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EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR P-LAPLACIAN STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

D. D. HAI

Abstract. We prove the existence of positive solutions of the Sturm-Liouville boundary value problem

$$-(r(t)\phi(u'))' = \lambda g(t)f(t,u), \quad t \in (0,1),$$
$$au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0,$$

where $\phi(u') = |u'|^{p-2}u'$, $p > 1$, $f : (0,1) \times (0,\infty) \to \mathbb{R}$ satisfies a p-sublinear condition and is allowed to be singular at $u = 0$ with semipositone structure. Our results extend previously known results in the literature.

1. Introduction

We consider the boundary-value problem

$$\begin{gathered} -(r(t)\phi(u'))' = \lambda g(t)f(t,u), \quad t \in (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0, \end{gathered} \tag{1.1}$$

where $\phi(u') = |u'|^{p-2}u'$, $p > 1$, a, b, c, d are nonnegative constants with $ac+ad+bc > 0$, $f : (0,1) \times (0,\infty) \to \mathbb{R}$ is allowed to be singular at $u = 0$, and λ is a positive parameter.

When $p = 2$ and $f : [0,1] \times [0,\infty) \to \mathbb{R}$ is continuous, Yang and Zhou [13] prove the existence of a positive solution to (1.1) under the assumption

$$\lim_{u\to\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u} < \frac{\lambda_1}{\lambda} < \lim_{u\to 0^+} \inf_{t\in[0,1]} \frac{f(t,u)}{u},$$

where $\lambda_1 > 0$ denotes the first eigenvalue of $-(r(t)u')' = \lambda g(t)u$ in $(0,1)$ with Sturm-Liouville boundary conditions. Their result allows $\lim_{u\to\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u} = -\infty$, which complements previous existence results in [1, 4, 7, 8, 9, 10, 12, 14].

In this article, we shall extend the result in [13] to the general case $p > 1$ and also allow f to be singular at $u = 0$. We also establish the existence of a positive solution to (1.1) for λ large allowing $\lim_{u\to 0^+} \inf_{t\in(0,1)} f(t,u)/u^{p-1} = -\infty$ and $\lim_{u\to\infty} \inf_{t\in(0,1)} f(t,u) = 0$, which does not seem to have been considered in the literature even when $p = 2$. Note that the approach in [13] depends on the Green function and can not apply to the nonlinear case $p > 1$ or the case when f is

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singular at $u = 0$. Our approach depends on a new sub- and super solutions type argument and comparison principle.

Let g satisfy condition (A2) below. Then the eigenvalue problem $-(r(t)\phi(u'))' = \lambda g(t)\phi(u)$ in $(0,1)$ with the Sturm-Liouville boundary conditions in (1.1) has a positive first eigenvalue λ_1 with corresponding positive eigenfunctions (see e.g. [3, 11]).

We shall make the following assumptions:

(A1) $r : [0,1] \to (0,\infty)$ and $f : (0,1) \times (0,\infty) \to \mathbb{R}$ are continuous.

(A2) $g \in L^1(0,1)$ with $g \geq 0, g \not\equiv 0$ and there exists a constant $\gamma \geq 0$ such that
$$\int_0^1 \frac{g(t)}{q^\gamma(t)} dt < \infty,$$
where $q(t) = \min(b + at, d + c(1-t))$.

(A3) For each $r > 0$, there exists a constant $K_r > 0$ such that
$$|f(t,u)| \leq \frac{K_r}{u^\gamma}$$
for $t \in (0,1), u \in (0,r]$, where γ is defined in (A2).

(A4) $\lim_{u\to\infty} \sup \frac{f(t,u)}{\phi(u)} < \frac{\lambda_1}{\lambda} < \lim_{u\to 0^+} \inf \frac{f(t,u)}{\phi(u)}$, where the limits are uniform in $t \in (0,1)$.

(A5) $\lim_{u\to\infty} \sup \frac{f(t,u)}{\phi(u)} < \frac{\lambda_1}{\lambda}$ uniformly in $t \in (0,1)$.

(A6) There exist positive constants A, L such that
$$f(t,u) \geq \frac{L}{u^\gamma}$$
for $t \in (0,1)$ and $u \geq A$.

By a solution of (1.1), we mean a function $u \in C^1[0,1]$ with $r(t)\phi(u')$ absolutely continuous on $[0,1]$ and satisfying (1.1).

