Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 299, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

UNIQUENESS OF A VERY SINGULAR SOLUTION TO NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH ABSORPTION FOR DIRICHLET BOUNDARY CONDITION

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Abstract. We prove the existence and uniqueness of singular solutions (fundamental solution, very singular solution, and large solution) of quasilinear parabolic equations with absorption for Dirichlet boundary condition. We also show the short time behavior of singular solutions as t tends to 0.

1. Introduction

This article concerns the nonnegative singular solutions of the degenerated parabolic equation

$$\begin{gathered}\partial_t u - \Delta(u^m) + u^q = 0, \quad \text{in } \Omega \times (0,\infty),\\ u = 0, \quad \text{on } \partial\Omega \times (0,\infty),\end{gathered} \tag{1.1}$$

where $q > m > 1$, and Ω is a smooth bounded domain in $\mathbb{R}^N$. Here, singular solutions refer to the large solution, the very singular solution, and the solution with initial Dirac measure.

Our main purpose is to consider the uniqueness of very singular solution (in short VSS) of equation (1.1), which has not been proved before for any bounded domain. Roughly speaking, a VSS of (1.1) is a solution which is more singular than solutions with initial Dirac measures. This terminology is introduced first by Brezis et al. [4]. This solution has been intensively studied during last decades.

In the sequel, we assume without loss of generality that $0 \in \Omega$, and such a VSS has a singularity at $x = 0$. Most of papers have studied the existence and uniqueness of VSS for the Cauchy problems, i.e: $\Omega = \mathbb{R}^N$, see e.g. [4, 7, 13, 14, 15, 18, 2, 3], and references therein. Note that this kind of solution plays a crucial role in studying the long time behavior of solutions of the Cauchy problem corresponding to equation (1.1), see [13, 16].

Let us mention the results involving our problem. Peletier and Terman [18] showed that there exists a self-similar VSS of equation (1.1) in $\mathbb{R}^N \times (0,\infty)$, which is of the form

$$W(x,t) = t^{-\frac{1}{p-1}} f(|x| t^{(p-m)/2(p-1)}),$$

2010 *Mathematics Subject Classification.* 35K65, 35K15.
Key words and phrases. Degenerate parabolic equations; large solution; very singular solution; Dirac measure.

Submitted August 14, 2016. Published November 25, 2016.

provided $1 < m < p < m + \frac{2}{N}$. In order for W to fulfill the singular condition above, f must satisfy the condition

$$
\begin{gathered}
(f^m)'' + \frac{N-1}{\tau}(f^m)' + \frac{p-m}{2(p-1)}\tau f' + \frac{f}{p-1} - f^p = 0, \\
\tau = |x|t^{(p-m)/2(p-1)}, \quad f'(0) = 0, \\
\lim_{\tau\to+\infty} \tau^{2/(p-m)} f(\tau) = 0, \quad f(\tau)\begin{cases} > 0, & \text{if } 0 \le \tau < \tau_0, \\ = 0, & \text{if } \tau_0 \le \tau < \infty. \end{cases}
\end{gathered} \tag{1.2}
$$

for some $\tau_0 > 0$, see also Leoni, [12] for the case $0 < m < 1$. The uniqueness result of self-similar solutions of (1.2) was proved by Kamin and Veron, [15] (see also [17, 14]). The proof of the uniqueness result is intensively based on the self-similarity in order to lead to solving the ODE (1.2). It is of course that this method does not work for such a bounded domain Ω.

In this article, we show that (1.1) has a unique VSS. Our idea is to construct a maximal VSS, and a minimal VSS. Then we show that both solutions are equal. It is well known that the minimal VSS is the convergence of a non-decreasing sequence of solutions with initial Dirac measures. While, we construct the maximal VSS, which is the decreasing convergence of large solutions. This leads to consider large solutions of (1.1).

Let us discuss large solutions. Crandall, Lions, and Souganidis [8] considered nonnegative solutions of the equation

$$
\begin{gathered}
\partial_t u - \Delta u + |\nabla u|^q = 0, \quad \text{in } \Omega \times (0,\infty), \\
u = 0, \quad \text{on } \partial\Omega \times (0,\infty),
\end{gathered} \tag{1.3}
$$

with initial data

$$
u(0) = \begin{cases} +\infty, & \text{in } \mathcal{O}, \\ 0, & \text{in } \Omega\backslash\overline{\mathcal{O}}, \end{cases} \tag{1.4}
$$

where $\mathcal{O}$ is an open subset of Ω. The initial data is understood as follows: $u(x,t) \to +\infty$, for any $x \in \mathcal{O}$, and $u(x,t) \to 0$, for any $x \in \overline{\Omega}\backslash\overline{\mathcal{O}}$.

