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SOLVABILITY OF SINGULAR SECOND-ORDER INITIAL-VALUE PROBLEMS

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Abstract. This article concerns the solvability of the initial-value problem $x'' = f(t, x, x')$, $x(0) = A$, $x'(0) = B$, where the scalar function f may be unbounded as $t \to 0$. Using barrier strip type arguments, we establish the existence of monotone and/or positive solutions in $C^1[0,T] \cap C^2(0,T]$.

1. Introduction

In this article we study the solvability of the initial value problem (IVP)

$$\begin{gathered} x'' = f(t, x, x'), \\ x(0) = A, \quad x'(0) = B, \end{gathered} \tag{1.1}$$

where the scalar function $f(t, x, p)$ is defined for $(t, x, p) \in D_t \times D_x \times D_p$, and $D_t, D_x, D_p \subseteq R$, but there may be sets $X \subseteq D_x$ and $P \subseteq D_p$ such that f is unbounded as $t \to 0$ and $(x, p) \in X \times P$.

The solvability of various nonsingular and singular second order IVPs has been studied by Aslanov [3], Agarwal and O'Regan [1, 2], Bobisud and O'Regan [4], Bobisud and Lee [5], Cabada et al. [6, 7, 8], Cid [9], Maagli and Masmoudi[13], Rachůnková and Tomeček [14, 15, 16], Yang [17, 18] and Zhao [19]. Yang [17, 18], for example, established the solvability in $C^1[0,1]$ and $C[0,1] \times C^2(0,1)$ in the case $A = B = 0$. In these works the function $f(t, x, p) \in C((0,1), (0,\infty)^2)$ is allowed to be singular at $t = 0, t = 1, x = 0$ or $p = 0$ and is such that

$$0 < f(t, x, p) \le k(t)F(x)G(p) \quad \text{for } (t, x, p) \in (0,1) \times (0,\infty)^2,$$

where k, F and G are suitable functions.

Here we present sufficient conditions guaranteeing monotone and/or positive solutions to (1.1) in $C^1[0,T] \times C^2(0,T]$. They are established by adapting ideas from Kelevedjiev and Popivanov [10] and Kelevedjiev et al. [11] (sse also Kelevedjiev [12]), where (1.1) may be singular at $x = A$ and/or $p = B$. The results in these works rely on a combination of a barrier type condition with the assumption that there is a constant $k < 0$ such that

$$f(t, x, p) \le k \tag{1.2}$$

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on a suitable bounded subset of the domain of f. It turned out, however, that (1.2) is not necessary when (1.1) is singular only at $t=0$, that is why we pay a special attention to this case.

In our considerations we use two results from [11] for the nonsingular problem

$$\begin{gathered} x''=f(t,x,x'),\\ x(a)=A,\ x'(a)=B, \end{gathered} \tag{1.3}$$

where $f:D_t\times D_x\times D_p\to R,\ D_t,D_x,D_p\subseteq R$. They are based on the assumption

(A1) There are constants $T>a,\ m_1,\overline{m}_1,M_1,\overline{M}_1$ and a sufficiently small $\tau>0$ such that

$$\overline{M}_1-\tau\geq M_1\geq B\geq m_1\geq \overline{m}_1+\tau,$$
$$[a,T]\subseteq D_t,[m_0-\tau,M_0+\tau]\subseteq D_x,\quad [\overline{m}_1,\overline{M}_1]\subseteq D_p,$$

where $M_0=\max\{|m_1|,|M_1|\}(T-a)+|A|$, and $m_0=-M_0$,

$$\begin{gathered} f(t,x,p)\in C\big([a,T]\times[m_0-\tau,M_0+\tau]\times[m_1-\tau,M_1+\tau]\big),\\ f(t,x,p)\leq 0\quad \text{for } (t,x,p)\in[a,T]\times D_x\times[M_1,\overline{M}_1],\\ f(t,x,p)\geq 0\quad \text{for } (t,x,p)\in[a,T]\times D_{M_0}\times[\overline{m}_1,m_1], \end{gathered}$$

where $D_{M_0}=D_x\cap(-\infty,M_0]$.

So, we need the following result.

