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STABILIZATION OF THE WAVE EQUATION WITH VARIABLE COEFFICIENTS AND A DYNAMICAL BOUNDARY CONTROL

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Abstract. In this article we consider the boundary stabilization of the wave equation with variable coefficients and a dynamical Neumann boundary control. The dynamics on the boundary comes from the acceleration terms which can not be ignored in some physical applications. It has been known that addition of dynamics to the boundary may change drastically the stability properties of the underlying system. In this paper by applying a boundary feedback control we obtain the exponential decay for the solutions. Our proof relies on the Geometric multiplier skills and the energy perturbed approach.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary Γ. We assume $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We consider the following wave equation with dynamical Neumann boundary condition

$$
\begin{gathered}
u_{tt} - \operatorname{div} A(x)\nabla u = 0, \quad \text{in } \Omega \times (0,\infty), \\
u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \\
m(x)u_{tt}(x,t) + \partial_{\nu_A} u(x,t) = C(t), \quad \text{on } \Gamma_1 \times (0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{gathered} \tag{1.1}
$$

where $\operatorname{div} X$ denotes the divergence of the vector field X in the Euclidean metric, $A(x) = (a_{ij}(x))$ are symmetric and positive definite matrices for all $x \in \mathbb{R}^N$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^N$. Let $\partial_{\nu_A} u = \sum_{i,j=1}^n a_{ij}(x)\partial_{x_j} u \nu_i$, where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^T$ denotes the outward unit normal vector of the boundary and $\nu_A = A\nu$. Here $C(t)$ is the boundary feedback control.

We say that equation (1.1) is with dynamical boundary conditions when $m(x) \neq 0$. In some physical applications one has to take the acceleration terms into account on the boundary. Usually in this case to describe what really happened we need the models with dynamical boundary conditions. They are not only important theoretically but also have strong backgrounds for physical applications. There are numerous of these applications in the bio-medical domain ([7, 20]) as well as in

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applications related to noise suppression and control of elastic structures ([1, 3, 18, 21]). Moreover, in this article we assume that

$$m(x) \in L^\infty(\Gamma_1); \quad m(x) \geq m_0 > 0, \quad x \in \Gamma_1.$$

Our motivating example is the hybrid wave equation

$$\begin{gathered}
u_{tt} - \Delta u = 0, \quad \text{in } \Omega \times (0, \infty),\\
u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty),\\
mu_{tt}(x,t) + \partial_\nu u(x,t) = -\partial_\nu u_t, \quad \text{on } \Gamma_1 \times (0,\infty),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega.
\end{gathered} \tag{1.2}$$

It is shown that the above system with $m > 0$ is not uniformly stable, while the case when $m = 0$ the system is exponentially stable under suitable geometrical conditions on Ω. That is to say, the dynamical terms on the boundary may change the stability property of the system. For details, see [10, 12, 17]. The purpose of this paper is to study how the dynamic boundary conditions on Γ_1 affect the decay of the system. In this article, we shall design a collocated boundary feedback to obtain the exponential stabilization of the system (1.1). We set

$$C(t) = -\beta u_t(x,t) - \gamma \partial_{\nu_A} u_t\,, \tag{1.3}$$

where the constants β and γ are positive numbers such that $\beta\gamma < m$.

Before we state and prove our results, let us first recall some works related to the problem we address. Wave equations with acceleration terms in the dynamical boundary conditions have been studied within the framework of the model

$$\begin{gathered}
u_{tt} - \Delta u + g(u_t) = f, \quad \text{in } \Omega \times (0,\infty),\\
K(u)u_{tt}(x,t) + \partial_\nu u(x,t) = C(t), \quad \text{on } \Gamma \times (0,\infty),\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega.
\end{gathered} \tag{1.4}$$

In the one dimensional case, problem (1.4) with $C(t) = h(u_t)$ has been studied in [5, 8], and with $C(t) = -\alpha u + f(u_{xt} - \alpha u_t)$ has been considered in [6]. For N-dimensional case with $N \geq 2$ [4, 9] discussed the asymptotic stabilization and the existence of the solutions, respectively. In [25], the case when $C(t)$ involves an unknown disturbance was considered, again in the one dimensional case. For equations with other type of dynamical boundary conditions, see [2, 11, 16] for wave equations with acoustic boundary conditions and [10] for wave equations with viscoelastic damping and dynamic boundary conditions acting on a surface of local reaction, and the references therein.