This problem is motivated by studying the theory of large deviations of Markov diffusion processes. The authors showed that there is a unique solution of problem (1.3), (1.4) when $q > 1$. Such a solution with initial data (1.4) is called a large solution. Inspired by their work, and also for our purpose later, we would like to prove the existence and uniqueness of large solution of problem (1.1).

In the next section, we give the definitions of large solution and VSS, and give our results.

2. Some definitions and main results

Notation: We denote by $B(x, r)$ the open ball with center at x and radius $r > 0$. We also denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(\lambda)$ means that C depends on λ.

Let us first define a VSS of equation (1.1).

Definition 2.1. V is called a VSS of equation (1.1) if $V \in \mathcal{C}(\overline{\Omega} \times [0,\infty)\backslash\{(0,0)\})$ satisfies (1.1) in the sense of distributions, and V has the following properties:

$$\begin{gathered} V(x,0) = 0, \quad \forall x \in \Omega\backslash\{0\}, \\ \lim_{t\to 0} \int_{B(0,r)} V(x,t)dx = \infty, \quad \text{for all } r > 0. \end{gathered} \tag{2.1}$$

Definition 2.2. u is called a large solution of (1.1) if $u \in \mathcal{C}(\overline{\Omega} \times (0,\infty))$ satisfies (1.1) in the sense of distributions, and u fulfills condition (1.4).

Our first results are the existence and uniqueness of large solutions.

Theorem 2.3. *Let $q > m > 1$. Then, there exists a unique large solution of* (1.1).

Concerning VSS, we have the following result.

Theorem 2.4. *Let $m > 1$, and $m < p < m + \frac{2}{N}$. Then, there exists a unique VSS of* (1.1).

Now, we state a result of the short time behavior of the VSS.

Theorem 2.5. *Let u be the unique VSS of equation* (1.1) *in Theorem 2.4. Then*

$$\lim_{t\to 0} t^{\frac{1}{p-1}} u(0,t) = f(0). \tag{2.2}$$

Remark 2.6. The result of Theorem 2.5 implies that the short time behavior of VSS of equation (1.1) for a bounded domain and the one in $\mathbb{R}^N$ are the same.
Of course our results above also hold for $m = 1$.

In the next section, we give the proof of Theorem 2.3. The proof of Theorem 2.4, and Theorem 2.5 will be given in the last section.

3. Proof of Theorem 2.3

(i) Existence. For any $n \geq 1$, we set $\mathcal{O}_n = \{x \in \mathcal{O} : \operatorname{dist}(x, \partial\mathcal{O} > \frac{1}{n})\}$, and construct a nondecreasing sequence of Lipschitz functions, ψ_n such that

$$\psi_n = \begin{cases} n, & \text{if } x \in \mathcal{O}_n, \\ 0, & \text{if } x \in \Omega\backslash\overline{\mathcal{O}}. \end{cases}$$

Now, we consider equation (1.1) with initial data $u_0 = \psi_n$. By the classical results (see [19]), there exists a unique solution $u_n \in \mathcal{C}(\overline{\Omega} \times [0,\infty))$. Clearly, $z(t) = (q-1)^{\frac{-1}{q-1}} t^{1-q}$, is a solution of the ODE:

$$\begin{gathered} z'(t) + z^q(t) = 0, \quad t > 0, \\ z(0) = +\infty, \end{gathered}$$

By the strong comparison principle (see [1]), we obtain

$$u_n(x,t) \leq z(t), \quad \forall (x,t) \in \Omega \times (0,\infty). \tag{3.1}$$

It is obvious that $\{u_n\}_{n\geq 1}$ is non-decreasing. By (3.1), there is a function u such that $u_n \uparrow u$, and $u(x,t)$ is also bounded by $z(t)$ in $\Omega \times (0,\infty)$.

