Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 300, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC PROPERTIES OF THE VON FOERSTER-LASOTA EQUATION AND INDICES OF ORLICZ SPACES

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Abstract. This article concerns the asymptotic behaviour of the dynamical systems induced by the von Foerster-Lasota equation. We study chaoticity of the system in the sense of Devaney and its strong stability in Orlicz spaces generated by any φ-function. We apply Matuszewska-Orlicz indices to a description of asymptotic properties considered semigroup.

1. Introduction

The aim of this article is to show chaos and stability criteria for a dynamical system induced by the von Foerster-Lasota equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1,\ \gamma \in \mathbb{R} \tag{1.1}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leqslant x \leqslant 1 \tag{1.2}$$

where v belongs to some function space. In 1926 McKendrick [14] proposed the first age-dependent model of the dynamics of a population where the state of a population in time t is described by a function $u(\cdot,t)$. From this model follows the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \lambda(x)u,$$

which is called the McKendrick equation or more often as the von Foerster equation. McKendrick's model was generalized on many ways. Its generalized form appeared in paper by Lasota and Ważewska [19], as the part of mathematical description of a particular population, such as population of red blood cells. Because of biological application, the equation is still the matter of interest of many mathematicians. The Lasota equation in its basic form

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = f(x,u)$$

is the element of so-called precursor cells model [9] and is studied in different function spaces, from Lasota [8] onwards [1, 2, 17, 5, 18] (with references therein). Equation (1.1) with initial condition (1.2) generates a semigroup $(T_t)_{t\geqslant 0}$ acting on some function space V. This paper is devoted to study asymptotic properties of

2010 *Mathematics Subject Classification.* 35B10, 35B35, 35B40.
Key words and phrases. von Foerster-Lasota equation; stability; chaos; Orlicz space; Matuszewska-Orlicz indices.

Submitted October 25, 2016. Published November 25, 2016.

the semigroup $(T_t)_{t\geqslant 0}$ in the Orlicz space generating by any φ-function. In Section 2 we introduce some definitions, notation and basic properties of the Orlicz spaces appearing subsequently. We next study, among others, chaotic behaviour $(T_t)_{t\geqslant 0}$ in the sense of Devaney [6], i.e criteria when the set of all periodic points of $(T_t)_{t\geqslant 0}$ is dense in V and $(T_t)_{t\geqslant 0}$ is transitive. We also consider strong stability $(T_t)_{t\geqslant 0}$ in a space V, which is equivalent to the condition $\lim_{t\to\infty} T_t v = 0$ in V. It turns out that asymptotic behaviour of the solution of the von Foerster-Lasota equation depends on the coefficient γ values. These decisive values are strictly connected with certain numerical description of considered Orlicz space, i.e. so called indices of Orlicz space (more in Section 3). The novel contribution of our research is latitude in the choice of the φ-function generated the Orlicz space. Furthermore we apply Matuszewska-Orlicz indices to the estimation of the decisive value of the coefficient γ.

2. Preliminaries

In this section we list the principal definitions, notation and symbols (cf. [11, 16]). Let X be a real vector space. A functional $\rho : X \to [0,\infty]$ is called s-convex modular, if it satisfies the following conditions:

- $\rho(x) = 0$ iff $x = 0$,
- $\rho(-x) = \rho(x)$,
- $\rho(\alpha x + \beta y) \leqslant \alpha^s \rho(x) + \beta^s \rho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$.

1-convex modulars are called convex. The modular space generated by ρ is the subspace

$$X_\rho = \big\{x \in X : \lim_{\beta\to 0} \rho(\beta x) = 0\big\}.$$

A sequence (x_k) of elements of X_ρ is called modular convergent to $x \in X_\rho$ if there exists $\lambda > 0$ such that $\rho(\lambda(x_k - x)) \to 0$, as $k \to \infty$. Let (Ω, Σ, μ) be measure space, where Ω is a nonempty set, Σ is a σ-algebra of subset of Ω and μ is a nonnegative, complete measure not vanishing identically. A real function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$, is called φ-function if it is nondecreasing, continuous and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, $\varphi(u) \to \infty$ as $u \to \infty$. We will say that a φ-function satisfies $\triangle_2$-condition if for some $\omega > 0$ we have $\varphi(2u) \leqslant \omega\varphi(u)$ for all $0 \leqslant u < \infty$. Let X be the set of all real-valued, Σ-measurable and finite μ-almost everywhere functions on Ω, with equality μ-almost everywhere. Then for every $x \in X$