Our main results read as follows:

Theorem 1.1. *Let* (A1)–(A4) *hold. Then* (1.1) *has a positive solution* u *with* $\inf_{(0,1)}(u/q) > 0$.

Theorem 1.2. *Let* (A1)–(A3), (A5), (A6) *hold. Then there exists a constant* $\lambda_0 > 0$ *such that for* $\lambda > \lambda_0$*, Equation* (1.1) *has a positive solution* u_λ *with* $\inf_{(0,1)}(u_\lambda/q) \to \infty$ *as* $\lambda \to \infty$.

Let $\bar{\lambda} < \lambda_1$ and consider the problem
$$\begin{gathered} -(r(t)\phi(u'))' - \bar{\lambda} g(t)\phi(u) = \lambda g(t) f(t,u), \quad t \in (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0, cu(1) + d\phi^{-1}(r(1))u'(1) = 0. \end{gathered} \tag{1.2}$$
Then, as an immediate consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.3. *Let* (A1)–(A3) *hold and suppose that*
$$\lim_{u\to\infty} \sup \frac{f(t,u)}{\phi(u)} < \frac{\lambda_1 - \bar{\lambda}}{\lambda} < \lim_{u\to 0^+} \inf \frac{f(t,u)}{\phi(u)}.$$
Then (1.2) *has a positive solution.*

Remark 1.4. When $p = 2$ and $f : [0,1] \times [0,\infty) \to \mathbb{R}$ is continuous, [13, Theorem 3.1] follows from Theorem 1.1 with $\gamma = 0$.

Example 1.5. Let $g(t) \equiv 1 \equiv r(t)$ and consider the BVP

$$\begin{gathered} -(\phi(u'))' = \lambda f(t,u), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{1.3}$$

Note that $\lambda_1 = \pi_p^p$, where

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}$$

is the first eigenvalue of $-(\phi(u'))'$ with zero boundary conditions (see [5, 6]).

(i) Let $f(t,u) = u^{p-1}\big(\frac{e^t}{u^\gamma} - u^\beta\big)$, where $\gamma \in [0,1)$, and $\beta > 0$. Suppose $\lambda > \lambda_1$ if $\gamma = 0$, and λ is any positive constant if $\gamma > 0$. Then (A1)–(A4) hold and therefore Theorem 1.1 gives the existence of a positive solution to (1.3).

(ii) Let $f(t,u) = -\frac{1}{u^\gamma} + \frac{1}{u^\beta}$, where $0 < \beta < \gamma < 1$. Then it is easy to see that the assumptions of Theorem 1.2 are satisfied and therefore (1.3) has a positive solution for λ large. Note that since $\lim_{u\to 0^+} \inf_{t\in(0,1)} \frac{f(t,u)}{u^{p-1}} = -\infty$ and $\lim_{u\to\infty} \inf_{t\in(0,1)} f(t,u) = 0$, the results in [1, 4, 7, 8, 9, 10, 12, 13, 14] do not apply here.

(iii) Let $f(t,u) = (1-u^{p-1})\cos t$. Then

$$\lim_{u\to\infty} \sup \frac{f(t,u)}{\phi(u)} < 0 \quad \text{and} \quad \lim_{u\to 0^+} \inf \frac{f(t,u)}{\phi(u)} = \infty$$

uniformly in $t \in (0,1)$ and so (1.2) has a positive solution for all $\lambda > 0$, by Corollary 1.3.

2. Preliminaries

We shall denote the norms in $C^1[0,1]$ and $L^q(0,1)$ by $|\cdot|_1$ and $\|\cdot\|_q$ respectively. Here $|u|_1 = \max(\|u\|_\infty, \|u'\|_\infty)$. We first recall the following results in [8].

Lemma 2.1. *Let $h \in L^1(0,1)$. Then the problem*

$$\begin{gathered} -(r(t)\phi(u'))' = h, \quad t \in (0,1) \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0 \end{gathered}$$

has a unique solution $u = Sh \in C^1[0,1]$. Furthermore, S is completely continuous and there exists a constant $m > 0$ such that

$$|u|_1 \le m\phi^{-1}(\|h\|_1).$$

Lemma 2.2. *Suppose $u \in C^1[0,1]$ satisfies*

$$\begin{gathered} -(r(t)\phi(u'))' \ge 0, \quad t \in (0,1) \\ au(0) - b\phi^{-1}(r(0))u'(0) \ge 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) \ge 0. \end{gathered}$$

Then there exists a constant $m_0 > 0$ independent of u such that

$$u(t) \ge m_0\|u\|_\infty q(t)$$

for $t \in [0,1]$, where q is defined by (A2).

