

MULTIPLE SOLUTIONS FOR SCHRÖDINGER-POISSON SYSTEMS WITH SIGN-CHANGING POTENTIAL AND CRITICAL NONLINEARITY

LIUYANG SHAO, HAIBO CHEN

ABSTRACT. In this article, we study the Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + V(x)u + k(x)\phi(x)u &= h_1(x)|u|^4u + \mu h_2(x)u + h_3(x), \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi(x) &= k(x)u^2, \quad \text{in } \mathbb{R}^3, \end{aligned}$$

where $h_1(x), h_2(x), h_3(x)$ and $V(x)$ are allowed to be sign-changing and $\mu > 0$ is a parameter. Under some appropriate assumptions on $V(x)$, we obtain the existence of two different solutions for the above system via variational methods.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In this article we consider the Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + V(x)u + k(x)\phi(x)u &= h_1(x)|u|^4u + \mu h_2(x)u + h_3(x), \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi(x) &= k(x)u^2, \quad \text{in } \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $h_1(x), h_2(x), h_3(x)$ and $V(x)$ are allowed to be sign-changing and $\mu > 0$ is a parameter. Moreover, $h_3(x)$ is a perturbed term. System (1.1) is a modified version of the classical Schrödinger-Poisson system (also called Schrödinger-Maxwell equation), which has a strong physical backgrounds because of its appearances in quantum mechanical models and in semiconductor theory. For more details, we refer the readers to [3, 4, 12] and the references therein.

In recent years, with the aid of variational methods, there have been many results on existence, nonexistence and multiplicity of solutions for such system depending on the assumptions of the potential $V(x)$. According to the conditions imposed on the potential $V(x)$, these results can be roughly classified into four cases. Case 1: Many articles deal with the case when $V(x)$ is a positive constant or radially symmetric function, see for example [1, 2, 5, 7, 15, 19, 25] and the references therein. Case 2: There are also a great number of articles devoted to the case when $V(x)$ is nonradial, see for instance [8, 9, 14, 20]. Case 3: Many articles deal with the case when $V(x)$ possesses some kind of periodicity see [10, 16, 17, 18, 21, 22, 23]. Case 4: We know that [11, 22] treat the case when $V(x)$ is sign-changing. Here

2010 *Mathematics Subject Classification.* 35B38, 35G99.

Key words and phrases. Schrödinger-Poisson system; variational methods; mountain pass theorem; Ekeland's variational principle.

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Submitted March 12, 2016. Published October 17, 2016.

we emphasize problem (1.1) not only has sign-changing potential $V(x)$ but also possesses critical nonlinearity.

Motivated by the above facts, the goal of this paper is to consider the multiplicity of nontrivial solutions for (1.1) when $V(x)$ is sign-changing. Under some natural assumptions, by using Mountain Pass Theorem in combination with Ekeland's variational principle, the existence results of at least two nontrivial solutions are obtained. Actually, one positive solution and one negative solution.

Before stating our main results, we give the following assumptions on $V(x)$.

(A1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^3} V(x) > -\infty$. Moreover, there exists a constant d_0 such that

$$\lim_{|y| \rightarrow +\infty} \text{meas}\{x \in \mathbb{R}^3 : |x - y| \leq d_0, V(x) \leq M\} = 0, \quad \forall M > 0.$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in \mathbb{R}^3 .

Inspired by Zhang and Xu [27], we can find a constant $V_0 > 0$ such that $\tilde{V}(x) := V(x) + V_0 \geq a$, where $a > 0$ is a constant, and let $\tilde{\mu}\tilde{h}_2(x) = V_0 + \mu h_2(x)$, for all $x \in \mathbb{R}^3$.

Throughout this paper, instead of (A1) we make the following assumptions:

(A2) $\tilde{V} \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^3} \tilde{V}(x) \geq a > 0$, where $a > 0$ is a constant, and there exists a constant $d_0 > 0$ such that

$$\lim_{|y| \rightarrow +\infty} \text{meas}\{x \in \mathbb{R}^3 : |x - y| \leq d_0, \tilde{V}(x) \leq M\} = 0, \quad \forall M > 0.$$

Then it is easy to verify the following lemma.