By the boundedness of u_n, the classical argument allows us to pass to the limit as $n \to \infty$, in order to obtain u, a weak solution of equation (1.1). The regularity $u \in \mathcal{C}(\overline{\Omega} \times (0,\infty))$ follows from the regularity results in [9] (see also [19, 10, 11]).

It remains to show that $u(0)$ fulfills condition (1.4). Indeed, for any $x \in \mathcal{O}$, there is a natural number $n_x \in \mathbb{N}$ such that $x \in \mathcal{O}_n$, for all $n \geq n_x$. Then the monotonicity of the sequence $\{u_n\}_{n\geq 1}$ implies

$$\liminf_{t\to 0} u(x,t) \geq \liminf_{t\to 0} u_n(x,t) = n.$$

The last inequality holds for any $n \geq n_x$, thereby proves $u(x,0) = +\infty$ in $\mathcal{O}$.

Next, we claim that $u(t)$ converges to 0 in $\Omega\backslash\overline{\mathcal{O}}$ as $t \to 0$. Let

$$\begin{gathered} -\Delta\alpha = 1, \quad \text{in } B(x_0,r), \\ \alpha = 0, \quad \text{on } \partial B(x_0,r), \end{gathered} \tag{3.2}$$

for any $x_0 \in \Omega\backslash\overline{\mathcal{O}}$, and $r > 0$ is small enough such that $B(x,r) \subset \overline{\Omega}\backslash\overline{\mathcal{O}}$.

Let $w(x,t) = \lambda e^{Ct} e^{\frac{1}{\alpha(x)}}$, for any $\lambda \in (0,1)$, and constant $C > 0$ is chosen later such that

$$\partial_t w - \Delta(w^m) + w^q \geq 0. \tag{3.3}$$

After this, the comparison principle yields

$$u_n(x,t) \leq w(x,t), \quad \text{in } B(x_0,r) \times (0,\infty),$$

since $w = +\infty$ on $\partial B(x_0,r)$, and $u_n(x,0) = 0$ in $B(x_0,r)$. The above inequality implies

$$u(x,t) \leq w(x,t), \quad \text{in } B(x_0,r) \times (0,\infty), \tag{3.4}$$

hence

$$0 \leq \limsup_{t\to 0} u(x,t) \leq \lambda e^{\frac{1}{\alpha(x)}}, \quad \text{in } B(x_0,r).$$

Thus, the claim follows as $\lambda \to 0$.

Now, we show (3.3). Indeed, computations yield

$$w_t = Cw, \quad \Delta(w^m) = mw^m\Big(\frac{m|\nabla\alpha|^2}{\alpha^4} + \frac{2|\nabla\alpha|^2}{\alpha^3} + \frac{-\Delta\alpha}{\alpha^2}\Big)$$

Note that $-\Delta\alpha = 1$, so we obtain

$$w_t - \Delta(w^m) + w^q = Cw - mw^m\Big(\frac{m|\nabla\alpha|^2}{\alpha^4} + \frac{2|\nabla\alpha|^2}{\alpha^3} + \frac{1}{\alpha^2}\Big) + w^q.$$

One hand, $|\nabla\alpha|$ is bounded on $\overline{B(x_0,r)}$. Other hand, $w(x,t) \to +\infty$ faster than $\alpha^{-l}(x)$, for any $l \geq 1$, as $x \to \partial B(x_0,r)$. Thus

$$-mw^m\Big(\frac{m|\nabla\alpha|^2}{\alpha^4} + \frac{2|\nabla\alpha|^2}{\alpha^3} + \frac{1}{\alpha^2}\Big) + w^q > 0, \quad \text{on the set } \big\{x \in \Omega : |x-x_0| > r-\delta\big\},$$

for some $\delta > 0$. Note that one can choose $\delta > 0$ so that it is independent of C. Hence,

$$w_t - \Delta(w^m) + w^q > 0, \quad \text{on the set } \big\{x \in \Omega : |x - x_0| > r - \delta\big\}.$$

It remains to choose $C = C(\lambda) > 0$ large enough such that

$$w_t - \Delta(w^m) + w^q > 0, \quad \text{on the set } \big\{x \in \Omega : |x - x_0| \leq r - \delta\big\}.$$

Combining the last two inequalities yields (3.3).

