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GENERALIZED NONLINEAR PICONE'S IDENTITY FOR THE P-LAPLACIAN AND ITS APPLICATIONS

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Abstract. In this article we derive a generalized version of nonlinear Picone's identity for the p-Laplacian. We use this identity to obtain a Hardy-type inequality and a Sturm comparison result. We also establish the relationship between the components of the solution of nonlinear elliptic systems.

1. Introduction

In recent years, qualitative problems related to Picone's identity, Sturm comparison theorem and the relationship between the components of the solution of elliptic systems have been extensively studied. We know that Picone's identity plays an important role in the qualitative theory of elliptic equations.

Let u and v be differentiable functions in a domain $\Omega \subset \mathbb{R}^n$ and $v(x) \neq 0$ in Ω. The classical Picone's identity reads

$$\|\nabla u\|^2 - \langle \nabla v, \nabla\Big(\frac{u^2}{v}\Big)\rangle = \|\nabla u - \frac{u}{v}\nabla v\|^2, \tag{1.1}$$

where $\langle , \rangle$, $\| \cdot \|$, and ∇ denote the inner product, the Euclidean norm, and the gradient in $\mathbb{R}^n$, respectively [17, 18].

Allegretto and Huang extended (1.1) for the nonlinear p-Laplace operator $\Delta_p v := \operatorname{div}\big(\|\nabla v\|^{p-2}\nabla v\big)$ with $p > 1$ as follows.

Theorem 1.1 ([1])**.** *Let $u \geq 0$ and $v > 0$ be differentiable functions. Denote*

$$L(u,v) = \|\nabla u\|^p + (p-1)\frac{u^p}{v^p}\|\nabla v\|^p - p\frac{u^{p-1}}{v^{p-1}}\langle \|\nabla v\|^{p-2}\nabla v, \nabla u\rangle$$

and

$$R(u,v) = \|\nabla u\|^p - \langle \|\nabla v\|^{p-2}\nabla v, \nabla\Big(\frac{u^p}{v^{p-1}}\Big)\rangle. \tag{1.2}$$

Then $L(u,v) = R(u,v)$. Moreover, $L(u,v) \geq 0$ and $L(u,v) = 0$ in Ω if and only if $\nabla\big(\frac{u}{v}\big) = 0$ in Ω.

By using Theorem 1.1, Allegretto and Huang, obtained a wide range of applications of the eigenvalue problem

$$\begin{gathered} -\Delta_p v = \lambda g(x)|v|^{p-2}v, \quad \text{in } \Omega \\ v = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

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where g is a weight function.

In a recent paper Tyagi [14] proved a generalized version of nonlinear Picone's identity for the problem

$$\begin{gathered} -\Delta v = a(x)f(v) \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

where $a \in L^\infty(\Omega)$. Tyagi's result is the following.

Theorem 1.2 ([14]). *Let v be a differentiable function in Ω such that $v \neq 0$ in Ω and u be a nonconstant differentiable function in Ω. Let $f(y) \neq 0$, for $0 \neq y \in R$ and suppose that there exists $\alpha > 0$ such that $f'(y) \geq \frac{1}{\alpha}$, for $0 \neq y \in R$. Denote*

$$L(u,v) = \alpha\|\nabla u\|^2 - \frac{\|\nabla u\|^2}{f'(v)} + \|\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}}\|^2, \tag{1.5}$$

$$R(u,v) = \alpha\|\nabla u\|^2 - \langle \nabla\Big(\frac{u^2}{f(v)}\Big), \nabla v\rangle. \tag{1.6}$$

Then $L(u,v) = R(u,v)$. Moreover, $L(u,v) \geq 0$ and $L(u,v) = 0$ in Ω if and only if $f'(v) = \frac{1}{\alpha}$ and $u = c_1 v + c_2$ for some arbitrary constants c_1, c_2.

Bal [3] extended the nonlinear Picone's identity by Tyagi to include the p-Laplace operator $\Delta_p v$, as stated in the following theorem.