$$\rho(x) = \int_\Omega \varphi(|x(t)|)d\mu$$

is a modular in X. Moreover, if φ is a s-convex function, then ρ is a s-convex modular in X. The respective modular space X_ρ will be called an Orlicz space and denoted by $L^\varphi(\Omega, \Sigma, \mu)$ (or briefly L^φ):

$$L^\varphi = \big\{x \in X : \int_\Omega \varphi(\beta|x(t)|)d\mu \to 0 \text{ as } \beta \to 0^+\big\}.$$

Moreover, the set

$$L_0^\varphi = \big\{x \in X : \int_\Omega \varphi(|x(t)|)d\mu < \infty\big\}$$

will be called the Orlicz class. L_0^φ is a convex subset of L^φ. In a modular space

$$|x|^F = \inf\big\{s > 0 : \int_\Omega \varphi\big(|\frac{x(t)}{s}|\big)d\mu \leqslant s\big\}$$

is a F-norm. If φ is convex then the functional

$$\|x\|^L = \inf\big\{s > 0 : \int_\Omega \varphi\big(|\frac{x(t)}{s}|\big)d\mu \leqslant 1\big\}$$

is a norm in L^φ, called the Luxemburg norm. For a φ-function φ we can define (see [11])

$$\begin{gathered} M_a(t,\varphi) = \sup_{u>0} \frac{\varphi(tu)}{\varphi(u)}, \\ M_0(t,\varphi) = \limsup_{u\to 0^+} \frac{\varphi(tu)}{\varphi(u)}, \\ M_\infty(t,\varphi) = \limsup_{u\to\infty} \frac{\varphi(tu)}{\varphi(u)}. \end{gathered}$$

All the above functions are non-decreasing, submultiplicative and are equal to 1 at the point 1. Matuszewska and Orlicz [12, 13] introduced certain numerical descriptions for a φ-function i.e.

$$\begin{gathered} p^i = \lim_{t\to 0^+} \frac{\ln M_i(t,\varphi)}{\ln t}, \\ q^i = \lim_{t\to\infty} \frac{\ln M_i(t,\varphi)}{\ln t} \end{gathered}$$

where $i = a, 0, \infty$. These numbers are called indices of Orlicz spaces or Matuszewska-Orlicz indices (lower and upper index, respectively). We quote modified definition of the indices after Montgomery-Smith [15].

Definition 2.1. For a φ-function φ, we define the lower Matuszewska-Orlicz index to be

$$\begin{aligned} \underline{p} = \sup\big\{&p : \text{ for some } C > 0 \text{ we have } \varphi(at) \geqslant Ca^p\varphi(t) \\ &\text{for } 0 \leqslant t < \infty \text{ and } a \geqslant 1\big\}. \end{aligned}$$

We define the upper Matuszewska-Orlicz index to be

$$\begin{aligned} \overline{q} = \inf\big\{&q : \text{for some } C < \infty \text{ we have } \varphi(at) \leqslant Ca^q\varphi(t) \\ &\text{for } 0 \leqslant t < \infty \text{ and } a \geqslant 1\big\}. \end{aligned}$$

The above definition of the indices is consistent with the notation used in [7] and [10].

It is obvious that we have the inequalities $0 \leqslant \underline{p} \leqslant \overline{q} \leqslant \infty$ for the Matuszewska-Orlicz indices. Moreover, a φ-function satisfies the $\triangle_2$-condition if and only if $\overline{q} < \infty$.