(ii) Uniqueness. We use a scaling argument as in [8]. For any $\lambda > 0$, we set

$$u_\lambda(x,t) = \lambda u(\lambda^{\frac{q-m}{2}} x, \lambda^{q-1} t),$$

Clearly, u_λ is a large solution of problem (1.1) corresponding to $(\lambda^{\frac{q-m}{2}}\Omega, \lambda^{\frac{q-m}{2}}\mathcal{O})$ instead of $(\Omega, \mathcal{O})$. Then, by the routine argument we have for any large solution v of (1.1),

$$u_\lambda(x,t) \geq v(x,t) \geq u_{\lambda'}(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty), \text{ for } \lambda > 1 > \lambda' > 0. \tag{3.5}$$

Letting $\lambda \to 1^+$ and $\lambda' \to 1^-$ in (3.5) yields

$$u = v, \quad \text{in } \Omega \times (0,\infty).$$

This completes the proof of Theorem 2.3.

4. Uniqueness of VSS, and short time behavior

Now we give the proof of Theorems 2.4 and 2.5.

Proof. **Step 1:** First, we construct a maximal VSS. Let u_ε be a unique large solution of (1.1) for $(\Omega, B(0,\varepsilon))$. It is clear that $\{u_\varepsilon\}_{\varepsilon>0}$ is a non-decreasing sequence. Then, there is a function u such that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0$. We will show that u is a maximal VSS. Indeed, u is bounded by $z(t)$ in $\Omega \times (0,\infty)$, so the classical argument implies that u is a weak solution of (1.1).

Next, for any $x_0 \in \Omega \backslash \{0\}$, from (3.4) we have

$$u(x_0, 0) \leq u_\varepsilon(x_0, 0) \leq \lambda e^{\frac{1}{\alpha(x_0)}},$$

Therefore, u fulfills the first condition in (2.1) as $\lambda \to 0$.

It remains to prove that u is the maximal solution. This is equivalent to show that for any $\varepsilon > 0$, and for any VSS v of equation (1.1), it holds

$$v \leq u_\varepsilon, \quad \text{in } \Omega \times (0,\infty), \tag{4.1}$$

On the one hand, since $v(x,0) = 0$ for any $x \neq 0$, then proceeding as in the proof of (3.4) yields that for any $\tau > 0$,

$$v(x,\tau) \leq \lambda e^{C\tau} e^{\frac{1}{\alpha(x)}}, \quad \forall x \in \Omega, |x| \geq \varepsilon/2.$$

where $\alpha(x)$ is the solution of (3.2) in $B(x, \varepsilon/4)$. Thereby,

$$v(x,\tau) \leq m_\varepsilon \lambda e^{C\tau}, \quad \forall x \in \Omega, \ |x| \geq \varepsilon/2, \tag{4.2}$$

with $m_\varepsilon = \sup_{y \in B(x,\varepsilon/4)} \{e^{\frac{1}{\alpha(y)}}\}$.

On the other hand, since $u_\varepsilon(x,t) \to \infty$ uniformly on any compact of $B(0,\varepsilon)$ as $t \to 0$, there exists then a time $s(\tau) > 0$ such that

$$v(x,\tau) \leq u_\varepsilon(x,s), \quad \text{for any } x \in B(0,\varepsilon/2), \ \forall s \in (0, s(\tau)). \tag{4.3}$$

By (4.2) and (4.3), we obtain

$$v(x,\tau) \leq m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(x,s), \quad \text{for any } x \in \Omega, \ \forall s \in (0, s(\tau)),$$

From the comparison principle it follows that

$$v(x, t+\tau) \leq m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(x, t+s), \quad \forall (x,t) \in \Omega \times (0,\infty). \tag{4.4}$$

Inequality (4.4) holds for any $s \in (0, s(\tau))$. Then letting $s \to 0$ yields

$$v(x, t+\tau) \leq m_\varepsilon \lambda e^{C\tau} + u_\varepsilon(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

The above inequality holds for any $\tau > 0$, thereby we obtain after passing to the limit $\tau \to 0$,

$$v(x,t) \leq \lambda + u_\varepsilon(x,t), \quad \forall (x,t) \in \Omega \times (0,\infty).$$

Finally, passing $\lambda \to 0$ yields conclusion (4.1). In the sequel, we denote by u^{Ω}_{max}, the maximal VSS of (1.1) in $\Omega \times (0, \infty)$. By the construction, the sequence $\{u^{B_R}_{max}\}_{R>0}$ is non-decreasing. Note that this sequence is also bounded by $z(t)$. Thus,

$$u^{B_R}_{\max}(x, t) \uparrow W(x, t), \tag{4.5}$$

for any $(x, t) \in \mathbb{R}^N \times (0, \infty)$, as $R \to \infty$. It is not difficult to verify that W is a self-similar VSS of (1.1) in $\mathbb{R}^N \times (0, \infty)$.