Remark 2.3. Lemma 2.2 is a special case of [8, Lemma 3.4] when $h = 0$. Note that the proof of [8, Lemma 3.4] is incorrect for $1 < p < 2$ when $h \not\equiv 0$ since it uses the inequality

$$|\phi^{-1}(x) - \phi^{-1}(y)| \le 2\phi^{-1}(|x-y|) \quad \text{for all } x, y \in \mathbb{R},$$

which is not true when $1 < p < 2$. However, when $h = 0$, this inequality is not needed in [8, Proof of Lemma 3.4], which guarantees the validity of Lemma 2.2.

Lemma 2.4. *There exists a constant $k > 0$ such that $|u| \le k|u|_1 q$ in $[0,1]$ for all $u \in C^1[0,1]$ satisfying the Sturm-Liouville boundary conditions in (1.1).*

Proof. Let $u \in C^1[0,1]$. Then, if $b > 0$,
$$u(t) = u(0) + \int_0^t u' \le 2|u|_1 \le \frac{2}{b}|u|_1(b+at)$$
for $t \in [0,1]$, while if $b = 0$ then $a > 0$, this implies $u(0) = 0$ and $u(t) \le |u|_1 t$ for $t \in [0,1]$. Hence
$$u(t) \le k_0|u|_1(b+at), \tag{2.1}$$
for $t \in [0,1]$, where $k_0 = 2/b$ if $b > 0$, and $1/a$ if $b = 0$. Similarly, using
$$u(t) = u(1) - \int_t^1 u',$$
we obtain
$$u(t) \le k_1|u|_1(d + c(1-t)) \tag{2.2}$$
for $t \in [0,1]$, where $k_1 = 2/d$ if $d > 0$, and $1/c$ if $d = 0$.

Combining (2.1) and (2.2), we see that $u \le k|u|_1 q$ in $(0,1)$, where $k = \max(k_0, k_1)$. By replacing u by $-u$, we see that Lemma 2.4 holds. □

Lemma 2.5. *Let $h_0, h_1 \in L^1(0,1)$. Suppose $u_0, u_1 \in C^1[0,1]$ satisfy*
$$-(r(t)\phi(u_i'))' = h_i, \quad t \in (0,1),$$
$$au_i(0) - b\phi^{-1}(r(0))u_i'(0) = 0, \quad cu_i(1) + d\phi^{-1}(r(1))u_i'(1) = 0,$$
for $i = 0,1$. Then there exists a constant $M_0 > 0$ depending on p, a, b, c, d, and C such that
$$|u_1 - u_0|_1 \le M_0 \max\{\|h_1 - h_0\|_1, \|h_1 - h_0\|_1^{\frac{1}{p-1}}\}, \tag{2.3}$$
where $C > 0$ is such that $\|h_i\|_1 < C$ for $i = 0,1$.

Proof. By integrating, we obtain
$$u_i(t) = C_i + \int_0^t \phi^{-1}\Big(\frac{D_i - \int_0^s h_i}{r(s)}\Big) ds \tag{2.4}$$
for $i = 0,1$, where C_i, D_i are constants satisfying
$$aC_i - b\phi^{-1}(D_i) = 0,$$
$$c\Big(C_i + \int_0^1 \phi^{-1}\big(\frac{D_i - \int_0^s h_i}{r(s)}\big)ds\Big) + d\phi^{-1}\big(D_i - \int_0^1 h_i\big) = 0.$$
Suppose first that $a = 0$. Then $b, c > 0$, $D_i = 0$, and
$$C_i = \frac{d}{c}\phi^{-1}\Big(\int_0^1 h_i\Big) + \int_0^1 \phi^{-1}\Big(\frac{\int_0^s h_i}{r(s)}\Big)ds,$$
and so
$$u_i(t) = \frac{d}{c}\phi^{-1}\Big(\int_0^1 h_i\Big) + \int_t^1 \phi^{-1}\Big(\frac{\int_0^s h_i}{r(s)}\Big)ds.$$
For $p \ge 2$, using the inequality
$$|\phi^{-1}(x) - \phi^{-1}(y)| \le 2\phi^{-1}(|x-y|) \quad \text{for } x, y \in \mathbb{R},$$