In equation (1.1) we adopt the feedback law given in (1.3) to obtain the exponential stabilization of the following closed loop system:

$$\begin{gathered}
u_{tt} - \operatorname{div} A(x)\nabla u = 0, \quad \text{in } \Omega \times (0,\infty),\\
u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty),\\
m(x)u_{tt}(x,t) + \partial_{\nu_A} u(x,t) = -\beta u_t(x,t) - \gamma\partial_{\nu_A} u_t, \quad \text{on } \Gamma_1 \times (0,\infty),\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad \text{in } \Omega.
\end{gathered} \tag{1.5}$$

We define

$$g = A^{-1}(x), \quad x \in \Omega$$

as a Riemannian metric on Ω and consider the couple (Ω, g) as a Riemannian manifold. Let D denote the Levi-Civita connection of the metric g. For each

$x \in \Omega$, the metric g induces an inner product on R^n_x by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|_g^2 = \langle X, X\rangle_g\,, \; X, Y \in R^n_x\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard metric of the Euclidean space R^n.

To obtain the stabilization of the problem (1.5), the following geometrical hypotheses are assumed.

Geometrical assumptions. There exists a vector field H on Riemannian manifold (Ω, g) such that the following properties hold:

(A1) $DH(\cdot,\cdot)$ is strictly positive definite on $\overline{\Omega}$: there exists a constant $\rho > 0$ such that for all $x \in \overline{\Omega}$, for all $X \in M_x$(the tangent space at x):

$$DH(X, X) \equiv \langle D_X H, X\rangle_g \geq \rho |X|^2. \tag{1.6}$$

(A2)

$$H \cdot \nu \leq 0 \quad \text{on } \Gamma_0\,. \tag{1.7}$$

Remark 1.1. For any Riemannian manifold M, the existence of such a vector field H in (A1) has been proved in [22], where some examples are given. See also [24]. For the Euclidean metric, taking the vector field $H = x - x_0$ and we have $DH(X, X) = |X|^2$, which means assumption (A1) always holds true with $\rho = 1$ for the Euclidean case.

Energy of the system (1.1)**.** Before we go to the stabilization of the system, we should first define an energy connected with the natural energy of the hybrid system. We set

$$\eta(x, t) = mu_t(x, t) + \gamma \partial_{\nu_A} u,\; x \in \Gamma_1\,.$$

Let u be a regular solution of system (1.5). Then we associate to system (1.5) the energy functional E(t) as

$$E(t) = \int_\Omega \big(u_t^2 + |\nabla_g u|_g^2\big)dx + \int_{\Gamma_1} \frac{1}{m - \beta\gamma}\eta^2 d\Gamma\,.$$

The main result of this paper reads as follows.

Theorem 1.2. *Let the geometrical assumptions* (A1) *and* (A2) *hold. Then there exist constants* $C > 0$ *and* $\omega > 0$ *such that*

$$E(t) \leq Ce^{-\omega t}E(0)\,, \quad t \geq 0\,. \tag{1.8}$$

This article is organized as follows. In the next section, we discuss the well-posedness of the nonlinear close-loop system by semigroup theory. Section 3 devotes to the proof of the exponential stability. We construct two auxiliary functions to estimate the energy, thus to obtain the main result finally.