Step 2: Now, we construct a minimal VSS, which is the convergence of the increasing sequence of solutions with initial Dirac measures. It is convenient for us to construct a Dirac solution first. Consider problem (1.1) with the initial data ρ_n, $\rho_n(x) = n^N \rho(nx)$, and

$$\rho(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

where C is the constant such that $\int_{\mathbb{R}^N} \rho(x)dx = 1$. It is clear that ρ_n converges to Dirac δ_0. By the classical result (see [19]), there exists a unique continuous solution v_n. It is not difficult to show that v_n converges to u, a unique solution of equation (1.1) with initial data δ_0, see [14, 5].

At the moment, let u^{Ω}_k be the unique solution of (1.1) with initial data $k\delta_0$ in $\Omega \times (0, \infty)$. Clearly, $\{u^{\Omega}_k\}_{k>0}$ is the non-decreasing sequence, and it is bounded by $z(t)$. Thus, there is a function, say $u^{\Omega}_{\min}$ such that u^{Ω}_k converges to $u^{\Omega}_{\min}$ as $k \to \infty$. Note that $u^{\Omega}_{\min}$ is the minimal VSS of (1.1), see [14].

By its construction, the sequence $\{u^{B_R}_{\min}\}_{R>0}$ is non-decreasing, and it converges to V as $R \to \infty$, a self-similar VSS of equation (1.1) in $\mathbb{R}^N \times (0, \infty)$. Since W and V are two self-similar solutions of the Cauchy equation (1.1), they must satisfy equation (1.2). It follows from the uniqueness solution of equation (1.2) (see [15, 17]) that

$$W = V, \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{4.6}$$

Next, we claim that for any $k > 0$, and for $\varepsilon > 0$ (small)

$$u^{B_R}_k(x, t) \leq u^{\Omega}_k(x, t) + m_\varepsilon \lambda e^{Ct}, \quad \text{in } B_R \times (0, \infty), \tag{4.7}$$

for any $R > 0$ large enough such that $\Omega \subset\subset B(0, R)$, and m_ε is as in (4.2).

To prove (4.7), it suffices to consider the case $k = 1$. By the uniqueness of fundamental solutions, we only need to show

$$v^{B_R}_n(x, t) \leq v^{\Omega}_n(x, t) + m_\varepsilon \lambda e^{Ct}, \quad \text{in } B_R \times (0, \infty). \tag{4.8}$$

Recall that v^{Ω}_n (resp. $v^{B_R}_n$) is the unique solution of (1.1) with initial data ρ_n in $\Omega \times (0, \infty)$ (resp. $B(0, R) \times (0, \infty)$). In fact, for any n large enough such that $\frac{1}{n} < \frac{\varepsilon}{8}$, we note that $Supp(v^{B_R}_n(., 0) = \rho_n) \subset \overline{B(0, 1/n)}$. By the same analysis as (4.2), we also obtain

$$v^{B_R}_n(x, t) \leq m_\varepsilon \lambda e^{Ct}, \quad \forall x \in B(0, R), |x| \geq \varepsilon/2, \ t > 0, \tag{4.9}$$

On the one hand, $(v^{\Omega}_n + m_\varepsilon \lambda e^{Ct})$ is a super-solution of (1.1) in $\Omega \times (0, \infty)$. On the other hand, from (4.9) it follows that $v^{B_R}_n(x, t) \leq m_\varepsilon \lambda e^{Ct}$, for any $x \in \partial\Omega$, and for $t > 0$.

Note that $v^{B_R}_n(x, 0) = v^{\Omega}_n(x, 0) = \rho_n(x)$. Thus, the strong comparison result implies

$$v^{B_R}_n \leq v^{\Omega}_n + m_\varepsilon \lambda e^{Ct}, \quad \text{in } \Omega \times (0, \infty). \tag{4.10}$$

By combining (4.9) and (4.10), we obtain

$$v_n^{B_R} \le v_n^{\Omega} + m_\varepsilon \lambda e^{Ct}, \quad \text{in } B(0,R)\times(0,\infty).$$

Letting $n \to \infty$ we obtain (4.8), and proves (4.7).