Lemma 1.1 ([11]). *Let* (A1) *hold and $x\in C^2[a,T]$ be a solution to* (1.3). *Then*

$$m_0\leq x(t)\leq M_0,\quad m_1\leq x'(t)\leq M_1,\quad m_2\leq x''(t)\leq M_2\quad \textit{for } t\in[a,T],$$

where $m_2=\min f(t,x,p)$ and $M_2=\max f(t,x,p)$ for $(t,x,p)\in[a,T]\times[m_0,M_0]\times[m_1,M_1]$.

This lemma was used in the proof of the following theorem.

Theorem 1.2 ([11]). *Let* (A1) *hold. Then nonsingular IVP* (1.3) *has at least one solution in $C^2[a,T]$.*

2. Existence results

Returning our attention to singular problem (1.1), we assume that

(A2) There are constants $T>0,\ m_1,\overline{m}_1,M_1,\overline{M}_1$ and a sufficiently small $\tau>0$ such that

$$\overline{M}_1-\tau\geq M_1\geq B\geq m_1\geq \overline{m}_1+\tau,$$
$$(0,T]\subseteq D_t,[\tilde{m}_0-\tau,\tilde{M}_0+\tau]\subseteq D_x,\quad [\overline{m}_1,\overline{M}_1]\subseteq D_p,$$

where $\tilde{M}_0=\max\{|m_1|,|M_1|\}T+|A|$, and $\tilde{m}_0=-\tilde{M}_0$,

$$\begin{gathered} f(t,x,p)\in C\big((0,T]\times[\tilde{m}_0-\tau,\tilde{M}_0+\tau]\times[m_1-\tau,M_1+\tau]\big),\\ f(t,x,p)\leq 0\quad \text{for } (t,x,p)\in(0,T]\times D_x\times[M_1,\overline{M}_1],\\ f(t,x,p)\geq 0\quad \text{for } (t,x,p)\in(0,T]\times D_{\tilde{M}_0}\times[\overline{m}_1,m_1], \end{gathered} \tag{2.1}$$

where $D_{\tilde{M}_0}=D_x\cap(-\infty,\tilde{M}_0]$.

We are now in a position to state our first existence theorem.

Theorem 2.1. *Let* (A2) *hold. Then* (1.1) *has at least one solution in* $C^1[0,T] \cap C^2(0,T]$ *such that*

$$m_1 t + A \le x(t) \le M_1 t + A \quad \text{for } t \in [0,T],$$
$$m_1 \le x'(t) \le M_1 \quad \text{for } t \in [0,T].$$

Proof. We will do the proof in several steps considering the family of nonsingular problems

$$\begin{gathered} x'' = f(t,x,x'), \\ x(n^{-1}) = A, \quad x'(n^{-1}) = B, \end{gathered} \tag{2.2}$$

where $n \in N_T = \{n \in N : n^{-1} < T\}$.

Step 1 Construction of a sequence $\{x_n\}$ of $C^2[n^{-1},T]$-solutions to (2.2). It is not hard to check that each problem of (2.2) satisfies (A1) for $a = n^{-1}$, $M_0 = \max\{|m_1|,|M_1|\}(T-n^{-1})+|A| < \tilde{M}_0$, and $m_0 = -M_0$. Thus, according to Theorem 1.2, (2.2) has a solution

$$x_n \in C^2[n^{-1},T] \quad \text{for each } n \in N_T.$$

In addition, for each $n \in N_T$ Lemma 1.1 guarantees the bounds

$$\tilde{m}_0 < m_0 \le x_n(t) \le M_0 < \tilde{M}_0 \quad \text{for } t \in [n^{-1},T],$$
$$m_1 \le x_n'(t) \le M_1 \quad \text{for } t \in [n^{-1},T].$$

Step 2 Construction of a $C^2(0,T]$-solution to the differential equation. Now, we introduce a numerical sequence $\{\theta_i\}, i \in N$, having the properties

$$\theta_i \in (0,T), \quad \theta_{i+1} < \theta_i \quad \text{for } i \in N \text{ and } \lim_{t\to\infty} \theta_i = 0,$$

and consider the sequence $\{x_n\}$ of $C^2[n^{-1},T]$-solutions of family (2.2) only for $n \in N_1 = \{n \in N_T : n^{-1} < \theta_1\}$. Clearly, the bounds

$$\tilde{m}_0 < x_n(t) < \tilde{M}_0 \quad \text{for } t \in [\theta_1,T], \tag{2.3}$$
$$m_1 \le x_n'(t) \le M_1 \quad \text{for } t \in [\theta_1,T], \tag{2.4}$$

