM_1T + \sqrt{T}M_2)s\Big\} < 0.
$$

Hence one gets

$$
\|\tilde{u}_k\|^2 \le \frac{a_3M_1^2T}{\lambda_1 - M_1}|P\bar{u}_k|^2 - \frac{2C_{14}}{\lambda_1 - M_1}. \tag{2.18}
$$

So

$$
\|\tilde{u}_k\| \le \frac{M_1\sqrt{a_3T}}{\sqrt{\lambda_1 - M_1}}|P\bar{u}_k| + C_{15} \tag{2.19}
$$

holds for all large k, where $C_{15} > 0$. It follows from the boundedness of $\varphi(u_k)$, (2.16), (2.18) and (2.19) that

$$
C_3 \ge \varphi(u_k) = \varphi(\hat{u}_k)
$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{t=1}^{T}|\Delta\hat{u}_k(t)|^2 + \sum_{t=1}^{T}[F(t,\hat{u}_k(t)) - F(t,P\bar{u}_k)] + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&\geq -\frac{\lambda_{[T/2]}}{2}\|\tilde{u}_k\|^2 - M_1MT|P\bar{u}_k| - \frac{a_3M_1^2T}{2}|P\bar{u}_k|^2 - \frac{1}{2a_3}\|\tilde{u}_k\|^2\\
&\quad - C_{13} - \frac{1}{2}M_1\|\tilde{u}_k\|^2 - M_1M\sqrt{T}\|\tilde{u}_k\| - M_2\sqrt{T}\|\tilde{u}_k\| + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&\geq -\Big(\frac{\lambda_{[T/2]}}{2} + \frac{M_1}{2} + \frac{1}{2a_3}\Big)\Big(\frac{a_3M_1^2T}{\lambda_1 - M_1}|P\bar{u}_k|^2 - \frac{2C_{14}}{\lambda_1 - M_1}\Big) - M_1MT|P\bar{u}_k|\\
&\quad - \frac{a_3M_1^2T}{2}|P\bar{u}_k|^2 - C_{13} - (M_1M + M_2)\sqrt{T}\Big(\frac{M_1\sqrt{a_3T}}{\sqrt{\lambda_1 - M_1}}|P\bar{u}_k| + C_{15}\Big)\\
&\quad + \sum_{t=1}^{T}F(t,P\bar{u}_k)\\
&= |P\bar{u}_k|^2\Big\{\frac{\sum_{t=1}^{T}F(t,P\bar{u}_k)}{|P\bar{u}_k|^2} - \Big[\Big(\frac{\lambda_{[T/2]} + M_1}{2} + \frac{1}{2a_3}\Big)\frac{a_3}{\lambda_1 - M_1} + \frac{a_3}{2}\Big]M_1^2T\\
&\quad - C_{16}|P\bar{u}_k|^{-1}\Big\} - C_{17},
\end{aligned}
$$

for all large k, where $C_{16}, C_{17} > 0$. This inequality, with (2.15), implies that $\{P\bar{u}_k\}$ is bounded. Then from (2.19) it follows that $\{\tilde{u}_k\}$ is bounded. Thus, in a way similar to the proof of Theorem 1.1, one obtains φ satisfies (PS) condition.

Subsequently, we verify the linking conditions (a) and (b) of Lemma 2.3. Part (a) is easy to verify by (A6) and the same method in the proof in Theorem 1.1. By (A2'), there exists a constant $C_{18} > 0$, such that for all $t \in \mathbb{Z}[1,T]$ and $x \in \mathbb{R}^N$,

$$
|F(t,x)| \leq \frac{M_1}{2}|x|^2 + M_2|x| + C_{18}. \tag{2.20}
$$

By (2.4) and (2.20), for any $\pi(u) \in W \times V$, $u = Q\bar{u} + \tilde{u}$, one obtains

$$
\begin{aligned}
f(\pi(u)) &= f(\pi(Q\bar{u} + \tilde{u})) = \varphi(Q\bar{u} + \tilde{u})\\
&= -\frac{1}{2}\sum_{t=1}^{T}|\Delta\tilde{u}(t)|^2 + \sum_{t=1}^{T}F(t,Q\bar{u} + \tilde{u}(t))\\
&\leq -\frac{1}{2}\lambda_1\|\tilde{u}\|^2 + \sum_{t=1}^{T}(\frac{M_1}{2}|Q\bar{u} + \tilde{u}(t)|^2 + M_2|Q\bar{u} + \tilde{u}(t)| + C_7)\\
&\leq -\frac{1}{2}(\lambda_1 - M_1)\|\tilde{u}\|^2 + (MM_1 + M_2)\sqrt{T}\|\tilde{u}\| + C_{18}.
\end{aligned}
$$