3. Chaotic and stable solutions of the von Foerster-Lasota equation

We consider the partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1,\ \gamma \in \mathbb{R} \tag{3.1}$$

with the initial condition

$$u(0,x) = v(x), \quad 0 \leqslant x \leqslant 1\,, \tag{3.2}$$

where v belongs to some normed vector space V of functions defined on $[0,1]$. The function T_t is given by the formula (see [4])

$$(T_t v)(x) = u(t,x) = e^{\gamma t} v(xe^{-t}), \quad x \in [0,1] \tag{3.3}$$

where u is the unique solution of (1.1) and (1.2). In paper [5] we studied the asymptotic properties of the von Foerster-Lasota equation in the Orlicz space $L^{\varphi}(0,1)$ with homogeneous φ-function $\varphi(x) = x^p$, $0 < p < 1$, for which both lower and upper Matuszewska-Orlicz indices equal p. In such space the equation displays chaotic behaviour in the sense of Devaney for $\gamma > -\frac{1}{p}$ and is strongly stable for $\gamma \leqslant -\frac{1}{p}$. In this section we generalize these results. We consider an Orlicz space $L^{\varphi}(0,1)$ and in the sequel we assume that a φ-function satisfies the $\triangle_2$-condition. This clearly forces the followings: the separability of the L^{φ} space, the continuity of the modular ρ, the condition $\overline{q} < \infty$ and $L^{\varphi}_0 = L^{\varphi}$.

Theorem 3.1. *If $\gamma > -1/\overline{q}$, then for any $t_0 > 0$ there exists a periodic point $v_0 \in L^{\varphi}$ of the dynamical system $(T_t)_{t \geqslant 0}$.*

Proof. Let w be an arbitrary function belonging to $L^{\varphi}(e^{-t_0}, 1)$. We can define a function v_0 on the interval $(0,1] = \cup_{n=0}^{\infty} (e^{-(n+1)t_0}, e^{-nt_0}]$ by squeezing the graph of the function w into the intervals $(e^{-(n+1)t_0}, e^{-nt_0}]$. We put

$$v_0(x) = \begin{cases} e^{-n\gamma t_0} w(xe^{nt_0}) & \text{for } x \in (e^{-(n+1)t_0}, e^{-nt_0}] \\ w(x) & \text{for } x \in (e^{-nt_0}, 1]. \end{cases} \tag{3.4}$$

It is sufficient to prove that v_0 belongs to the $L^{\varphi}(0,1)$ space.

$$\begin{aligned} \rho_{[0,1]}(\beta v_0) &= \int_0^1 \varphi(\beta |v_0(x)|) dx = \sum_{n=0}^{\infty} \int_{e^{-(n+1)t_0}}^{e^{-nt_0}} \varphi(\beta |v_0(x)|) dx \\ &= \sum_{n=0}^{\infty} \int_{e^{-(n+1)t_0}}^{e^{-nt_0}} \varphi(\beta e^{-n\gamma t_0} |w(xe^{nt_0})|) dx \\ &= \sum_{n=0}^{\infty} e^{-nt_0} \int_{e^{-t_0}}^{1} \varphi(\beta e^{-n\gamma t_0} |w(x)|) dx. \end{aligned}$$

According to Definition 2.1, if $-1/\overline{q} < \gamma < 0$ then there exists constant C that

$$\begin{aligned} \rho_{[0,1]}(\beta v_0) &\leqslant C \sum_{n=0}^{\infty} e^{-nt_0(1+\overline{q}\gamma)} \int_{e^{-t_0}}^{1} \varphi(\beta |w(x)|) dx \\ &= C \rho_{[e^{-t_0},1]}(\beta w) \sum_{n=0}^{\infty} e^{-nt_0(1+\overline{q}\gamma)}. \end{aligned}$$

Whereas $\gamma \geqslant 0$, we obtain

$$\begin{aligned} \rho_{[0,1]}(\beta v_0) &\leqslant \sum_{n=0}^{\infty} e^{-nt_0} \int_{e^{-t_0}}^{1} \varphi(\beta |w(x)|) dx \\ &= C \rho_{[e^{-t_0},1]}(\beta w) \sum_{n=0}^{\infty} e^{-nt_0}. \end{aligned}$$

In the both cases we obtain the geometric convergent series. It gives the conclusion $\rho_{[0,1]}(\beta v_0) \to 0$ as $\beta \to 0^+$ because of the assumption $w \in L^{\varphi}$. $\square$

Theorem 3.2. *If $\gamma > -1/\overline{q}$ then the set of periodic points of* (1.1) *is dense in the* $L^\varphi(0,1)$ *space.*