independent of $n \in N_1$. In view of (2.1), $f(t,x,p)$ is continuous on the set $[\theta_1,T] \times [\tilde{m}_0,\tilde{M}_0] \times [m_1,M_1]$ and so there is a constant $M_{1,2}$, independent on n, such that

$$|x_n''(t)| \le M_{1,2} \quad \text{for } t \in [\theta_1,T].$$

The obtained bounds for $x_n(t), x_n'(t)$ and $x_n''(t)$ on the interval $[\theta_1,T]$ allows us to apply the Arzela-Ascoli theorem on the sequence $\{x_n\}$ to conclude that there are a subsequence $\{x_{1,n_k}\}$, $k \in N$, $n_k \in N_1$, and a function $x_{\theta_1} \in C^2[\theta_1,T]$ such that

$$\|x_{1,n_k} - x_{\theta_1}\|_1 \to 0 \quad \text{on } t \in [\theta_1,T];$$

that is, the sequences $\{x_{1,n_k}\}$ and $\{x_{1,n_k}'\}$ converge uniformly on $[\theta_1,T]$ to x_{θ_1} and x_{θ_1}', respectively. Since (2.3) and (2.4) are valid in particular for the elements of $\{x_{1,n_k}\}$ and $\{x_{1,n_k}'\}$, letting $k \to \infty$, we obtain

$$\tilde{m}_0 \le x_{\theta_1}(t) \le \tilde{M}_0 \quad \text{for } t \in [\theta_1,T], \tag{2.5}$$
$$m_1 \le x_{\theta_1}'(t) \le M_1 \quad \text{for } t \in [\theta_1,T]. \tag{2.6}$$

On the other hand, on using that the functions $x_{1,n_k}(t), n_k \in N_1$, are solutions of the differential equation (2.2), we have

$$x'_{1,n_k}(t) = x'_{1,n_k}(\theta_1) + \int_{\theta_1}^{t} f(s, x_{1,n_k}(s), x'_{1,n_k}(s))ds,\ t \in (\theta_1, T].$$

Next, we need to show that the sequence $\{f(s, x_{1,n_k}(s), x'_{1,n_k}(s))\}$, $n_k \in N_1$, converges uniformly on the interval $[\theta_1, T]$. To this aim we observe at first that since $f(t, x, p)$ is uniformly continuous on the compact set $[\theta_1, T] \times [\tilde{m}_0, \tilde{M}_0] \times [m_1, M_1]$, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(t_0, x_0, p_0) - f(t_1, x_1, p_1)| < \varepsilon \tag{2.7}$$

if $(t_0, x_0, p_0), (t_1, x_1, p_1) \in [\theta_1, T] \times [\tilde{m}_0, \tilde{M}_0] \times [m_1, M_1]$ and

$$\sqrt{(t_0 - t_1)^2 + (x_0 - x_1)^2 + (p_0 - p_1)^2} < \delta.$$

Now, from the uniform convergence of $\{x_{1,n_k}\}$ and $\{x'_{1,n_k}\}$ on $[\theta_1, T]$ it follows that there is a $N_{\delta(\varepsilon)}$ with the properties

$$|x_{1,n_k} - x_{\theta_1}| < \frac{\delta}{\sqrt{2}} \quad \text{and} \quad |x'_{1,n_k} - x'_{\theta_1}| < \frac{\delta}{\sqrt{2}} \quad \text{for } t \in [\theta_1, T]$$

and each $n_k > N_{\delta(\varepsilon)}$. As a result, for $t \in [\theta_1, T]$ we obtain

$$\sqrt{(t-t)^2 + (x_{1,n_k} - x_{\theta_1})^2 + (x'_{1,n_k} - x'_{\theta_1})^2} < \delta. \tag{2.8}$$

Finally, for $t \in [\theta_1, T]$ and $n_k > N_{\delta(\varepsilon)}$ from (2.3)-(2.6) we obtain

$$(t, x_{1,n_k}(t), x'_{1,n_k}(t)), (t, x_{\theta_1}(t), x'_{\theta_1}(t)) \in [\theta_1, T] \times [\tilde{m}_0, \tilde{M}_0] \times [m_1, M_1]. \tag{2.9}$$