2. Well-posedness of the closed loop system

In this section, we study well-posedness results for system (1.5) using semigroup theory. Denote $H^1_{\Gamma_0} = \{u \in H^1(\Omega)\big| u|_{\Gamma_0} = 0\}$. We consider the unknown

$$U = \big(u, v = u_t|_\Omega, \eta\big)^T\,,$$

in the state space, denoted by

$$\Upsilon := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1)\,, \tag{2.1}$$

with the norm defined by

$$\|U\|^2 = \|(u,v,\eta)^T\|^2 = \int_\Omega \big(|\nabla_g u|_g^2 + v^2\big)dx + \int_{\Gamma_1} \frac{1}{m-\beta\gamma}\eta^2 d\Gamma\,. \tag{2.2}$$

The closed-loop system (1.5) can be rewritten in the abstract form

$$\begin{gathered} U' = \mathcal{A}U,\\ U_0 = \big(u_0, u_1, \eta_0\big)^T, \end{gathered} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u\\ v\\ \eta \end{pmatrix} = \begin{pmatrix} v\\ \operatorname{div}\nabla_g u\\ -\frac{1}{\gamma}\eta + (\frac{m}{\gamma} - \beta)v(x,t) \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}) := \big\{&(u,v,\eta)^T \in H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1):\\ &\operatorname{div}\nabla_g u \in L^2(\Omega), \eta = mv|_{\Gamma_1} + \gamma\partial_{\nu_A}u\big\}\,. \end{aligned} \tag{2.4}$$

We will show that $\mathcal{A}$ generates a C_0 semigroup on Υ. Now we state the well-posedness result.

Theorem 2.1. *For any initial datum $U_0 \in \Upsilon$, there exists a unique solution $U \in C([0,\infty),\Upsilon)$ of system* (2.3)*. Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), D(\mathcal{A}))$.*

Proof. **Step 1.** We prove that $\mathcal{A}$ is dissipative. We know Υ is a Hilbert space equipped with the adequate scalar product $\langle\cdot,\cdot\rangle_\Upsilon$ and norm $\|U\|$ defined by (2.2). For $U \in D(\mathcal{A})$, a simple computation leads to

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_\Upsilon &= \frac{1}{2}\frac{d}{dt}\|U\|^2\\ &= \int_\Gamma u_t \partial_{\nu_A} u d\Gamma + \frac{1}{m-\beta\gamma}\int_{\Gamma_1} \eta\eta_t d\Gamma\\ &= \int_{\Gamma_1} u_t\partial_{\nu_A} u d\Gamma - \frac{1}{(m-\beta\gamma)\gamma}\int_{\Gamma_1}\eta^2 d\Gamma + \int_{\Gamma_1}\frac{1}{\gamma}\eta u_t(x,t)d\Gamma\\ &= \int_{\Gamma_1} u_t\partial_{\nu_A} u d\Gamma - \frac{1}{2m\gamma}\int_{\Gamma_1}\eta^2 d\Gamma\\ &\quad - \Big(\frac{1}{(m-\beta\gamma)\gamma} - \frac{1}{2m\gamma}\Big)\int_{\Gamma_1}\eta^2 d\Gamma + \int_{\Gamma_1}\frac{1}{\gamma}\eta u_t(x,t)d\Gamma\\ &= -\int_{\Gamma_1}\big(\frac{m}{2\gamma}u_t^2 + \frac{\gamma}{2m}\partial_{\nu_A}^2 u\big)d\Gamma\\ &\quad - \Big(\frac{1}{(m-\beta\gamma)\gamma} - \frac{1}{2m\gamma}\Big)\int_{\Gamma_1}\eta^2 d\Gamma + \int_{\Gamma_1}\frac{1}{\gamma}\eta u_t(x,t)d\Gamma. \end{aligned} \tag{2.5}$$

Now we handle the items in (2.5) by applying Hölder inequality

$$\int_{\Gamma_1}\frac{1}{\gamma}\eta u_t(x,t)d\Gamma \le \frac{1}{k_1 m\gamma}\int_{\Gamma_1}\eta^2 d\Gamma + \frac{k_1 m}{4\gamma}\int_{\Gamma_1} u_t^2 d\Gamma\,, \tag{2.6}$$

where $k_1 > 0$ is to be chosen later. Substituting the inequality (2.6) to (2.5) yields