Lemma 1.1. *System (1.1) is equivalent to the problem*

$$\begin{aligned} -\Delta u + \tilde{V}(x)u + k(x)\phi(x)u &= h_1(x)|u|^4u + \tilde{\mu}\tilde{h}_2(x)u + h_3(x), & \text{in } \mathbb{R}^3, \\ -\Delta\phi(x) &= k(x)u^2, & \text{in } \mathbb{R}^3. \end{aligned} \quad (1.2)$$

We also assume that

(A3) $k \in L^\infty(\mathbb{R}^3, \mathbb{R})$, and $k(x) \geq 0$ for any $x \in \mathbb{R}^3$.

(A4) $h_1, h_3 \in L^2 \cap C_0^\infty(\mathbb{R}^3, \mathbb{R})$ and $\tilde{h}_2 \in L^6 \cap C_0^\infty(\mathbb{R}^3, \mathbb{R})$.

(A5) $0 < \tilde{\mu} < \mu_0$, where μ_0 is defined by

$$\mu_0 := \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)u^2) dx : \int_{\mathbb{R}^3} \tilde{h}_2(x)|u|^2 dx = 1 \right\}. \quad (1.3)$$

Proposition 1.2 ([6, Lemma 2.5]). *Assume that (A3) and (A4) hold. Then the infimum μ_0 is achieved.*

Now we state our main results.

Theorem 1.3. *Suppose that (A2)–(A5) hold. Then there exists $m_0 > 0$ such that (1.2) admits at least two nontrivial weak solutions when $\|h_3\|_2 \leq m_0$. Actually, one solution is positive and one is negative.*

Remark 1.4. It is not difficult to find the functions $V(x)$ satisfying the above conditions. For example, let $V(x)$ be a zig-zag function with respect to $|x|$ defined as

$$V(x) = \begin{cases} 2n|x| - 2n(n-1) + a_0, & n-1 \leq |x| \leq \frac{(2n-1)}{2}, \\ -2n|x| + 2n^2 + a_0, & \frac{(2n-1)}{2} \leq |x| \leq n, \end{cases}$$

where $n \in \mathbb{N}$ and $a_0 \in \mathbb{R}$. Set $V_0 := \sup_{x \in \mathbb{R}^3} |V(x)|$, it is not difficult to verify that $V(x)$ satisfies the conditions (A1) and (A2).

Remark 1.5. The nonlinear growth of $|u|^4 u$ reaches the Sobolev critical exponent since the critical exponent $2^* = 6$ in three spatial dimensions, which is why we call it critical nonlinearity in the title.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our main results.

Hereafter, we use the following notation.

- $H^1(\mathbb{R}^3)$ denotes the usual sobolev space endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx, \quad \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}} := (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{1/2}$.
- Denote the space $E = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)|u|^2) dx < \infty\}$, with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)|u|^2) dx.$$

- E^* denotes the dual space of E .
- For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_{\rho(z)}$ denotes the ball of radius ρ centered at z . $|B_{\rho(z)}|$ denotes its lebesgue measure of the ball.

2. VARIATIONAL SETTING AND PRELIMINARIES

In this section, we recall some basic notation and preliminaries. From [3], we know that under the assumption (A2), the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s \in [2, 6)$.

It is easy to show that system (1.2) can be reduced to a single equation with a nonlocal term. For $u \in E$, we define a linear functional L_u in $D^{1,2}(\mathbb{R}^3)$ as follows:

$$L_u : v \rightarrow \int_{\mathbb{R}^3} k(x)u^2 v dx.$$

One can check that the functional L_u is continuous in $D^{1,2}(\mathbb{R}^3)$. Indeed, by using the Hölder's inequality and Sobolev inequality, we obtain

$$\left| \int_{\mathbb{R}^3} k(x)u^2 v dx \right| \leq \|k\|_\infty \|u^2\|_{\frac{6}{5}} \|v\|_6 = \|k\|_\infty \|u\|_{\frac{12}{5}}^2 \|v\|_6 \leq C \|u\|_E^2 \|v\|_{D^{1,2}}. \quad (2.1)$$

Given $u \in E$, by the Lax-Milgram Theorem, there exists a unique solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ of the equation

$$-\Delta \phi(x) = k(x)u^2.$$

Moreover, ϕ_u has the integral expression

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} k(y) \frac{u^2(y)}{|x-y|} dy \geq 0,$$

combining this with (2.1), we obtain

$$\|\phi\|_{D^{1,2}}^2 < C \|u\|_E^2 \|\phi_u\|_{D^{1,2}};$$

that is, $\|\phi_u\|_{D^{1,2}} \leq c\|u\|_E^2$. And then we have

$$\begin{aligned} \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} k(y) \frac{u^2(x)u^2(y)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} u^2 \phi_u(x) dx \\ &\leq C\|u\|_E^4. \end{aligned} \quad (2.2)$$