On combining (2.8) and (2.9) with (2.7), we establish that for an arbitrary $\varepsilon > 0$ there exists $N_{\delta(\varepsilon)}$ such that for $n_k > N_{\delta(\varepsilon)}$ we have

$$|f(s, x_{1,n_k}(s), x'_{1,n_k}(s)) - f(s, x_{\theta_1}(s), x'_{\theta_1}(s))| < \varepsilon \quad \text{for } t \in [\theta_1, T],$$

i.e. the sequence $\{f(s, x_{1,n_k}(s), x'_{1,n_k}(s))\}, n_k \in N_1$, converges uniformly on the interval $[\theta_1, T]$ to $f(s, x_{\theta_1}(s), x'_{\theta_1}(s))$. Then, returning to the integral equation and letting $k \to \infty$ yield

$$x'_{\theta_1}(t) = x'_{\theta_1}(t) + \int_{\theta_1}^{t} f(s, x_{\theta_1}(s), x'_{\theta_1}(s))ds, \quad t \in (\theta_1, T],$$

from where it follows that $x_{\theta_1}(t)$ is a $C^2[\theta_1, T]$-solution to the differential equation $x'' = f(t, x, x')$ on $[\theta_1, T]$.

Further, we consider the sequence $\{x_{1,n_k}\}$ on the new interval $[\theta_2, T]$ and for $n_k \in N_2 = \{n_k \in N_T, k \in N : n_k^{-1} < \theta_2\}$. Obviously, for $n_k \in N_2$ we have

$$\tilde{m}_0 \le x_{1,n_k}(t) \le \tilde{M}_0 \quad \text{for } t \in [\theta_2, T],$$
$$m_1 \le x'_{1,n_k}(t) \le M_1 \quad \text{for } t \in [\theta_2, T].$$

Besides, there is a constant $M_{2,2}$, independent on n_k, such that

$$|x''_{1,n_k}(t)| \le M_{2,2} \quad \text{for } t \in [\theta_2, T].$$

Having obtained bounds, we apply the Arzela-Ascoli theorem on the sequence $\{x_{1,n_k}\}$ to conclude that there exist a subsequence $\{x_{2,n_k}\}, k \in N, n_k \in N_2$, and a function $x_{\theta_2} \in C^2[\theta_2, T]$ such that

$$\|x_{2,n_k} - x_{\theta_2}\|_1 \to 0 \quad \text{on the new interval } [\theta_2, T].$$

As above we establish also that $x_{\theta_2}(t)$ is a $C^2[\theta_2,T]$-solution to the differential equation $x''=f(t,x,x')$ on $[\theta_2,T]$ and

$$\begin{gathered}\tilde m_0 \le x_{\theta_2}(t) \le \tilde M_0 \quad \text{for } t\in[\theta_2,T],\\ m_1 \le x'_{\theta_2}(t) \le M_1 \quad \text{for } t\in[\theta_2,T].\end{gathered}$$

In addition, since $\{x_{2,n_k}\}$ is a subsequence of $\{x_{1,n_k}\}$, then $\{x_{2,n_k}\}$ converges uniformly to x_{θ_1} on the interval $[\theta_1,T]$ which means

$$x_{\theta_2}(t)\equiv x_{\theta_1}(t) \quad \text{for } t\in[\theta_1,T].$$

Applying the same procedure repeatedly for $\theta_i\to 0$, we establish that for each $i\in N$ there exists a function $x_{\theta_i}(t)$ which is a $C^2[\theta_i,T]$-solution to the equation $x''=f(t,x,x')$ on the interval $[\theta_i,T]$,

$$\|x_{i,n_k}-x_{\theta_i}\|_1\to 0 \quad \text{on the interval } [\theta_i,T] \tag{2.10}$$

as $k\to\infty$ and $n_k\in N_i=\{n_k\in N_T, k\in N: n_k^{-1}<\theta_i\}$,

$$\begin{gathered}\tilde m_0 \le x_{\theta_i}(t) \le \tilde M_0 \quad \text{for } t\in[\theta_i,T],\\ m_1 \le x'_{\theta_i}(t) \le M_1 \quad \text{for } t\in[\theta_i,T],\\ x_{\theta_{i+1}}(t)\equiv x_{\theta_i}(t) \quad \text{for } t\in[\theta_i,T].\end{gathered}$$