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_\Upsilon &\le -\int_{\Gamma_1} \frac{\gamma}{2m}\partial^2_{\nu_A} u d\Gamma - \int_{\Gamma_1} (\frac{m}{2\gamma} - \frac{k_1 m}{4\gamma}) u_t^2 d\Gamma \\ &\quad - (\frac{1}{(m-\beta\gamma)\gamma} - \frac{1}{2m\gamma} - \frac{1}{k_1 m\gamma}) \int_{\Gamma_1} \eta^2 d\Gamma \\ &\le -\int_{\Gamma_1} \frac{\gamma}{2m}\partial^2_{\nu_A} u d\Gamma - \frac{2\beta\gamma}{m+\beta\gamma}\int_{\Gamma_1} u_t^2 d\Gamma - \frac{\beta(m+\beta\gamma)}{2m^2(m-\beta\gamma)}\int_{\Gamma_1} \eta^2 d\Gamma\,,\end{aligned} \tag{2.7}$$

by choosing $k_1 = \frac{2m}{m+\beta\gamma}$, which is negative noticing that $\beta\gamma < m$.

Step 2. We will show that $\lambda I - \mathcal{A}$ is surjective for a fixed $\lambda > 0$. Given $(\bar a, \bar b, \bar c)^T \in \Upsilon$, we seek a solution $U = (u, v, \eta)^T \in D(\mathcal{A})$ of

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} \bar a \\ \bar b \\ \bar c \end{pmatrix};$$

that is, satisfying

$$\begin{gathered} \lambda u - v = \bar a, quad \text{in } \Omega, \\ \lambda v - \operatorname{div} A(x)\nabla u = \bar b, \quad \text{in } \Omega, \\ \lambda\eta + \frac{1}{\gamma}\eta + (\beta - \frac{m}{\gamma}) v(x,t) = \bar c, \quad \text{on } \Gamma_1, \\ \eta = m v|_{\Gamma_1} + \gamma\partial_{\nu_A} u, \ \text{on } \Gamma_1\,, \end{gathered} \tag{2.8}$$

where we take t as a parameter for granted. Suppose that we have found u with the appropriate regularity, then from the equation (2.8)-1, (2.8)-3 and (2.8)-4 we have

$$v := \lambda u - \bar a\,,\ \eta = (\lambda + \frac{1}{\gamma})^{-1}\big(\bar c - (\beta - \frac{m}{\gamma})(\lambda u - \bar a)\big).$$

Now we state the process on how to get u. Eliminating v and noticing $\eta = mv|_{\Gamma_1} + \gamma\partial_{\nu_A} u(x,t)$, we find that the function u satisfies

$$\begin{gathered} \lambda^2 u - \operatorname{div} A(x)\nabla u = \lambda\bar a + \bar b, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \Gamma_0, \\ \partial_{\nu_A} u = \frac{1}{\gamma}\eta - \frac{m}{\gamma}(\lambda u - \bar a), \quad \text{on } \Gamma_1. \end{gathered} \tag{2.9}$$

We obtain a weak formulation of system (2.9) by multiplying the equation by ψ and using Green's formula

$$\begin{aligned} &\int_\Omega (\lambda^2 u\psi + \langle A(x)\nabla u, \nabla\psi\rangle) dx + \int_{\Gamma_1} \frac{\lambda(m\lambda+\beta)}{\lambda\gamma+1} u\psi \\ &= \int_\Omega (\bar b + \lambda\bar a)\psi dx + \int_{\Gamma_1} (\frac{1}{\lambda\gamma+1}\bar c + \frac{m\lambda+\beta}{\lambda\gamma+1}\bar a)\psi\,, \end{aligned} \tag{2.10}$$

for any $\psi \in H^1_{\Gamma_0}(\Omega) = \{\psi \in H^1(\Omega) | \psi|_{\Gamma_0} = 0\}$. As the left hand side of (2.10) is coercive on $H^1(\Omega)$, Lax-Milgram Theorem guarantees the existence and uniqueness of a solution $u \in H^1(\Omega)$ of (2.9).