Theorem 1.3 ([3]). *Let $v > 0$ and $u \geq 0$ be two nonconstant differentiable functions in Ω. Also let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 function and $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$, $p > 1$ for all y. Define*

$$L(u,v) = \|\nabla u\|^p - \frac{pu^{p-1}}{f(v)}\langle \|\nabla v\|^{p-2}\nabla u, \nabla v\rangle + \frac{u^p f'(v)}{f^2(v)}\|\nabla v\|^p \tag{1.7}$$

$$R(u,v) = \|\nabla u\|^p - \langle \|\nabla u\|^{p-2}\nabla\Big(\frac{u^p}{f(v)}\Big), \nabla v\rangle.$$

Then $L(u,v) = R(u,v) \geq 0$. Moreover, $L(u,v) = 0$ in Ω if and only if $f'(v) = (p-1)\big(f(v)\big)^{\frac{p-2}{p-1}}$ and $\nabla\big(\frac{u}{v}\big) = 0$ in Ω.

In the late 1990's several authors derived Picone-type identities for a variety of equations which include the p-Laplace operator and gave various applications. (See for example [5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein).

In this article, motivated by the ideas in [1, 3, 4, 9, 11, 14], we obtain a new nonlinear analogue of (1.1), and give some applications which extend and improve Tyagi's [14] and Bal's [3] results.

2. Nonlinear analogue of Picone's identity

Define $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$ and $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in \mathbb{R}^n$, for $\alpha > 0$. We begin with the following lemma.

Lemma 2.1 ([9]). *For $X, Y \in \mathbb{R}^n$, we have*

$$F(X,Y) := \langle X, \Phi(X)\rangle + \alpha\langle Y, \Phi(Y)\rangle - (\alpha+1)\langle X, \Phi(Y)\rangle \geq 0, \tag{2.1}$$

where the equality holds if and only if $X = Y$.

Next we present a nonlinear Analogue of Picone's identity.

Theorem 2.2. *Assume $f \in C^1(R,R)$ and $f(v) \neq 0$ for all $0 \neq v \in R$. Let v be a differentiable function in Ω such that $v \neq 0$ in Ω and u be a nonconstant differentiable function in Ω. Define*

$$L(u,v) = \frac{|f(v)|^{\alpha-1}}{(f'(v))^{\alpha}} F\Big(\frac{u\nabla v f'(v)}{f(v)}, \alpha \nabla u\Big),$$
$$R(u,v) = \frac{|f(v)|^{\alpha-1}}{(f'(v))^{\alpha}} \|\alpha \nabla u\|^{\alpha+1} - \alpha \langle \nabla\Big(\frac{\varphi(u)u}{f(v)}\Big), \Phi(\nabla v)\rangle.$$

Then $L(u,v) = R(u,v)$. Moreover $L(u,v) \geq 0$ and $L(u,v) = 0$ in Ω if and only if $|u|^{\alpha} = |Kf(v)|$ where $K \neq 0$ is a constant.

Proof. Expanding $R(u,v)$ by direct calculation we obtain $L(u,v)$. From Lemma (2.1), $L(u,v) \geq 0$. $L(u,v) = 0$ in Ω if and only if

$$\frac{u\nabla v f'(v)}{f(v)} = \alpha \nabla u \quad \text{or} \quad \nabla\Big(\frac{|u|^{\alpha}}{|f(v)|}\Big) = 0 \quad \text{in } G.$$

Since u is a nonconstant continuous function in Ω, there exists a nonzero constant K such that $|u|^{\alpha} = |Kf(v)|$. $\square$

Note that when $\alpha = 1$, $p(x) = 1$ and $f(v) = v$, we obtain the classical Picone's identity (1.1).

3. Applications

Picone's identity plays a significant role in eigenvalue problems, establishing Sturmian comparison and oscillation theorems for partial differential equations, deriving Hardy-Sobolev inequalities, determining the Morse index, proving the nonexistence of the positive solution, etc. In this section, motivated by the ideas in [1, 3, 11, 14], we will give some applications of Theorem 2.2 in the nonlinear framework.

For the rest of this paper, we impose the following hypotheses on f:

- (H1) $f \in C^1(R,R)$ and there exist $\alpha_0, \alpha_1 \in (0,\infty)$ such that $\alpha_0 |v|^{\alpha-1} \leq f'(v)$ and $\alpha_1 |v|^{\alpha-1} \geq f(v) \neq 0$ for all $0 \neq v \in R$;
- (H2) $f \in C^1(R,R)$ with $f(v) \neq 0$ for all $0 \neq v \in R$ and there exists $k > 0$ such that $f'(v) \geq k|f(v)|^{\frac{\alpha-1}{\alpha}}$ for all $v \in R$.