Now we define a functional I on E by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \tilde{V}(x)u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} k(x)\phi_u u^2 dx \\ &\quad - \frac{1}{6} \int_{\mathbb{R}^3} h_1(x)u^6 dx - \frac{1}{2} \int_{\mathbb{R}^3} \tilde{\mu}\tilde{h}_2(x)u^2 dx - \int_{\mathbb{R}^3} h_3(x)u dx. \end{aligned}$$

It is easy to verify that the functional I is of class $C^1(E, \mathbb{R})$. Moreover,

$$\begin{aligned} \langle I'(u), v \rangle &= \int_{\mathbb{R}^3} (|\nabla u| \nabla v + \tilde{V}(x)uv) dx + \int_{\mathbb{R}^3} k(x)\phi_u uv dx \\ &\quad - \int_{\mathbb{R}^3} h_1(x)u^5 v dx - \int_{\mathbb{R}^3} \tilde{\mu}\tilde{h}_2(x)uv dx - \int_{\mathbb{R}^3} h_3(x)v dx. \end{aligned}$$

Hence, if $u \in E$ is a critical point of I , then the pair (u, ϕ_u) is a solution of (1.2).

Theorem 2.1 ([26, Mountain Pass Theorem]). *Let X be a real Banach space, suppose that $I \in C^1(X, \mathbb{R})$ satisfies the (PS) condition with $I(0) = 0$. In addition, suppose that*

- (i) *there are $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ when $\|u\|_X = \rho$;*
- (ii) *there is $e \in X$, $\|e\|_X > \rho$, such that $I(e) < 0$.*

Define

$$\Gamma = \{\gamma \in C^1([0, 1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

Then $c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t))$ is a critical value of I .

3. PROOF OF THEOREM 1.3

Lemma 3.1. *Assume that (A2)–(A5) hold. Then I satisfies the (PS) condition.*

Proof. We first prove that $\{u_n\}$ is bounded in E . Then

$$\begin{aligned} C + 1 + \|u_n\|_E &\geq I(u_n) - \frac{1}{6} \langle I'(u_n), u_n \rangle \\ &= \frac{1}{3} \|u_n\|_E^2 + \frac{1}{12} \int_{\mathbb{R}^3} k(x)\Phi u^2 dx \\ &\quad - \frac{1}{3} \tilde{\mu} \int_{\mathbb{R}^3} \tilde{h}_2(x)u^2 dx - \left(1 - \frac{1}{6}\right) \int_{\mathbb{R}^3} h_3(x)u_n dx \\ &\geq \frac{1}{3} \|u_n\|_E^2 + \frac{1}{12} \int_{\mathbb{R}^3} k(x)\Phi u_n^2 dx - \frac{\tilde{\mu}}{3\mu_0} \|u_n\|_E^2 \\ &\quad - \frac{5}{6} \left(\int_{\mathbb{R}^3} h_3^2(x) dx \right)^{1/2} \left(\int_{\mathbb{R}^3} u_n^2 dx \right)^{1/2} \\ &\geq \frac{1}{3} \left(1 - \frac{\tilde{\mu}}{\mu_0}\right) \|u_n\|_E^2 - C\|u_n\|_E, \end{aligned} \quad (3.1)$$

which implies that $\{u_n\}$ is bounded.

Next we show that $\{u_n\}$ possesses a strong convergent subsequence in E . In fact, in view of the boundedness of $\{u_n\}$, without loss of generality, we assume that

there exists $u_0 \in E$ such that $u_n \rightharpoonup u_0$ as $n \rightarrow \infty$. Since $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s \in [2, 6)$, $u_n \rightarrow u_0$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 6)$. We obtain

$$\begin{aligned} \|u_n - u\|_E^2 &= 2\langle I'(u_n) - I'(u), u_n - u \rangle - \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx \\ &\quad + \int_{\mathbb{R}^3} h_1(|u_n|^4 u - |u|^4 u)(u_n - u) dx + \tilde{\mu} \int_{\mathbb{R}^3} \tilde{h}_2(x)(u_n - u)^2 dx \quad (3.2) \\ &\quad + \int_{\mathbb{R}^3} h_3(x)(u_n - u) dx. \end{aligned}$$