we obtain

$$\max\{|u_1(t)-u_0(t)|\ ,|u_1'(t)-u_0'(t)|\}\le M_1\|h_1-h_0\|_1^{\frac{1}{p-1}}, \tag{2.5}$$

for $t\in[0,1]$, where $r_0=\min_{t\in[0,1]} r(t)>0$, $M_1=2\big(d/c+\phi^{-1}(1/r_0)\big)$.

For $1<p<2$, using the Mean Value Theorem, we obtain

$$|\phi^{-1}(x)-\phi^{-1}(y)|\le (p-1)^{-1}|x-y|(\max\{|x|,|y|\})^{\frac{2-p}{p-1}}$$

for $x,y\in\mathbb{R}$, which implies

$$\max\{|u_1(t)-u_0(t)|,|u_1'(t)-u_0'(t)|\}\le M_2\|h_1-h_0\|_1, \tag{2.6}$$

for $t\in[0,1]$, where $M_2=(p-1)^{-1}\big(dc^{-1}+r_0^{-1/(p-1)}\big)C^{\frac{2-p}{p-1}}$.

Suppose next that $a>0$. Then $C_i=(b/a)\phi^{-1}(D_i)$, and D_i satisfies

$$c\Big(\frac{b}{a}\phi^{-1}(D_i)+\int_0^1\phi^{-1}\Big(\frac{D_i\ -\int_0^s h_i}{r(s)}\Big)ds\Big)+d\phi^{-1}\Big(D_i-\int_0^1 h_i\Big)=0 \tag{2.7}$$

for $i=0,1$. Since ϕ^{-1} is increasing and $\phi^{-1}(0)=0$, it follows from (2.7) that $|D_i|\le\|h_i\|_1$, and

$$|D_1-D_0|\le\|h_1-h_0\|_1,$$

which, together with (2.4), imply

$$\max\{|u_1(t)-u_0(t)|,|u_1'(t)-u_0'(t)|\}\le M_3\max\{\|h_1-h_0\|_1,\|h_1-h_0\|_1^{\frac{1}{p-1}}\} \tag{2.8}$$

for $t\in[0,1]$, where $M_3=2(b/a+(2/r_0)^{\frac{1}{p-1}})$ if $p\ge 2$, and $M_3=(p-1)^{-1}(b/a+(2/r_0)^{1/(p-1)})C^{\frac{2-p}{p-1}}$ if $1<p<2$. Combining (2.5),(2.6), and (2.8), we obtain (2.3) with $M_0=\max_{1\le i\le 3}M_i$, which completes the proof. $\square$

3. Proofs of main results

Let $z_1\in C^1[0,1]$ be the normalized positive eigenfunction of $-(r(t)\phi(u'))'=\lambda g(t)\phi(u)$ in $(0,1)$ with Sturm-Liouville boundary conditions corresponding to λ_1 i.e. $z_1>0$ on $(0,1)$ and $\|z_1\|_\infty=1$. By Lemma 2.2, there exists a constant $m_0>0$ such that $z_1\ge m_0 q$ in $(0,1)$.

Proof of Theorem 1.1. Since $\lim_{z\to0^+}\inf\frac{f(t,z)}{\phi(z)}>\frac{\lambda_1}{\lambda}$ uniformly in $t\in(0,1)$, there exists a constant $c>0$ such that

$$\frac{f(t,z)}{\phi(z)}>\frac{\lambda_1}{\lambda} \tag{3.1}$$

for $z\in(0,c]$ and $t\in(0,1)$. Let $Z=cz_1$ and $Z_1=Mz_1$, where $M>c$ is a large constant to be determined later. In view of (3.1), Z satisfies

$$-(r(t)\phi(Z'))'=\lambda_1 g(t)\phi(Z)\le\lambda g(t)f(t,Z) \tag{3.2}$$

for $t\in(0,1)$. For $v\in C[0,1]$, let $\tilde v=\min\{\max\{v,Z\},Z_1\}$. Then $Z\le\tilde v\le Z_1\le M$ in $(0,1)$ and (A3) gives