Remark 3.1. Assumption (H1) motivates us to study the nonlinearities of the form

$$f(v) = |v|^{\alpha-1} v (1 \mp \text{a nonlinear part})$$

where the nonlinear part is decaying at ∞.

Assumption (H2) is a common condition in the literature for half-linear equations.

Hardy-type inequality. The following theorem can be applied to prove Hardy-type inequality using the same method as in [1].

Theorem 3.2. *Assume* (H1) *(or* (H2)*) holds. Also assume that there is a strictly positive $v \in W^{1,\alpha+1}(\Omega)$ satisfying*

$$-\Delta_{\alpha} v \geq \lambda g(x) f(v) \tag{3.1}$$

for some $\lambda > 0$ and nonnegative continuous function g. Then for $u \in W_0^{1,\alpha+1}(\Omega)$, we have
$$\int_\Omega \|\nabla u\|^{\alpha+1} dx \geq \lambda \int_\Omega g(x)|u|^{\alpha+1} dx.$$

Proof. Let $\Omega_0 \subset \Omega$, Ω_0 be compact and (H1) hold. Take $\Phi_1 \in W_0^{1,\alpha+1}(\Omega)$. Then we have
$$\begin{aligned}
0 &\leq \int_{\Omega_0} L(\Phi_1, v) dx \\
&\leq \int_\Omega L(\Phi_1, v) dx = \int_\Omega R(\Phi_1, v) dx \\
&= \int_\Omega \{ \frac{|f(v)|^{\alpha-1}}{f'(v)^\alpha} \|\alpha \nabla \Phi_1\|^{\alpha+1} - \alpha \langle \nabla \Big(\frac{|\Phi_1|^{\alpha+1}}{f(v)} \Big), \|\nabla v\|^{\alpha-1} \nabla v \rangle \} dx \\
&= \int_\Omega \frac{|f(v)|^{\alpha-1}}{f'(v)^\alpha} \|\alpha \nabla \Phi_1\|^{\alpha+1} \, dx + \alpha \int_\Omega \frac{|\Phi_1|^{\alpha+1}}{f(v)} \Delta_\alpha v \, dx.
\end{aligned}$$
Choosing $c_1 = \max\{ \big(\frac{\alpha_0}{\alpha_1 \alpha}\big)^\alpha \alpha_1, \alpha \}$ and using (3.1), we have
$$0 \leq c_1 \int_\Omega \|\nabla \Phi_1\|^{\alpha+1} dx - \lambda \int_\Omega g(x)|\Phi_1|^{\alpha+1} dx.$$
Letting $\varphi = u$, this completes the proof.

Under the hypothesis (H2), the proof is the same, but we chose $c_1^* = \max\{k^{-\alpha}, \alpha\}$ instead of c_1. □

Note that by using (H1) and (H2), the above theorem improves [3, Theorem 3.1].

Sturmian comparison results. Comparison results always play an important role in the qualitative study of partial differential equations. Now, we give the following nonlinear version of the Sturm comparison theorem which can easily be proven by means of the Picone's identity given in Theorem 2.2.

Theorem 3.3. *Let* (H1) *(or* (H2)*) hold. Suppose that g_1 and g_2 are two weight functions which satisfy $g_1(x) \leq g_2(x)$, $x \in \Omega$ with $g_1(x) \not\equiv g_2(x)$ in Ω. If there is a positive solution of*
$$-\Delta_\alpha u = g_1(x) f(u) \tag{3.2}$$
in Ω such that $u = 0$ on $\partial\Omega$, then any solution of the equation
$$-\Delta_\alpha v = g_2(x) f(v), \quad x \in \Omega \tag{3.3}$$
must change sign in Ω.