Thanks to the properties of the functions of $\{x_{\theta_i}\}$, we conclude that there is some function $x_0(t)$ which is a $C^2(0,T]$-solution to the equation $x''=f(t,x,x')$ on the interval $(0,T]$,

$$\tilde m_0 \le x_0(t) \le \tilde M_0 \quad \text{for } t\in(0,T],$$
$$m_1 \le x'_0(t) \le M_1 \quad \text{for } t\in(0,T], \tag{2.11}$$
$$x_0(t)\equiv x_{\theta_i}(t) \quad \text{for } t\in[\theta_i,T]. \tag{2.12}$$

Step 3 Construction of a $C^1[0,T]\cap C^2(0,T]$-solution to (1.1). To define a $C[0,T]$-solution to (1.1) we need to show that

$$\lim_{t\to 0^+} x_0(t)=A. \tag{2.13}$$

To this aim, we assume firstly on the contrary that for some $\varepsilon>0$ there exists $\delta>0$ such that $(0,\delta)\subset[0,T]$ and

$$x_0(t)\notin(A-\varepsilon,A+\varepsilon) \quad \text{for } t\in(0,\delta). \tag{2.14}$$

Returning our attention to the sequence $\{x_n\}$, from $x_n\in C[0,T]$ and $x_n(n^{-1})=A$ deduce that there is a number n_δ such that for each $n\ge n_\delta, n\in N$, there exists a sufficiently small $\delta_n>n^{-1}$ with the properties $(n^{-1},\delta_n)\subset(0,\delta)$ and

$$x_n(t)\in(A-\varepsilon/2,A+\varepsilon/2) \quad \text{for } t\in(n^{-1},\delta_n).$$

On the other hand, there exists a number n^* such that for each $n\ge n^*, n\in N$, there exists some $i^*\in N$ for which

$$[\theta_{i^*},\theta_{i^*-1}]\subset(n^{-1},\delta_n)\subset(0,\delta);$$

the assumption that the interval $[\theta_{i^*},\theta_{i^*-1}]$ does not exist contradicts to the fact that $t=0$ is an accumulation point of the sequence $\{\theta_i\}$. As a result, for each $n\ge\max\{n_\delta,n^*\}$ there exists $i^*\in N$ such that

$$A-\varepsilon/2<x_n(t)<A+\varepsilon/2 \quad \text{for } t\in[\theta_{i^*},\theta_{i^*-1}]\subset(0,\delta). \tag{2.15}$$

It is easy to see, for every i^* there is a number n_{i^*} such that (2.15) holds for each $n_k \in N_{i^*}, k \in N$, with $n_k \geq \max\{n_{i^*}, n_\delta, n^*\}$, that is,

$$A - \varepsilon/2 < x_{i^*,n_k}(t) < A + \varepsilon/2 \quad \text{for } t \in [\theta_{i^*}, \theta_{i^*-1}] \subset (0, \delta). \tag{2.16}$$

Further, from (2.10) and (2.12) for each $i \in N$ we obtain

$$\|x_{i,n_k} - x_0\|_1 \to 0 \quad \text{on } [\theta_i, T] \text{ when } k \to \infty \text{ and } n_k \in N_i, \tag{2.17}$$

which means that for each $i \in N$ there is a number $\overline{n}_i$ such that for each $n_k \in N_i$ with $n_k \geq \overline{n}_i$ we have

$$-\varepsilon/2 < x_{i,n_k}(t) - x_0(t) < \varepsilon/2 \quad \text{for } t \in [\theta_i, T]$$

or

$$x_{i,n_k}(t) - \varepsilon/2 < x_0(t) < x_{i,n_k}(t) + \varepsilon/2 \quad \text{for } t \in [\theta_i, T].$$

In particular, for each $n_k \in N_{i^*}$ with $n_k \geq \max\{n_{i^*}, \overline{n}_{i^*}, n_\delta, n^*\}$, $k \in N$, we obtain

$$x_{i^*,n_k}(t) - \varepsilon/2 < x_0(t) < x_{i^*,n_k}(t) + \varepsilon/2 \quad \text{for } t \in [\theta_i^*, T].$$

This combined with (2.16) yields

$$A - \varepsilon < x_0(t) < A + \varepsilon \quad \text{for } t \in [\theta_{i^*}, \theta_{i^*-1}] \subset (0, \delta),$$

which contradicts to (2.15) and so (2.13) holds.