Step 3. Finally, the well-posedness result follows from Lummer-Phillips Theorem. □

3. Exponential Stability

In this section, we show the exponential stability of the system by energy perturbed approach. Here we define two auxiliary functions

$$V_1(t) = \int_\Omega H(u)u_t dx\,,$$
$$V_2(t) = \frac{1}{2}\int_\Omega (\operatorname{div}_0 H - \rho)uu_t.$$

To estimate the functions, we need some lemmas from [24] and [13].

Lemma 3.1 ([24, Theorem 2.1])**.** *Suppose that $u(x,t)$ is a solution of the equation $u_{tt} + \operatorname{div} A\nabla u = 0$. Then*

$$\dot V_1(t) = B_1 + I_1\,,$$

where we denote the boundary term

$$B_1(\Gamma) = \int_\Gamma \partial_{\nu_A} u H(u) d\Gamma + \frac{1}{2}\int_\Gamma (u_t^2 - |\nabla_g u|_g^2) H\cdot \nu d\Gamma, \tag{3.1}$$

and the internal term

$$I_1 = -\int_\Omega D_g H(\nabla_g u, \nabla_g u)dx - \frac{1}{2}\int_\Omega (u_t^2 - |\nabla_g u|_g^2)\operatorname{div}_0 H\,dx,$$

here $\operatorname{div}_0 H$ is the divergence of H in the Euclidean metric of R^n.

Lemma 3.2 ([24, Theorem 2.2])**.** *Suppose that $u(x,t)$ is a solution of the equation $u_{tt} + \operatorname{div} A\nabla u = 0$. Then*

$$\dot V_2(t) = B_2 + I_2\,,$$

where we denote the boundary term by

$$B_2(\Gamma) = -\frac{1}{4}\int_\Gamma u^2 \partial_{\nu_A}(\operatorname{div}_0 H)d\Gamma + \frac{1}{2}\int_\Gamma (\operatorname{div}_0 H - \rho)u\partial_{\nu_A} u d\Gamma,$$

and the internal term

$$I_2 = \frac{1}{2}\int_\Omega (\operatorname{div}_0 H - \rho)(u_t^2 - |\nabla_g u|_g^2)dx + \frac{1}{4}\int_\Omega u^2 \operatorname{div}(\partial_{\nu_A}\operatorname{div}_0 H)dx\,.$$

Lemma 3.3 ([13, Lemma 7.2])**.** *Let $\varepsilon > 0$ be given small. Let u solves the problem* (1.1)*. Then*

$$\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1} |\nabla_g u|_g^2 d\Gamma dt \le C_{T,\varepsilon}\Big\{\int_0^T\int_{\Gamma_1}\big((\partial_{\nu_A} u)^2 + u_t^2\big)d\Gamma dt + \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega\times(0,T))}\Big\}\,. \tag{3.2}$$

According to Lemmas 3.1, 3.2 and 3.3 we obtain:

Lemma 3.4. *Suppose that the geometrical assumptions* (A1) *and* (A2) *hold. Let u solve* (1.1)*. Then there exist constants $c_1, c_2, c_3 > 0$ such that*

$$\frac{\rho}{2}E(t) + \dot V_1(t) + \dot V_2(t) \le \frac{\rho}{2}\int_{\Gamma_1}\frac{1}{m-\beta\gamma}\eta^2 d\Gamma + c_1\int_{\Gamma_1}(u_t^2 + |\nabla_g u|_g^2)d\Gamma + c_1\int_\Omega u^2 dx\,, \tag{3.3}$$

$$|V_1(t)| \le c_2 E(t),\ |V_2(t)| \le c_3 E(t)\,. \tag{3.4}$$

Proof. Obviously the estimate (3.4) is true. Now we prove the inequality (3.3). First we estimate the boundary terms B_1, B_2 given in Lemma 3.1 and Lemma 3.2. Since $u|_{\Gamma_0} = 0$, we have

$$\nabla_g u = \partial_{\nu_A} u \frac{\nu_A}{|\nu_A|_g^2}\,, \tag{3.5}$$

which induces

$$|\nabla_g u|_g^2 = |\partial_{\nu_A} u|_g^2 \frac{1}{|\nu_A|_g^2}. \tag{3.6}$$

Similarly we have

$$H(u) = \langle H, \nabla_g u\rangle_g = \partial_{\nu_A} u \frac{1}{|\nu_A|_g^2} H\cdot \nu. \tag{3.7}$$