Proof. Let w be an arbitrary function from the $L^\varphi(0,1)$ space and let $\varepsilon > 0$. Define v by (3.4). Fix t_0 so large that $|w|^F_{[0,e^{-t_0}]} < \frac{\varepsilon}{2}$ and $|v|^F_{[0,e^{-t_0}]} < \frac{\varepsilon}{2}$. For $x \in [e^{-t_0}, 1]$ $v(x) = w(x)$ so finally we have

$$|v-w|^F_{[0,1]} = |v-w|^F_{[0,e^{-t_0}]} \leqslant |v|^F_{[0,e^{-t_0}]} + |w|^F_{[0,e^{-t_0}]} < \varepsilon.$$

This completes the proof. $\square$

Theorem 3.3. *If $\gamma > -1/\overline{q}$ then the dynamical system $(T_t)_{t\geqslant 0}$ is transitive in the* $L^\varphi(0,1)$ *space.*

Proof. Let

$$B(v_1,\varepsilon_1) = \{\sigma \in L^\varphi(0,1) : |v_1 - \sigma|^F_{[0,1]} < \varepsilon_1\},$$
$$B(v_2,\varepsilon_2) = \{\sigma \in L^\varphi(0,1) : |v_2 - \sigma|^F_{[0,1]} < \varepsilon_2\}$$

be two open balls with centers in $v_1, v_2 \in L^\varphi(0,1)$. Let us define the function

$$w(x) = \begin{cases} e^{-\gamma t} v_2(xe^t) & \text{for } x < e^{-t} \\ v_1(x) & \text{for } x \geqslant e^{-t} \end{cases}$$

at the suitable choice of t. We should show that the above function w belongs to the space $L^\varphi(0,1)$.

$$\rho_{[0,e^{-t}]}(\beta w) = \int_0^{e^{-t}} \varphi(\beta|e^{-\gamma t} v_2(xe^t)|)dx = e^{-t}\int_0^1 \varphi(\beta|e^{-\gamma t}v_2(x)|)dx.$$

If $-1/\overline{q} < \gamma < 0$ then for $C > 0$ we have

$$\rho_{[0,e^{-t}]}(\beta w) \leqslant Ce^{-t(1+\overline{q}\gamma)} \int_0^1 \varphi\left(\beta|v_2(x)|\right)dx = Ce^{-t(1+\overline{q}\gamma)}\rho_{[0,1]}(\beta v_2),$$

hence

$$\rho_{[0,1]}(\beta w) \leqslant \rho_{[0,e^{-t}]}(\beta w) + \rho_{[e^{-t},1]}(\beta w) \leqslant Ce^{-t(\gamma\overline{q}+1)}\rho_{[0,1]}(\beta v_2) + \rho_{[0,1]}(\beta v_1).$$

When $\gamma \geqslant 0$ we obtain

$$\rho_{[0,e^{-t}]}(\beta w) \leqslant e^{-t}\int_0^1 \varphi\left(\beta|v_2(x)|\right)dx = e^{-t}\rho_{[0,1]}(\beta v_2),$$
$$\rho_{[0,1]}(\beta w) \leqslant \rho_{[0,e^{-t}]}(\beta w) + \rho_{[e^{-t},1]}(\beta w) \leqslant e^{-t}\rho_{[0,1]}(\beta v_2) + \rho_{[0,1]}(\beta v_1).$$

In the both cases we have $\rho_{[0,1]}(\beta w) \to 0$, as $\beta \to 0^+$. It turns out from the fact that $v_1, v_2 \in L^\varphi(0,1)$ and in the first case from the assumption $\gamma\overline{q} + 1 > 0$. So $w \in L^\varphi(0,1)$. Besides, from the above equality we can draw the following conclusion $|w|^F_{[0,e^{-t}]} \leqslant K(t)$, where $K(t)$ can be made arbitrarily small. Then

$$\begin{aligned} |v_1 - w|^F_{[0,1]} &= |v_1 - w|^F_{[0,e^{-t}]} \\ &\leqslant |v_1|^F_{[0,e^{-t}]} + |w|^F_{[0,e^{-t}]} \\ &= |v_1|^F_{[0,e^{-t}]} + K(t). \end{aligned}$$