It is clear that

$$\langle I'(u_n) - I'(u), u_n - u \rangle \rightarrow 0.$$

By using the Hölder's inequality and Sobolev inequality, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u) dx \right| &\leq \|\phi_{u_n} u_n\|_2 \|u_n - u\|_2 \\ &\leq \|\phi_{u_n}\|_6 \|u_n\|_3 \|u_n - u\|_2 \\ &\leq C \|\phi_{u_n}\|_{D^{1,2}} \|u_n\|_3 \|u_n - u\|_3 \\ &\leq C \|u_n\|_{\frac{12}{5}} \|u_n\|_3 \|u_n - u\|_2 \rightarrow 0. \end{aligned} \quad (3.3)$$

Similarly, we obtain that $\int_{\mathbb{R}^3} \phi_u u (u_n - u) dx \rightarrow 0$ as $n \rightarrow \infty$.

From the Brézis-Lieb Lemma [26], we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} h_1(x)(|u_n|^5 - |u|^5)(u_n - u) dx \right| \\ &\leq \|h_1\|_\infty \left| \int_{\mathbb{R}^3} (|u_n|^5 - |u|^5)(u_n - u) dx \right| \\ &\leq \|h_1\|_\infty \int_{\mathbb{R}^3} |u_n - u|^6 dx + o(1) \\ &\leq C \|h_1\|_\infty \|u_n - u\|_E^6 + o(1) \rightarrow 0. \end{aligned} \quad (3.4)$$

By using the Hölder's inequality, we obtain

$$\int_{\mathbb{R}^3} h_3(x)(u_n - u) dx \leq \|h_3\|_2 \|u_n - u\|_2 \rightarrow 0, \quad (3.5)$$

$$\int_{\mathbb{R}^3} \tilde{h}_2(x)(u_n - u)^2 dx \leq \|\tilde{h}_2\|_6 \|u_n - u\|_{12/5}^2 \rightarrow 0. \quad (3.6)$$

We obtain $\|u_n - u\|_E^2 \rightarrow 0$, which completes the proof. □

Lemma 3.2. *Assume that (A2)–(A5) are satisfied. Then there exists $m_0 > 0$ such that (1.2) has a positive solution when $\|h_3\|_2 \leq m_0$.*

Proof. It follows from Lemma 1.1 and the Sobolev inequality that

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \tilde{V}(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} k(x) \phi_u u^2 dx - \frac{1}{6} \int_{\mathbb{R}^3} h_1(x) u^6 dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} \tilde{\mu} \tilde{h}_2(x) u^2 dx - \int_{\mathbb{R}^3} h_3(x) u dx \\ &= \frac{1}{2} \|u\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} k(x) \phi_u u^2 dx - \frac{1}{6} \int_{\mathbb{R}^3} h_1(x) u^6 dx \\ &\quad - \frac{1}{2} \tilde{\mu} \int_{\mathbb{R}^3} \tilde{h}_2(x) u^2 dx - \int_{\mathbb{R}^3} h_3(x) u dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\|u\|_E^2 - \frac{C}{6}\|h_1\|_\infty\|u\|_E^6 - \frac{1}{2}\tilde{\mu}\int_{\mathbb{R}^3}|\tilde{h}_2(x)|u^2dx - \|h_3\|_2\|u\|_2 \\
&\geq \frac{1}{2}\left(1 - \frac{\tilde{\mu}}{\mu_0}\right)\|u\|_E^2 - \frac{C}{6}\|h_1\|_\infty\|u\|_E^6 - C_1\|h_3\|_2\|u\|_E \\
&= \|u\|_E\left(\frac{1}{2}\left(1 - \frac{\tilde{\mu}}{\mu_0}\right)\|u\|_E - \frac{C}{6}\|u\|_E^5 - C_1\|h_3\|_2\right).
\end{aligned}$$