Substituting the equalities (3.6) and (3.7) to (3.1) yields

$$B_1(\Gamma_0) = \frac{1}{2}\int_\Gamma (u_t^2 - |\nabla_g u|_g^2) H\cdot \nu d\Gamma \le 0\,,$$

where we notice the geometrical assumption (A2) is true. It is obvious that

$$B_1(\Gamma) = B_1(\Gamma_0) + B_1(\Gamma_1) \le c_1 \int_{\Gamma_1} (u_t^2 + |\nabla_g u|_g^2) d\Gamma\,. \tag{3.8}$$

Since $u|_{\Gamma_0} = 0$, we have $B_2(\Gamma_0) = 0$. And we have

$$B_2(\Gamma) = B_2(\Gamma_0) + B_2(\Gamma_1) \le c_1 \int_{\Gamma_1} |\nabla_g u|_g^2 d\Gamma. \tag{3.9}$$

Next, we estimate the internal terms I_1 and I_2. By the geometrical assumption (A1), we have

$$I_1 \le -\rho \int_\Omega |\nabla_g u|_g^2 dx - \frac{1}{2}\int_\Omega (u_t^2 - |\nabla_g u|_g^2) operatorname div_0 H\, dx. \tag{3.10}$$

It is obvious that

$$I_2 \le \frac{1}{2}\int_\Omega (\operatorname{div}_0 H - \rho)(u_t^2 - |\nabla_g u|_g^2) dx + c_1 \int_\Omega u^2 dx. \tag{3.11}$$

Combining the above inequalities (3.8), (3.9), (3.10) and (3.11) we complete the proof. □

The following is the observability inequality for the system (1.1).

Lemma 3.5. *Suppose that the geometrical assumptions* (A1) *and* (A2) *hold. Let u solve problem* (1.1)*. Then for any given $\varepsilon > 0$, there exists a time $T_0 > 0$ and a positive constant $C_{T,\varepsilon,\rho}$ such that*

$$E(0) \le C_{T,\varepsilon,\rho}\Big\{ \int_0^T \int_{\Gamma_1} \big(u_t^2 + (\partial_{\nu_A} u)^2 + \eta^2\big) d\Gamma dt + \|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}\Big\}, \tag{3.12}$$

for all $T > T_0$.

Proof. For any ε small enough, integrating the inequality (3.3) on the interval $(\varepsilon, T-\varepsilon)$ yields

$$\begin{aligned}
&\frac{\rho}{2}\int_\varepsilon^{T-\varepsilon} E(t)dt + V_1(T-\varepsilon) - V_1(\varepsilon) + V_2(T-\varepsilon) - V_2(\varepsilon)\\
&\le \frac{\rho}{2}\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1} \frac{1}{m-\beta\gamma}\eta^2 d\Gamma + c_1 \int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1} (u_t^2 + |\nabla_g u|_g^2) d\Gamma + c_1 \int_\varepsilon^{T-\varepsilon}\int_\Omega u^2 dx\,.
\end{aligned}$$

Then we use inequality (3.2) in Lemma 3.3 and inequality (3.4) in Lemma 3.4 to obtain

$$\begin{aligned}\int_{\varepsilon}^{T-\varepsilon} E(t)dt \le C_{T,\varepsilon,\rho}\Big\{ &\int_0^T \int_{\Gamma_1} (\eta^2 + u_t^2 + |\partial_{\nu_A} u|_g^2) d\Gamma \\ &+ \|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}\Big\} dx + c_0\big(E(T-\varepsilon) + E(\varepsilon)\big)\,,\end{aligned} \tag{3.13}$$

where the constant $C_{T,\varepsilon,\rho}$ depends on c_1, ρ, $\frac{1}{m-\beta\gamma}$, $meas(\Omega)$ and the constant $c_0 = \frac{4}{\rho}\max\{c_2, c_3\}$. Here c_2, c_3 are the constants given in inequality (3.4).