It turns out that for t large enough we obtain $|v_1 - w|^F_{[0,1]} < \varepsilon_1$, hence $w \in B(v_1,\varepsilon_1)$. Therefore $T_t w \in T_t\left(B(v_1,\varepsilon_1)\right)$ and $v_2 = T_t w \in B(v_2,\varepsilon_2)$. We learn from the above that the intersection two sets $B(v_2,\varepsilon_2)$ and $T_t\left(B(v_1,\varepsilon_1)\right)$ is not empty. So we obtain the conclusion about transitivity of the dynamical system $(T_t)_{t\geqslant 0}$ in the space $L^\varphi(0,1)$. $\square$

Corollary 3.4. *If $\gamma > -1/\overline{q}$ then the dynamical system $(T_t)_{t\geqslant 0}$ is chaotic in the sense of Devaney in the $L^\varphi(0,1)$ space.*

Theorem 3.5. *If $\gamma \leqslant -1/\underline{p}$ then the semigroup $(T_t)_{t\geqslant 0}$ is strongly stable in the $L^\varphi(0,1)$ space.*

Proof. Let $v \in L^\varphi(0,1)$ be an arbitrary function and let $\lambda > 0$. We obtain

$$\begin{aligned}
\rho_{[0,1]}\left(\lambda(T_t v)\right) &= \int_0^1 \varphi\left(\lambda\left|T_t v(x)\right|\right) dx \\
&= \int_0^1 \varphi\left(\lambda\left|e^{\gamma t} v(xe^{-t})\right|\right) dx \\
&= e^t \int_0^{e^{-t}} \varphi\left(\lambda\left|e^{\gamma t} v(x)\right|\right) dx \\
&\leqslant Ce^{t(1+\gamma\underline{p})} \int_0^{e^{-t}} \varphi\left(\lambda\left|v(x)\right|\right) dx \\
&= Ce^{t(1+\gamma\underline{p})} \rho_{[0,e^{-t}]}\left(\lambda v\right)
\end{aligned}$$

for some $C < \infty$. It is obvious that $\rho_{[0,e^{-t}]}(\lambda v) \to 0$ as $t \to \infty$, for any $\lambda > 0$. Moreover, $e^{1+\gamma\underline{p}} \leqslant 1$. It proves the strong stability of the system $(T_t)_{t\geqslant 0}$ in the $L^\varphi(0,1)$ space, that is $|T_t v|^F_{[0,1]} \to 0$ in $L^\varphi(0,1)$. $\square$

It is important to notice that if $\mu(\Omega) < \infty$ then $L_0^\varphi(\Omega) \subset L_0^\psi(\Omega)$ if and only if $\limsup_{u\to\infty} \frac{\psi(u)}{\varphi(u)} < \infty$ (see for example [11]). It follows that two φ-functions, satisfying the $\triangle_2$-condition, generate the same Orlicz space if they differ only on any finite subset of Ω. For example $L^\psi = L^\varphi$ where

$$\psi(t) = \begin{cases} \varphi(t) & \text{for } t \geqslant 1 \\ \varphi(1)t^p & \text{for } t < 1\,, \end{cases}$$

$p \geqslant 0$ and the φ-function φ satisfies the $\triangle_2$-condition. Therefore the replacing the φ-function by another on any finite subset of Ω has not influence on asymptotic behaviour in the L^φ space. According to the above remark, we can consider only the function M_∞ and the indices $\underline{p} = p^\infty$, $\overline{q} = q^\infty$. We give some example showing that the dynamical system $(T_t)_{t\geqslant 0}$ is not stable for the value of the coefficient γ from the interval $(-\frac{1}{\underline{p}}, 0)$.

Example 3.6. Let us consider Lasota equation (1.1) with initial condition (1.2) where

$$v(x) = \varphi^{-1}\left(\alpha x^{\alpha-1}\right),$$

$-1/\underline{p} < \gamma < 0$ and $\alpha = 1 + \gamma\underline{p} > 0$. Note that v is positive function and

$$\rho_{[0,1]}(v) = \int_0^1 \varphi(|v(x)|)dx = \int_0^1 \alpha x^{\alpha-1} dx = 1 < \infty.$$