Taking $\|u\|_E = t$, and letting

$$g(t) = \frac{1}{2}\left(1 - \frac{\tilde{\mu}}{\mu_0}\right)t - \frac{C}{6}t^5,$$

we have $g'(t_0) = 0$ when

$$t_0 = \left(\frac{3(1 - \frac{\tilde{\mu}}{\mu_0})}{5C}\right)^{1/4}.$$

Hence, there exists

$$m_0 = \|h_3\|_2 < \frac{2}{5C_1}\left(1 - \frac{\tilde{\mu}}{\mu_0}\right)\left(\frac{3(1 - \frac{\tilde{\mu}}{\mu_0})}{5C}\right)^{1/4},$$

such that

$$\frac{2}{5}\left(1 - \frac{\tilde{\mu}}{\mu_0}\right)\left(\frac{3(1 - \frac{\tilde{\mu}}{\mu_0})}{5C}\right)^{1/4} - C_1\|h_3\|_2 > 0.$$

Let $\rho = \|u\|_E$ small enough, there exists $\alpha > 0$, such that $I(u) \geq \alpha$. Thus, (i) in Theorem 2.1 is true.

Choose $\psi_0 \in C_0^\infty(\mathbb{R}^3)$, $\psi_0 \geq 0$ and $\psi_0 \not\equiv 0$ in \mathbb{R}^3 . Then

$$\begin{aligned}
I(t\psi_0) &= \frac{1}{2}t^2\|\psi_0\|^2 + \frac{1}{4}t^4\int_{\mathbb{R}^3}k(x)\phi_{\psi_0}\psi_0^2dx - \frac{1}{6}t^6\int_{\mathbb{R}^3}h_1(x)\psi_0^6dx \\
&\quad - \frac{1}{2}t^2\tilde{\mu}\int_{\mathbb{R}^3}h_2(x)\psi_0^2dx - t\int_{\mathbb{R}^3}h_3(x)\psi_0dx,
\end{aligned} \tag{3.7}$$

so $I(t\psi_0) \rightarrow -\infty$ as $t \rightarrow +\infty$. Therefore, there exists t_0 large enough, such that $I(t_0\psi_0) < 0$. Taking $e = t_0\psi_0$ such that $\|t_0\psi_0\| > \rho$ and $I(e) < 0$, and (ii) in Theorem 2.1 is true. It is obvious that $I(0) = 0$, by Theorem 2.1, problem (1.2) has a positive solution. \square

Lemma 3.3. *Assume that (A2)–(A4) hold. Then (1.2) has a negative solution.*

Proof. Since $h_3 \in L^2(\mathbb{R}^3) \setminus \{0\}$ and $h_3^+ \not\equiv 0$, we can choose a function $\phi_1 \in E$ such that

$$\int_{\mathbb{R}^3}h_3(x)\phi_1dx > 0.$$

For $t > 0$ small enough, we have

$$\begin{aligned}
I(t\phi_1) &= \frac{t^2}{2}\|\phi_1\|_E^2 + \frac{t^2}{4}k(x)\int_{\mathbb{R}^3}\phi_u\phi_1^2dx - \frac{t^6}{6}\int_{\mathbb{R}^3}h_1(x)\phi_1^6dx \\
&\quad - \frac{\tilde{\mu}t^2}{2}\int_{\mathbb{R}^3}\tilde{h}_2(x)\phi_1dx - t\int_{\mathbb{R}^3}h_3\phi_1dx \\
&\leq \frac{t^2}{2}\|\phi_1\|_E^2 + \frac{t^4}{4}c\|\phi_1\|_E^4 - \frac{t^6}{6}\int_{\mathbb{R}^3}h_3(x)\phi_1^6dx - \frac{\tilde{\mu}t^2}{2}\int_{\mathbb{R}^3}\tilde{h}_2(x)\phi_1dx \\
&\quad - t\int_{\mathbb{R}^3}h_3(x)\phi_1dx < 0.
\end{aligned}$$

Hence $\theta_0 := \inf\{I(u) : u \in \bar{B}_\rho\} < 0$. By the Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \bar{B}_\rho$, such that $I(u_n) \rightarrow 0$ and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Because the functional I satisfies the (PS) condition, there exists $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) = c_1 < 0$. The proof is complete. \square

Proof of Theorem 1.3. From Lemmas 3.2 and 3.3, we obtain the existence of at least two nontrivial weak solutions for the problem (1.2). Actually, one solution is positive and the other negative. \square

Acknowledgments. This work is partially supported by the Natural Science Foundation of China 11671403.

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LIUYANG SHAO
SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA, 410083
HUNAN, CHINA
E-mail address: `sluiyang316@163.com`

HAIBO CHEN (CORRESPONDING AUTHOR)
SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA, 410083
HUNAN, CHINA
E-mail address: `math_chb@163.com`