Next, passing to the limit as $k \to \infty$ in (4.7) yields

$$u_{\min}^{B_R} \le u_{\min}^{\Omega} + m_\varepsilon \lambda e^{Ct}, \quad \text{in } B(0,R)\times(0,\infty). \tag{4.11}$$

By (4.6), letting $R \to \infty$ in (4.11) we obtain

$$W = V \le u_{\min}^{\Omega} + m_\varepsilon \lambda e^{Ct}, \quad \text{in } \mathbb{R}^N \times(0,\infty). \tag{4.12}$$

By combining (4.5) and (4.12), we obtain

$$u_{\min}^{\Omega} \le u_{max}^{\Omega} \le W \le u_{\min}^{\Omega} + m_\varepsilon \lambda e^{Ct}, \quad \text{in } \Omega\times(0,\infty). \tag{4.13}$$

Thanks to the comparison result of Aronson et al. [1], we have

$$\begin{aligned}\int_\Omega \big(u_{max}^{\Omega}(t) - u_{\min}^{\Omega}(t)\big)^+ dx \le& \int_\Omega \big(u_{max}^{\Omega}(s) - u_{\min}^{\Omega}(s)\big)^+ dx \\ &+ \int_s^t \int_\Omega \Big(- \big(u_{max}^{\Omega}(\tau)\big)^q + \big(u_{\min}^{\Omega}(\tau)\big)^q \Big)^+ dx\, d\tau.\end{aligned}$$

Or

$$\int_\Omega \big(u_{max}^{\Omega}(t) - u_{\min}^{\Omega}(t)\big) dx \le \int_\Omega \big(u_{max}^{\Omega}(s) - u_{\min}^{\Omega}(s)\big) dx, \tag{4.14}$$

for any $0 < s < t$. From (4.13) and (4.14) it follows that

$$\int_\Omega \big(u_{max}^{\Omega}(t) - u_{\min}^{\Omega}(t)\big) dx \le \int_\Omega m_\varepsilon \lambda e^{Cs} dx = m_\varepsilon |\Omega| \lambda e^{Cs}.$$

The limit as $s \to 0$ yields

$$\int_\Omega \big(u_{max}^{\Omega}(t) - u_{\min}^{\Omega}(t)\big) dx \le |\Omega| m_\varepsilon \lambda.$$

The above inequality holds for any $\lambda > 0$ small enough, so the uniqueness result follows.

Finally, we prove the short time behavior result. From (4.13), we have

$$u_{max}^{\Omega}(0,t) \le W(0,t) = t^{\frac{-1}{p-1}} f(0) \le u_{max}^{\Omega}(0,t) + m_\varepsilon \lambda e^{Ct}, \quad \forall t > 0.$$

Or

$$t^{\frac{1}{p-1}} u_{max}^{\Omega}(0,t) \le f(0) \le t^{\frac{1}{p-1}} u_{max}^{\Omega}(0,t) + m_\varepsilon \lambda t^{\frac{1}{p-1}} e^{Ct}.$$

Then, the result follows by passing $t \to 0$ in the above inequality. □

As a consequence, we have the short time behavior of the unique large solution.

Corollary 4.1. *Let u_L be the unique large solution of problem* (1.1), (1.4). *Then, $u_L(x,t)$ has the rate $t^{-\frac{1}{p-1}}$ as $t \to 0$, for any $x \in \mathcal{O}$.*

Proof. It suffices to show that the result holds for $x = 0 \in \mathcal{O}$. Let u be the unique VSS. Then

$$u(0,t) \le u_L(0,t) \le (q-1)^{\frac{-1}{q-1}} t^{-\frac{1}{p-1}}.$$

Since $u(0,t)$ has the rate $t^{-\frac{1}{p-1}}$ as $t \to 0$, we obtain the conclusion. □

Remark 4.2. A potential alternative proof for the uniqueness result of VSS by using the finite speed of propagation suggested by Professor Kamin could be considered in the future for general nonlinear absorption term.

Acknowledgements. The author wants to thank Professor J. I. Diaz and Professor S. Kamin for their comments and encouragement.

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