With it and the fact $M_1 \in (0,\lambda_1)$,

$$
\sup_{\pi(u)\in W\times V} f(\pi(u)) \leq \gamma < \beta
$$

is implied for all large enough $\|\tilde{u}\|$. Thus part (b) of Lemma 2.3 holds and then the proof is complete. □

Proof of Theorem 1.4. Via (A7), one takes $a_4 > \frac{2}{\lambda_1 - M_1} > 0$ such that

$$
\limsup_{|x|\to+\infty}|x|^{-2}\sum_{t=1}^{T}F(t,x) < -\frac{a_4}{2}M_1^2T, \quad x \in \{0\}\times\mathbb{R}^{N-r}. \tag{2.21}
$$

In a way similar to (2.16), one obtains

$$\begin{aligned} \Big|\sum_{t=1}^{T}(F(t,\hat{u}(t))-F(t,P\bar{u}))\Big| &\le M_1MT|P\bar{u}|+\frac{a_4}{2}M_1^2T|P\bar{u}|^2+\frac{1}{2a_4}\|\tilde{u}\|^2 \\ &\quad +\frac{M_1}{2}\|\tilde{u}\|^2+(MM_1+M_2)\sqrt{T}\|\tilde{u}\|+C_{19}, \end{aligned} \tag{2.22}$$

where $C_{19}>0$. It is easy to know $\psi(u)$ is a G-invariant functional. Hence, from (2.22), we obtain

$$\begin{aligned} \psi(u) &= \psi(\hat{u}) \\ &= \frac{1}{2}\sum_{t=1}^{T}|\Delta\tilde{u}(t)|^2-\sum_{t=1}^{T}[F(t,\hat{u})-F(t,P\bar{u})]-\sum_{t=1}^{T}F(t,P\bar{u}) \\ &\ge \Big(\frac{\lambda_1}{2}-\frac{1}{2a_4}-\frac{M_1}{2}\Big)\|\tilde{u}\|^2-(MM_1+M_2)\sqrt{T}\|\tilde{u}\|-C_{19} \\ &\quad -|P\bar{u}|^2\Big(\frac{\sum_{t=1}^{T}F(t,P\bar{u})}{|P\bar{u}|^2}+\frac{a_4}{2}M_1^2T+M_1MT|P\bar{u}|^{-1}\Big), \end{aligned}$$

for all $u\in H_T$. With it and the fact $M_1\in(0,\lambda_1)$, one deduces that ψ is bounded from below.

We prove that the functional ψ satisfies the (PS) condition by the same method in the proof of Theorem 1.2. Then the proof is complete by using by Lemma 2.4. □

3. Examples

In this section, some examples illustrate our results.

Example 3.1. Let

$$\begin{aligned} F(t,x) &= \sin(\frac{2\pi t}{T})\Big(r+1+\sum_{j=1}^{r}\sin^2(x_j)+\frac{1}{2}\sum_{j=r+1}^{N}x_j^2\Big)^{7/8} \\ &\quad +\frac{2^{3/4}\cdot 49(2+\lambda_1)}{2\lambda_1^2}\Big(r+1+\sum_{j=1}^{r}\sin^2(x_j)+\frac{1}{2}\sum_{j=r+1}^{N}x_j^2\Big)^{3/4}, \end{aligned}$$

where $\lambda_1=2-2\cos(2\pi/T)>0$ and $x=(x_1,x_2,\dots,x_N)\in\mathbb{R}^N$. Then one has

$$|\nabla F(t,x)|\le\big(\frac{7\sqrt{2}}{8}+\varepsilon\big)|x|^{3/4}+A_1(\varepsilon),$$

where $A_1(\varepsilon)>0$ is a function in $\varepsilon\in(0,1)$.