$$|g(t)f(t,\tilde v)|\le\frac{K_M g(t)}{\tilde v^\gamma}\le\frac{K_M g(t)}{(cz_1)^\gamma}\le\frac{K_M g(t)}{(cm_0)^\gamma q^\gamma(t)} \tag{3.3}$$

for $t\in(0,1)$. Hence $g(t)f(t,\tilde v)\in L^1(0,1)$ by (A2). Define $Tv=u$, where u is the solution of

$$\begin{gathered}-(r(t)\phi(u'))'=\lambda g(t)f(t,\tilde v),\quad t\in(0,1),\\ au(0)-b\phi^{-1}(r(0))u'(0)=0,\quad cu(1)+d\phi^{-1}(r(1))u'(1)=0,\end{gathered} \tag{3.4}$$

whose existence follows from (3.3) and Lemma 2.1. Define $S_1 v = \lambda g(t) f(t, \tilde{v})$. Using (3.3) and the Lebesgue Dominated Convergence Theorem, we see that $S_1 : C[0,1] \to L^1(0,1)$ is continuous and bounded. Since $T = S \circ S_1$, where S is defined in Lemma 2.1, it follows that $T : C[0,1] \to C[0,1]$ is completely continuous and bounded. Hence, by the Schauder Fixed Point Theorem, T has a fixed point u. To complete the proof, we will first show that $u \geq Z$ in $(0,1)$. Indeed, suppose $u(t^*) < Z(t^*)$ for some $t^* \in (0,1)$. Let $(t_0, t_1) \subset (0,1)$ be the largest interval containing t^* such that $u < Z$ in (t_0, t_1). Then $\tilde{u} = Z$ in (t_0, t_1) and

$$au(t_0) - b\phi^{-1}(r(t_0))u'(t_0) \geq aZ(t_0) - b\phi^{-1}(r(t_0))Z'(t_0). \tag{3.5}$$

Indeed, if $t_0 > 0$ then $u(t_0) = Z(t_0)$ and $u'(t_0) \leq Z'(t_0)$, while if $t_0 = 0$ then we have equality in (3.5). Similarly,

$$cu(t_1) + d\phi^{-1}(r(t_1))u'(t_1) \geq cZ(t_1) + d\phi^{-1}(r(t_1))Z'(t_1). \tag{3.6}$$

Since

$$-(r(t)\phi(u'))' = \lambda g(t) f(t, Z), \quad t \in (t_0, t_1),$$

it follows from (3.2), (3.5), (3.6), and the comparison principle (see e.g. [8, Lemma 3.2]) that $u \geq Z$ in (t_0, t_1), a contradiction. Thus $u \geq Z$ in $(0,1)$ and so $\tilde{u} = \min\{u, Z_1\}$ in $(0,1)$.

Next, we show that $u \leq Z_1$ in $(0,1)$. Using (A3) and $\lim_{z\to\infty} \sup \frac{f(t,z)}{\phi(z)} < \frac{\lambda_1}{\lambda}$ uniformly in $t \in (0,1)$, we deduce the existence of constants $A, K_\lambda > 0$ and $\bar{\lambda} \in (0, \lambda_1)$ such that

$$\lambda f(t,z) \leq \bar{\lambda}\phi(z) + \frac{K_\lambda}{z^\gamma}$$

for $z > 0$ and $t \in (0,1)$. Hence

$$\begin{aligned}
-(r(t)\phi(u'))' = \lambda g(t) f(t, \tilde{u}) &\leq g(t)\Big(\bar{\lambda}\phi(\tilde{u}) + \frac{K_\lambda}{\tilde{u}^\gamma}\Big) \\
&\leq g(t)\Big(\bar{\lambda}(Mz_1)^{p-1} + \frac{K_\lambda}{(cz_1)^\gamma}\Big) \\
&\leq \bar{\lambda} g(t)(Mz_1)^{p-1} + \frac{K_\lambda g(t)}{(cm_0)^\gamma q^\gamma(t)}
\end{aligned}$$

for $t \in (0,1)$. Let $u_M = u/M$. Then u_M satisfies

$$-(r(t)\phi(u_M'))' \leq \bar{\lambda} g(t) {z_1}^{p-1} + \frac{K_\lambda g(t)}{(cm_0)^\gamma M^{p-1} q^\gamma(t)}$$