Proof. Suppose to the contrary that the conclusion of the theorem is not true. Let us assume $v > 0$ in Ω and (H1) holds. Then an easy calculation shows that
$$\begin{aligned}
0 &\leq \int_\Omega L(u, v) dx = \int_\Omega R(u, v) dx \\
&= \int_\Omega \{ \frac{|f(v)|^{\alpha-1}}{f'(v)^\alpha} \|\alpha \nabla u\|^{\alpha+1} - \alpha \langle \nabla \Big(\frac{\|u\|^{\alpha+1}}{f(v)} \Big), \|\nabla v\|^{\alpha-1} \nabla v \rangle \} dx
\end{aligned}$$
Choosing $c_1 = \max\{ \big(\frac{\alpha_0}{\alpha_1 \alpha}\big)^\alpha \alpha_1, \alpha \}$ and using (3.3) we obtain
$$0 \leq c_1 \int_\Omega \{ \|\nabla u\|^{\alpha+1} - g_2(x) \|u\|^{\alpha+1} \} dx.$$

Using (H1) and (3.2) and taking $c_2 = c_1 \max\{\alpha, 1\}$ the above inequality takes the form

$$0 \le \int_\Omega L(u,v)dx \le c_2 \int_\Omega (g_1(x) - g_2(x))\|u\|^{\alpha+1} dx \le 0.$$

Consequently we have $u^\alpha = |Kf(v)|$. But this is impossible since $g_1(x) \not\equiv g_2(x)$ in Ω. Under the hypothesis (H2), c_1 is replaced with c_1^*. This completes the proof. $\square$

We should note that [3, Theorem 3.3] can not be applied here. In [3], $f(v) > 0$ for $v > 0$, but there is no condition on $f(v)$ for $v < 0$, hence the proof cannot be completed. From Theorem 3.3, we can obtain the following result which is the corrected form of [3, Theorem 3.3].

Corollary 3.4. *Let* (H2) *hold. Also let g_1 and g_2 be the weight functions that satisfy $g_1(x) \le g_2(x)$, $x \in \Omega$ with $g_1(x) \not\equiv g_2(x)$ in Ω. If there is a positive solution of*

$$\begin{gathered} -\Delta_\alpha u = g_1(x)|u|^{\alpha-1}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

then any solution of (3.3) *must change sign in Ω.*

Coupled nonlinear elliptic systems. Now we establish a relationship between the components of the solution of the nonlinear elliptic systems. We begin with the problem

$$\begin{gathered} \Delta_\alpha u = v \quad \text{in } \Omega \\ -\Delta\Big(\frac{|u|^{\alpha+1}}{f(v)}\Big) = u \quad \text{in } \Omega \\ u \ne 0, \quad v \ne 0 \quad \text{in } \Omega \\ u = 0 = v \quad \text{on } \partial\Omega \end{gathered} \tag{3.4}$$

Theorem 3.5. *Let $(u,v) \in W_0^{1,\alpha+1}(\Omega) \times W_0^{1,\alpha+1}(\Omega)$ be a weak solution of* (3.4) *such that $v \ne 0$ in Ω and f satisfy* (H1) *(or* (H2)*). Then there exists a nonzero constant K such that $|u(x)|^\alpha = |Kf(v)|$.*

Proof. Assume (H1) holds. Since $(u,v) \in W_0^{1,\alpha+1}(\Omega) \times W_0^{1,\alpha+1}(\Omega)$ is a weak solution of (3.4), we have

$$\begin{gathered} \int_\Omega \langle \|\nabla u\|^{\alpha-1}\nabla u, \nabla\Phi_2\rangle dx = \int_\Omega v\Phi_2 dx, \\ \int_\Omega \langle \nabla\Big(\frac{\|u\|^{\alpha+1}}{f(v)}\Big)|\nabla v|^{\alpha-1}, \nabla\Phi_3\rangle dx = \int_\Omega v\Phi_3 dx \end{gathered} \tag{3.5}$$

for any $\Phi_2, \Phi_3 \in W_0^{1,\alpha+1}(\Omega)$.