Set $\alpha=3/4$, $M_1=7\sqrt{2}/8+\varepsilon$, $M_2=A_1(\varepsilon)$. Thus $F(t,x)$ satisfies (A1)–(A3) with $T_i=\pi$, $i=1,\dots,r$. Also, for $x\in\{0\}\times\mathbb{R}^{N-r}$, one obtains

$$|x|^2=\sum_{j=r+1}^{N}x_j^2$$

and

$$\begin{aligned} \liminf_{|x|\to\infty}\frac{\sum_{t=1}^{T}F(t,x)}{|x|^{2\cdot\frac{3}{4}}} &= \frac{49(2+\lambda_1)}{2\lambda_1^2}T>\frac{4(4+2\lambda_1)}{2\lambda_1^2}(\frac{7\sqrt{2}}{8}+\varepsilon)^2T \\ &> \frac{2\lambda_{[T/2]}+4\lambda_1}{\lambda_1^2}M_1^2T, \end{aligned}$$

for all $\varepsilon \in (0,1)$. This implies that such a function F satisfies (A4). By Theorem 1.1, problem (1.1) has at least $r+1$ geometrically distinct periodic solutions.

Example 3.2. Let $F(t,x)$ be defined as in (1.3). $\alpha = 3/4$, $M_1 = 7\sqrt{2}/8 + \varepsilon$, $M_2 = A(\varepsilon)$. Thus $F(t,x)$ satisfies (A1)–(A3) with $T_i = \pi$, $i = 1,\dots,r$. Also, for $x \in \{0\} \times \mathbb{R}^{N-r}$, one obtains

$$\limsup_{|x|\to\infty} \frac{\sum_{t=1}^{T} F(t,x)}{|x|^{2\cdot\frac{3}{4}}} = -\frac{49}{8\lambda_1}T < -\frac{(\frac{7\sqrt{2}}{8}+\varepsilon)^2}{\lambda_1}T = -\frac{M_1^2}{\lambda_1}T,$$

for all $\varepsilon \in (0,1)$. This implies that such a function F satisfies (A5). By Theorem 1.2, problem (1.1) has at least $r+1$ geometrically distinct periodic solutions.

Example 3.3. Let

$$F(t,x) = \sin(\frac{2\pi t}{T}) \sum_{j=1}^{r} \sin(x_j) + a \sum_{j=r+1}^{N} x_j^2,$$

where $a = \frac{\lambda_1+1-\sqrt{(\lambda_1+1)^2-\frac{\lambda_1^2}{2}}}{4}$. Then one has

$$|\nabla F(t,x)| \le 2a|x| + \sqrt{r}.$$

Set $M_1 = 2a$, $M_2 = \sqrt{r}$. A computation yields $M_1 < \lambda_1$ and $F(t,x)$ satisfies (A1), (A2') and (A3) with $T_i = \pi$, $i = 1,\dots,r$. On the other hand, for $x \in \{0\} \times \mathbb{R}^{N-r}$, one has

$$\liminf_{|x|\to\infty} \frac{\sum_{t=1}^{T} F(t,x)}{|x|^2} = aT > \frac{4+2\lambda_1-M_1}{2(\lambda_1-M_1)^2} M_1^2 T \ge \frac{\lambda_{[T/2]}+2\lambda_1-M_1}{2(\lambda_1-M_1)^2} M_1^2 T.$$

This implies that F satisfies (A6). By Theorem 1.3, problem (1.1) has at least $r+1$ geometrically distinct periodic solutions.

Example 3.4. Let

$$F(t,x) = \sin(\frac{2\pi t}{T}) \sum_{j=1}^{r} \sin(x_j) - \frac{\lambda_1}{8} \sum_{j=r+1}^{N} x_j^2.$$

Then one has

$$|\nabla F(t,x)| \le \frac{\lambda_1}{4}|x| + \sqrt{r}.$$

Set $M_1 = \lambda_1/4$, $M_2 = \sqrt{r}$. Obviously, $F(t,x)$ satisfies (A1),(A2') and (A3) with $T_i = \pi$, $i = 1,\dots,r$. On the other hand, for $x \in \{0\} \times \mathbb{R}^{N-r}$, one has

$$\limsup_{|x|\to\infty} \frac{\sum_{t=1}^{T} F(t,x)}{|x|^2} = -\frac{\lambda_1}{8}T < -\frac{\lambda_1}{12}T = -\frac{M_1^2}{\lambda_1 - M_1}T.$$

Then F satisfies (A7). By Theorem 1.4, problem (1.1) has at least $r+1$ geometrically distinct periodic solutions.

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