It follows that $v \in L^\varphi(0,1)$. Moreover,

$$\rho_{[0,1]}\left(T_t v\right) = \int_0^1 \varphi\left(\left|T_t v(x)\right|\right) dx = \int_0^1 \varphi\left(\left|e^{\gamma t} v(xe^{-t})\right|\right) dx$$

$$
\begin{aligned}
&= e^t \int_0^{e^{-t}} \varphi\left(\left|e^{\gamma t} v(x)\right|\right) dx = e^t \int_0^{e^{-t}} \varphi\left(v(x)\right) \frac{\varphi\left(e^{\gamma t} v(x)\right)}{\varphi\left(v(x)\right)} dx \\
&= e^t \int_0^{e^{-t}} \alpha x^{\alpha-1} \frac{\varphi\left(e^{\gamma t} v(x)\right)}{\varphi\left(v(x)\right)} dx \geqslant e^t \int_0^{e^{-t}} \alpha x^{\alpha-1} M_\infty(e^{\gamma t}, \varphi) dx \\
&\geqslant e^t \int_0^{e^{-t}} \alpha x^{\alpha-1} e^{\gamma t \underline{p}} dx = e^{\alpha t} \int_0^{e^{-t}} \alpha x^{\alpha-1} dx = 1.
\end{aligned}
$$

Hence, $T_t v \nrightarrow 0$. Therefore, the system $(T_t)_{t\geqslant 0}$ is not stable.

Remark 3.7. We can consider a more general form of equation (1.1), i.e.

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda(x)u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1 \tag{3.5}$$

with the initial condition

$$u(0, x) = v(x), \tag{3.6}$$

where $\lambda : [0, 1] \to \mathbb{R}$ is a continuous function. Brzeźniak and Dawidowicz prove in [3] that the asymptotic behaviour of the semigroup $(\widetilde{T}_t)_{t\geqslant 0}$ generated by (3.5) in a Banach space depends only on the behaviour of the function λ in the neighborhood of 0. Let the dynamical system $(\widehat{T}_t)_{t\geqslant 0}$ be generated by the equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \widehat{\lambda}(x)u, \quad t \geqslant 0,\ 0 \leqslant x \leqslant 1 \tag{3.7}$$

where

$$\widehat{\lambda}(x) = \lambda(x) \quad \text{for every } x \in [0, \delta]. \tag{3.8}$$

According to [3], there exist such $t_0 > 0$ and a continuous function $g : [0, 1] \to \mathbb{R}$ that

$$
\begin{gathered}
g(x) = 1 \quad \text{for } x \in [0, e^{-t_0}], \\
\widetilde{T}_t u = g \widehat{T}_t u \quad \text{for every } t \geqslant t_0.
\end{gathered}
$$

We have the same property for the dynamical systems $(\widetilde{T}_t)_{t\geqslant 0}$ and $(\widehat{T}_t)_{t\geqslant 0}$ in the Orlicz space $L^\varphi(0, 1)$. If λ is the continuous function satisfying the condition

(H1) there exist numbers $\delta > 0$ and $\gamma > -1/\overline{q}$ such that $\lambda(x) > \gamma$ for $x \in [0, \delta]$

then the function $\kappa : [0, 1] \to \mathbb{R}$ defined by

$$\kappa(x) = \exp\Big(-\int_x^1 \frac{\lambda(s) - \gamma}{s} ds\Big), \quad x \in [0, 1]$$

is well-defined, continuous and $\kappa(0) = 0$ (see [3]). The multiplication by κ defined a bounded, injective linear operator $\mathbf{R}$ on the space $L^\varphi(0, 1)$. If u is a solution to (1.1) then $\widetilde{u}$ defined by the formula

$$\widetilde{u}(t, x) = \kappa(x)u(t, x)$$

is the solution to (3.5) and the diagram

$$
\begin{tikzcd}
L^\varphi \arrow[r, "T_t"] \arrow[d, "\mathbf{R}"'] & L^\varphi \arrow[d, "\mathbf{R}"] \\
L^\varphi \arrow[r, "\widetilde{T}_t"'] & L^\varphi
\end{tikzcd}
$$

is commutative. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\kappa(x) \in [\delta, \frac{1}{\delta}]$ for all $x \in [\varepsilon, 1]$. Let us defined

$$u_n(x) = \begin{cases} u(x) & \text{for } x \in (\frac{1}{n}, 1] \\ 0 & \text{for } x \in [0, \frac{1}{n}] \end{cases}$$

where $u \in L^\varphi$. It is obvious that $\rho(u_n - u) \to 0$, as $n \to \infty$ and $\frac{u_n}{\kappa} \in L^\varphi$. Hence $u_n = \mathbf{R}\left(\frac{u_n}{\kappa}\right) \in \mathbf{R}(L^\varphi)$. It follows that the set $\mathbf{R}(L^\varphi)$ is dense in L^φ. Therefore the dynamical system $(\widetilde{T}_t)_{t\geqslant 0}$ is chaotic in L^φ under the assumption (H1). To prove the stability it is necessary to put the condition