By exactly the same reasoning applied on the sequence $\{x_n'\}$ we establish

$$\lim_{t \to 0^+} x_0'(t) = B.$$

Moreover, now we use that for each $i \in N$ and sufficiently large $n_k \in N_i, k \in N$, (2.17) yields

$$-\varepsilon/2 < x_{i,n_k}'(t) - x_0'(t) < \varepsilon/2 \quad \text{for } t \in [\theta_i, T].$$

Next, introduce the function

$$x(t) = \begin{cases} A & \text{for } t = 0, \\ x_0(t) & \text{for } t \in (0, T]. \end{cases}$$

Clearly, $x'(t) = x_0'(t)$ for $t \in (0, T]$. Besides,

$$x'(0) = \lim_{t \to 0^+} \frac{x(t) - x(0)}{t - 0} = \lim_{t \to 0^+} x'(t) = \lim_{t \to 0^+} x_0'(t) = B.$$

Thus, $x' \in C[0, T]$ and so $x(t)$ is a $C^1[0, T] \cap C^2(0, T]$-solution to (1.1).

The inequalities (2.11) give immediately

$$m_1 \leq x'(t) \leq M_1 \quad \text{for } t \in [0, T],$$

from where by integration from 0 to $t \in (0, T]$ we obtain the bounds for $x(t)$. $\square$

As an elementary consequence of Theorem 2.1 we obtain results guaranteeing important properties of the solutions.

Theorem 2.2. *Let $B \geq 0$ and let* (A2) *hold for $m_1 = 0$. Then problem* (1.1) *has at least one nondecreasing solution in $C^1[0, T] \cap C^2(0, T]$.*

Theorem 2.3. *Let $B > 0$ and let* (A2) *hold for $m_1 > 0$. Then problem* (1.1) *has at least one strictly increasing solution in $C^1[0, T] \cap C^2(0, T]$.*

Theorem 2.4. *Let $A > 0$ $(A = 0)$, $B \geq 0$ and let* (A2) *hold for $m_1 = 0$. Then problem* (1.1) *has at least one positive (nonnegative) nondecreasing solution in $C^1[0, T] \cap C^2(0, T]$.*

Theorem 2.5. *Let $A \geq 0, B > 0$ and let* (A2) *hold for $m_1 > 0$. Then problem* (1.1) *has at least one strictly increasing solution in $C^1[0,T] \cap C^2(0,T]$ having positive values for $t \in (0,T]$.*

3. Example

Consider the IVP

$$x'' = t^{-\frac{m}{n}} P_k(x'),$$
$$x(0) = A,\ x'(0) = B,$$

where $A \geq 0$, $B > 0$, $m, n \in N$, and the polynomial $P_k(p), k \geq 2$, has simple zeros p_1 and p_2 such that $P_k'(p_1) < 0$ and $0 < p_1 < B < p_2$.

Let $\theta > 0$ be so small that $p_1 - \theta > 0$, $p_1 + \theta < B < p_2 - \theta$ and

$$P_k(p) \neq 0 \quad \text{for } p \in [p_1 - \theta, p_1) \cup (p_1, p_1 + \theta) \cup [p_2 - \theta, p_2) \cup (p_2, p_2 + \theta].$$

Then $P_k'(p_1) < 0$ implies

$$P_k(p) > 0 \quad \text{for } p \in [p_1 - \theta, p_1) \quad \text{and} \quad P_k(p) < 0 \quad \text{for } p \in (p_1, p_1 + \theta].$$

Besides, we see easily that if

$$P_k(p) < 0 \quad \text{for } p \in [p_2 - \theta, p_2),$$

then (A2) holds for an arbitrary $T > 0$,

$$\overline{m}_1 = p_1 - \theta, \quad m_1 = p_1, M_1 = p_2 - \theta, \quad \overline{M_1} = p_2, \quad \tau = \theta/2,$$

moreover $\tilde{M}_0 = (p_2 - \theta)T + A$, and if

$$P_k(p) < 0 \quad \text{for } p \in (p_2, p_2 + \theta],$$

it is satisfied for an arbitrary $T > 0$,

$$\overline{m}_1 = p_1 - \theta, \quad m_1 = p_1, \quad M_1 = p_2, \quad \overline{M_1} = p_2 + \theta, \quad \tau = \theta/2,$$

moreover $\tilde{M}_0 = p_2 T + A$. So, it follows from Theorem 2.5 that for each $T > 0$ the considered problem has a strictly increasing solution in $C^1[0,T] \cap C^2(0,T]$ which is positive on $(0,T]$.

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