We notice that

$$\begin{aligned}
&E(0) + c_0\big(E(T-\varepsilon) + E(\varepsilon)\big) \\
&= \int_{\varepsilon}^{2c_0+\varepsilon+1} E(t)dt + \int_{\varepsilon}^{2c_0+\varepsilon+1} \big(E(0) - E(t)\big)dt \\
&\quad + c_0\big(E(\varepsilon) - E(0)\big) + c_0\big(E(T-\varepsilon) - E(0)\big) \\
&= \int_{\varepsilon}^{2c_0+\varepsilon+1} E(t)dt - \int_{\varepsilon}^{2c_0+\varepsilon+1} \big(\int_0^t \dot{E}(\tau)d\tau\big)dt + c_0\int_0^{\varepsilon} \dot{E}(\tau)d\tau \\
&\quad + c_0\int_0^{T-\varepsilon} \dot{E}(\tau)d\tau \\
&\le \int_{\varepsilon}^{2c_0+\varepsilon+1} E(t)dt + 3c_4\int_0^{\max\{T-\varepsilon, 2c_0+\varepsilon+1\}} \int_{\Gamma_1} \big(u_t^2 + (\partial_{\nu_A} u)^2 + \eta^2\big) d\Gamma dt\,,
\end{aligned} \tag{3.14}$$

where we used the following inequality known from (2.7),

$$\dot{E}(t) = \langle \mathcal{A}U, U\rangle_{\Upsilon} \le c_4 \int_{\Gamma_1} \big(u_t^2 + (\partial_{\nu_A} u)^2 + \eta^2\big) d\Gamma dt\,.$$

Now we shall take $T_0 = 2c_0 + 2\varepsilon + 1$ to guarantee that $T - \varepsilon > 2c_0 + \varepsilon + 1$, for all $T > T_0$. Substituting (3.14) in (3.13) completes the proof. $\square$

In what follows we use the compactness-uniqueness argument to absorb the lower order term in (3.12). We list the lemma and omit the proof, which could be found in [14, 15, 19, 23, 26] and many others.

Lemma 3.6. *Suppose that the geometrical assumptions* (A1) *and* (A2) *hold. Let u solve problem* (1.1). *Then for any $T > T_0$, there exists a positive constant C depending on T, ε, ρ,* meas(Ω) *such that*

$$E(0) \le C\int_0^T \int_{\Gamma_1} \big(u_t^2 + (\partial_{\nu_A} u)^2 + \eta^2\big) d\Gamma dt\,.$$

Proof of Theorem 1.2. From (2.7) we know that

$$\begin{aligned}
-\dot{E}(t) &= -\langle \mathcal{A}U, U\rangle_{\Upsilon} \\
&\ge \int_{\Gamma_1} \frac{\gamma}{2m}\partial_{\nu_A}^2 u d\Gamma + \frac{2\beta\gamma}{m+\beta\gamma}\int_{\Gamma_1} u_t^2 d\Gamma + \frac{\beta(m+\beta\gamma)}{2m^2(m-\beta\gamma)}\int_{\Gamma_1} \eta^2 d\Gamma \\
&\ge c_5\int_{\Gamma_1} \big(u_t^2 + (\partial_{\nu_A} u)^2 + \eta^2\big) d\Gamma dt\,,
\end{aligned} \tag{3.15}$$

where

$$c_5 = \min\{\frac{\gamma}{2m}, \frac{2\beta\gamma}{m+\beta\gamma}, \frac{\beta(m+\beta\gamma)}{2m^2(m-\beta\gamma)}\}.$$

By Lemma 3.6 and the above inequality (3.15), we have, that for all $T > T_0$,

$$E(0) \le C \int_0^T \int_{\Gamma_1} \left(u_t^2 + (\partial_{\nu_A} u)^2 + \eta^2\right) d\Gamma dt \le -\frac{C}{c_5} \int_0^T \dot{E}(t) = -\frac{C}{c_5}(E(T) - E(0)),$$

which yields

$$E(T) \le \frac{C - c_5}{C} E(0).$$

The exponential decay result (1.8) follows from the above inequality. □

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