Let us take $\Phi_2 = u$ and $\Phi_3 = v$ as test functions in (3.5). Then we obtain

$$\begin{gathered} \int_\Omega \langle \|\nabla u\|^{\alpha+1} dx = \int_\Omega vu dx, \\ \int_\Omega \langle \nabla\Big(\frac{\|u\|^{\alpha+1}}{f(v)}\Big), \|\nabla v\|^{\alpha-1}\nabla v\rangle dx = \int_\Omega uv dx. \end{gathered} \tag{3.6}$$

From Theorem 2.2 and (3.6) we can see that

$$0 = \int_\Omega \{\|\nabla u\|^{\alpha+1} - \langle \nabla\Big(\frac{\|u\|^{\alpha+1}}{f(v)}\Big), \|\nabla v\|^{\alpha-1}\nabla v\rangle\} dx$$

$$\geq c_1^{-1} \int_\Omega R(u,v)dx = c_1^{-1} \int_\Omega L(u,v)dx \geq 0.$$

Therefore $L(u,v) = 0$ and hence the conclusion follows by an application of Theorem (2.2). Under the hypothesis (H2), we c_1 with c_1^* and the proof is complete. □

Note that Theorem 3.5 is an extension of [14, Theorem 2.3].

Next, we consider the nonlinear system of elliptic equations

$$\begin{gathered} -\Delta_\alpha u = f^{\beta-1}(v) \quad \text{in } \Omega,\ \beta \in Z \\ -\Delta_\alpha v = \frac{f^\beta(v)}{u^\alpha} \quad \text{in } \Omega \\ u > 0, \quad v \neq 0 \quad \text{in } \Omega \\ u = 0 = v \quad \text{on } \partial\Omega \end{gathered} \tag{3.7}$$

Note that many problems in chemical heterogeneous catalyst dynamics are governed by system of nonlinear elliptic equations. For the applications of such nonlinear system of elliptic equations, we refer the reader to [2, 3] and the references therein.

Theorem 3.6. *Let $(u,v) \in W_0^{1,\alpha+1}(\Omega) \times W_0^{1,\alpha+1}(\Omega)$ be a weak solution of* (3.7) *with the first component $u > 0$ in Ω and f satisfy the hypothesis (H1) (or (H2)). Then there exists a nonzero constant K such that $u^\alpha = |Kf(v)|$.*

Proof. Since (u,v) is a weak solution of (3.7) for any $\Phi_4, \Phi_5 \in W_0^{1,\alpha+1}(\Omega)$, we have

$$\begin{gathered} \int_\Omega \langle \|\nabla u\|^{\alpha-1}\nabla u, \nabla\Phi_4\rangle dx = \int_\Omega f^{\beta-1}(v)\Phi_4 dx \\ \int_\Omega \langle \|\nabla u\|^{\alpha-1}\nabla u, \nabla\Phi_5\rangle dx = \int_\Omega \frac{f^\beta(v)}{u^\alpha}\Phi_5 dx. \end{gathered} \tag{3.8}$$

Choosing $\Phi_4 = u$ and $\Phi_5 = \frac{u^{\alpha+1}}{f(v)}$ in (3.8), we obtain

$$\int_\Omega \|\nabla u\|^{\alpha+1} = \int_\Omega f^{\beta-1}(v)u\, dx = \int_\Omega \langle \|\nabla v\|^{\alpha-1}\nabla v, \nabla\Big(\frac{u^{\alpha+1}}{f(v)}\Big)\rangle dx. \tag{3.9}$$

Using hypothesis (H1) and definition of $R(u,v)$ and (3.9) we obtain

$$0 \leq \int_\Omega \{R(u,v)dx \leq c_1 \int_\Omega \|\nabla u\|^{\alpha+1} - \langle \|\nabla v\|^{\alpha-1}\nabla v, \nabla\Big(\frac{u^{\alpha+1}}{f(v)}\Big)\rangle\}dx = 0$$

Note that we can replace c_1 with c_1^*, if we use the hypothesis (H2). From the above inequality, we have $R(u,v) = 0$ in Ω. By Theorem 2.2, $0 = R(u,v) = L(u,v)$ in Ω and $L(u,v) = 0$ in Ω if and only if $u^\alpha = |Kf(v)|$, where $K \neq 0$ is a constant. □

Remark 3.7. *If in problem* (3.7) *we take $\beta = 2$, $v > 0$ and $f(v) > 0$, using hypothesis (H2), we obtain the* [3, problem (3.4)]. *So our Theorem 3.3 generalizes* [3, Theorem 3.4].

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