for $t \in (0,1)$. Let $\bar{u}_M$ and $\bar{u}$ satisfy

$$-(r(t)\phi(\bar{u}_M'))' = \bar{\lambda} g(t) {z_1}^{p-1} + \frac{K_\lambda g(t)}{(cm_0)^\gamma M^{p-1} q^\gamma(t)} \equiv h_M, \quad t \in (0,1),$$

and

$$-(r(t)\phi(\bar{u}'))' = \bar{\lambda} g(t) z_1^{p-1} \equiv h, \quad t \in (0,1),$$

with Sturm-Liouville boundary conditions in (1.1). Note that $\bar{u} = (\bar{\lambda}/\lambda_1)^{\frac{1}{p-1}} z_1$. By the comparison principle, $u_M \leq \bar{u}_M$ in $(0,1)$. Let $\varepsilon > 0$ be such that $(\bar{\lambda}/\lambda_1)^{1/(p-1)} + \varepsilon < 1$. Since

$$\|h_M - h\|_1 = \frac{K_\lambda}{(cm_0)^\gamma M^{p-1}} \Big(\int_0^1 \frac{g(t)}{q^\gamma(t)} dt \Big) \to 0 \quad \text{as } M \to \infty,$$

it follows from Lemmas 2.4 and 2.5 that

$$\begin{aligned}\bar{u}_M - \bar{u} &\le k|\bar{u}_M - \bar{u}|_1 q \le k m_0^{-1} |\bar{u}_M - \bar{u}|_1 z_1 \\ &\le k m_0^{-1} M_0 \max\{\|h_M - h\|_1, \|h_M - h\|_1^{\frac{1}{p-1}}\} z_1 < \varepsilon z_1,\end{aligned}$$

provided that M is large enough. Consequently,

$$u_M \le \bar{u}_M \le \bar{u} + \varepsilon z_1 = \Big((\bar{\lambda}/\lambda_1)^{1/(p-1)} + \varepsilon\Big) z_1 \le z_1 \quad \text{in } (0,1),$$

i.e. $u \le M z_1 = Z_1$ in $(0,1)$. Hence $Z \le u \le Z_1$ in $(0,1)$ i.e. u is a positive solution of (1.1), which completes the proof. □

Proof of Theorem 1.2. By Theorem 1.1, there exists a positive solution w of the problem

$$\begin{gathered} -(r(t)\phi(w'))' = \frac{g(t)}{w^\gamma}, \quad t \in (0,1), \\ a w(0) - b\phi^{-1}(r(0)) w'(0)) = 0, \quad c w(1) + d\phi^{-1}(r(1)) w'(1) = 0 \end{gathered}$$

with $w \ge \alpha q$ in $(0,1)$ for some $\alpha > 0$. Let w_0 satisfy

$$-(r(t)\phi(w_0'))' = \begin{cases} \frac{L_1 g(t)}{w^\gamma} & \text{if } w > \frac{2AL_1^{-1/(p-1)}}{\lambda^\delta}, \\ -\frac{K_1 g(t)}{w^\gamma} & \text{if } w \le \frac{2AL_1^{-1/(p-1)}}{\lambda^\delta} \end{cases} \equiv h_\lambda \quad \text{in } (0,1),$$

with Sturm-Liouville boundary conditions, where $\delta = (\gamma + p - 1)^{-1}, L_1 = L^{\frac{p-1}{p-1+\gamma}}$ and $K_1 = 2^\gamma L_1^{-\gamma/(p-1)} K_{2A}$, and K_{2A} is defined in (A3). Let w_1 satisfy

$$-(r(t)\phi(w_1'))' = \frac{L_1 g(t)}{w^\gamma} \equiv h \quad \text{in } (0,1)$$

with Sturm-Liouville boundary conditions. Then $w_1 = L_1^{1/(p-1)} w$ and $w_0 \le w_1$ in $(0,1)$ by the comparison principle. Since

$$\|h_\lambda - h\|_1 = (L_1 + K_1) \int_{w \le \frac{2AL_1^{-1/(p-1)}}{\lambda^\delta}} \frac{g(t)}{w^\gamma(t)} dt \to 0 \quad \text{as } \lambda \to \infty,$$

it follows from Lemma 2.5 that

$$|w_0 - w_1|_1 \le M_0 \max\{\|h_\lambda - h\|_1, \|h_\lambda - h\|_1^{\frac{1}{p-1}}\} \to 0 \quad \text{as } \lambda \to \infty.$$