(H2) there exist numbers $\delta > 0$ and $\gamma \leqslant -1/\underline{p}$ such that $\lambda(x) < \gamma$ for $x \in [0, \delta]$

Under assumption (H2) the operator $\mathbf{R}$ is bounded on L^φ.

$$\rho_{[0,1]}(\widetilde{T}_t v) \leqslant C\rho_{[0,1]}(T_t v)$$

for some $C > 0$, which proves strong stability of the system $(\widetilde{T}_t)_{t\geqslant 0}$ in L^φ.

Acknowledgements. Anna Poskrobko was supported by the Bialystok University of Technology grant S/WI/1/2016, and by the resources for research by Ministry of Science and Higher Education.

References

[1] J. Aroza, T. Kalmes, E. Mangino; *Chaotic C_0-semigroups induced by semiflows in Lebesgue and Sobolev spaces*, J. Math. Anal. Appl., 412 (2014), 77–98.
[2] Z. Brzeźniak, A.L. Dawidowicz; *On periodic solutions to the von Foerster-Lasota equation*, Semigroup Forum, 78 (2009), 118–137.
[3] Z. Brzeźniak, A. L. Dawidowicz; *On the chaotic properties of the von Foerster-Lasota equation*, Semigroup Forum, 88 (2014), 287–299.
[4] A.L. Dawidowicz; *On the existence of an invariant measure for a quasi-linear partial differential equation*, Zeszyty Nauk. Uniw. Jagiello., Prace Mat., 23 (1982), 117–123.
[5] A. L. Dawidowicz, A. Poskrobko; *On chaotic and stable behaviour of the Foerster-Lasota equation in some Orlicz spaces*, Proc. Estonian Acad. Sci. Phys. Math., 57 (2008), 61–69.
[6] R.L. Devaney; *An introduction to chaotic dynamical systems*, Addison-Wesley, New York, 1989.
[7] D. Drasin, D. F. Shea; *Pólya peaks and the oscillation of positive functions*, Proc. Amer. Math. Soc., 34 (1972), 403–411.
[8] A. Lasota; *Stable and chaotic solutions of a first order partial differential equation*, Nonlinear Anal., 5 (1981), 1181–1193.
[9] A. Lasota, M. C. Mackey, M. Ważewska-Czyżewska; *Minimizing Therapeutically Induced Anemia*, J. Math. Biology, 13 (1981), 149–158.
[10] J. Lindenstrauss, L. Tzafriri; *Classical Banach spaces II*, Springer-Verlag, Berlin, 1979.
[11] L. Maligranda; *Orlicz spaces and interpolation*, Seminars in Mathematics, Vol. 5, Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.
[12] W. Matuszewska, W. Orlicz; *On certain properties of φ-functions*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 8 (1960), 439–443.
[13] W. Matuszewska, W. Orlicz; *On some classes of functions with regard to their orders of growth*, Studia Math., 26 (1965), 11–24.
[14] A.G. McKendrick; *Application of mathematics to medical problems*, Proc. Edin. Math. Soc., 44 (1926), 98–130.
[15] S. J. Montgomery-Smith; *Boyd Indices of Orlicz-Lorentz Spaces*, Lecture Notes in Pure and Appl. Math., 172 (1995), 321-334.
[16] J. Musielak; *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., Vol. 1034, Springer-Verlag, Berlin, 1983.
[17] R. Rudnicki; *Chaoticity and invariant measures for a cell population model*, J. Math. Anal. Appl., 393 (2012), 151–165.

[18] F. Takeo; *Chaos and hypercyclicity for solution semigroups to chaos and hypercyclicity for solution semigroups to some partial differential equations*, Nonlinear Anal., 63 (2005), e1943–e1953.

[19] M. Ważewska-Czyżewska, A. Lasota; *Matematyczne problemy dynamiki układu krwinek czerwonych*, Roczniki PTM, Mat. Stos., VI (1976), 23–40.

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