Hence by Lemma 2.4, there exists a constant $\lambda_0 > 0$ such that

$$w_0 \ge w_1 - k|w_0 - w_1|_1 q \ge \frac{L_1^{1/(p-1)} w}{2} \quad \text{in } (0,1) \tag{3.7}$$

for $\lambda > \lambda_0$. Let $Z = \lambda^\delta w_0$ and $Z_1 = M z_1$ where $M > \lambda^\delta k m_0^{-1} |w_1|_1$ (so that $Z_1 > Z$ in $(0,1)$). We shall verify that Z satisfies

$$-(r(t)\phi(Z'))' \le \lambda g(t) f(t, Z) \quad \text{in } (0,1). \tag{3.8}$$

Indeed,

$$-(r(t)\phi(Z'))' = \begin{cases} \frac{\lambda^{\delta(p-1)} L_1 g(t)}{w^\gamma} & \text{if } w > \frac{2AL_1^{-1/(p-1)}}{\lambda^\delta}, \\ -\frac{\lambda^{\delta(p-1)} K_1 g(t)}{w^\gamma} & \text{if } w \le \frac{2AL_1^{-1/(p-1)}}{\lambda^\delta}. \end{cases}$$

If $w > 2AL_1^{-1/(p-1)}/\lambda^\delta$ then by (3.7),

$$Z \geq \frac{\lambda^\delta L_1^{1/(p-1)} w}{2} \geq A,$$

from which (A6) gives

$$\begin{aligned} \lambda g(t) f(t, Z) &\geq \frac{\lambda L g(t)}{Z^\gamma} = \frac{\lambda^{1-\gamma\delta} L g(t)}{w_0^\gamma} \\ &\geq \frac{\lambda^{1-\gamma\delta} L g(t)}{w_1^\gamma} = \frac{\lambda^{\delta(p-1)} L g(t)}{L_1^{\gamma/(p-1)} w^\gamma} \\ &= \frac{\lambda^{\delta(p-1)} L_1 g(t)}{w^\gamma}. \end{aligned} \tag{3.9}$$

On the other hand, if $w \leq \frac{2AL_1^{-1/(p-1)}}{\lambda^\delta}$, then

$$Z \leq \lambda^\delta w_1 = L_1^{1/(p-1)} \lambda^\delta w \leq 2A,$$

from which (A3) and (3.7) give

$$\begin{aligned} \lambda g(t) f(t, Z) &\geq -\frac{\lambda K_{2A} g(t)}{Z^\gamma} = -\frac{\lambda^{1-\gamma\delta} K_{2A} g(t)}{w_0^\gamma} \\ &\geq -\frac{\lambda^{\delta(p-1)} K_{2A} g(t)}{\left(L_1^{1/(p-1)}/2\right)^\gamma w^\gamma} = -\frac{\lambda^{\delta(p-1)} K_1 g(t)}{w^\gamma}. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we see that (3.8) holds. Let T be the operator defined in the proof of Theorem 1.1 i.e. for $v \in C[0,1]$, $u = Tv$ satisfies (3.4); i.e.,

$$-(r(t)\phi(u'))' = \lambda g(t) f(t, \tilde{v}), \quad t \in (0,1),$$
$$au(0) - b\phi^{-1}(r(0))u'(0)) = 0, \quad cu(1) + d\phi^{-1}(r(1))u'(1) = 0,$$

where $\tilde{v} = \min\{\max\{v, Z\}, Z_1\}$. Then T has a fixed point u_λ in $C[0,1]$. Using the same arguments as in the proof of Theorem 1.1, we see that $u_\lambda \geq Z$ and, for M large enough $u_\lambda \leq Z_1$ in $(0,1)$; i.e., u_λ is a positive solution of (1.1) for $\lambda > \lambda_0$ with $u_\lambda \geq \lambda^\delta \left(L_1^{1/(p-1)}/2\right) w$ in $(0,1)$, which completes the proof. □

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Hai Dinh Dang
Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA
E-mail address: `dang@math.msstate.edu`