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EXACT CONTROLLABILITY OF THE EULER-BERNOULLI PLATE WITH VARIABLE COEFFICIENTS AND SIMPLY SUPPORTED BOUNDARY CONDITION

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Abstract. This article studies the exact controllability of an Euler-Bernoulli plate equation with variable coefficients, subject to the simply supported boundary condition. By the Riemannian geometry approach, the duality method, the multiplier technique, and the compactness-uniqueness argument, we establish the corresponding observability inequality and obtain the exact controllability results.

1. Introduction

Let $A(x) = (a_{ij}(x))$ be a symmetric, positive matrix for each $x \in \mathbb{R}^n$, where $a_{ij}(x)$ are C^∞ functions in $\mathbb{R}^n$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0, \quad \forall x \in \mathbb{R}^n,\ 0 \neq \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n.$$

We introduce

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n,$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. We denote by $g = \langle \cdot, \cdot \rangle_g$ the inner product. Then

$$\langle X, Y \rangle_g = \langle A^{-1}(x)X, Y \rangle \quad \text{for } X, Y \in \mathbb{R}^n_x, x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean product of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with a sufficient smooth boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$, where Γ_1 is nonempty. We consider the following Euler-Bernoulli plate model

$$\begin{gathered}
u_{tt} + \mathscr{A}^2 u = 0 \quad \text{in } Q = (0, T) \times \Omega, \\
u = 0 \quad \text{on } \Sigma_0 = (0, T) \times \Gamma_0, \\
u = \varphi \quad \text{on } \Sigma_1 = (0, T) \times \Gamma_1, \\
\mathscr{A} u + a(x)Bu = 0 \quad \text{on } \Sigma_0, \\
\mathscr{A} u + a(x)Bu = \psi \quad \text{on } \Sigma_1, \\
u(0) = u_0, u_t(0) = u_1 \quad \text{on } \Omega.
\end{gathered} \tag{1.1}$$

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with two controls φ and ψ, where u_{tt} stands for $\partial^2 u/\partial t^2$,

$$\mathscr{A} u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)$$

and B is a boundary operator, defined by

$$Bu = -\sum_{i=2}^{n} e_i \langle e_i, \nabla_{\Gamma_g} u\rangle_g + k u_{\nu_{\mathscr{A}}}.$$

Here ν is the outside normal along Γ, $\nu_{\mathscr{A}} = A(x)\nu$, and $u_{\nu_{\mathscr{A}}} = \langle \nabla_g u, \nu\rangle = \langle A(x)\nabla u, \nu\rangle$. For $2 \le i \le n$, e_i is the tangential vector fields on Γ such that $e_1 = \nu_{\mathscr{A}}/|\nu_{\mathscr{A}}|_g, e_2, \dots, e_n$ form a unit orthogonal basis of $(\mathbb{R}^n_x, g(x))$ for each $x \in \Gamma$, ∇_{Γ_g} is the gradient of Riemannian manifold (Γ, g). k and $a(x)$ are bounded positive functions on Γ and Ω respectively, which are related to the material. The boundary condition we consider here is known as the simply supported boundary condition of the plate (see [2, 8]), which arises from the physical models and includes moments of inertia realistically present in the system.

In the case of constant coefficients where $A(x)$ is the unit matrix and $n = 2$, exact controllability results of problem (1.1) have been obtained by Horn [6]. The objective of this paper is to generalize the exact controllability results to the case where $A(x)$ is a non-constant, symmetric, positive n-order matrix and represents some property of the materials, for example, the mass of the plate is not uniformly distributed with respect to spatial position. The problem is of practical and theoretical importance. From the physical point of view, the variable-coefficient model is more realistic. Meanwhile, this together with the simply supported boundary condition also introduces additional non-trivial complications for the mathematical analysis.

The high-dimensional Euler-Bernoulli equations ($n \ge 2$), as a kind of classical partial differential equation, are used to describe the vibration of elastic thin plates. Stimulated by the extensive applications in the architectural structures, automobile and aerospace industries, etc. (see [17, 18]), there have been a great amount of research on the control problems of Euler Bernoulli plates. We shall only cite the literature closely related to this paper, the exact controllability of the Euler Bernoulli plates with different choices of controls active in the varying boundary conditions. For the constant coefficient case, we refer the reader to [6, 7, 8, 10, 11, 14, 24], and the references therein. Particularly, in [14], Lions considered the exact controllability of the Euler-Bernoulli model with one control acting through Neumann boundary condition. Later, Lasiecka and Triggiani [10] studied the situation where control acts only on the Dirichlet boundary condition, in which they also managed to get rid of some geometrical conditions by further adding a Neumann control. And in [11], they discussed the exact controllability problem with boundary controls for displacement u and moment Δu, which act in the Dirichlet boundary conditions. Horn [6] derived the exact controllability of the Euler-Bernoulli plate with a simply supported boundary condition only via bending moments on the space of optimal regularity. For the variable coefficient case, Yao [22] used the Riemannian geometry approach to give checkable conditions for the exact controllability of two Euler-Bernoulli models with clamped and hinged boundary conditions respectively, which has been extended by many others like

[1, 4, 5, 12, 13]. In particular, Guo and Zhang [4] showed that the exact controllability of an Euler-Bernoulli plate with variable coefficients and partial boundary Neumann control is equivalent to the exponential stability of its closed-loop system under proportional output feedback.

The Riemannian geometry is a useful tool for the controllability of variable – coefficient systems mainly due to its two virtues: The Bochner technique can be used to simplify computation to obtain the multiplier identities, and the curvature theory provides the global information on the existence of an escape vector field which guarantees the exact controllability. Given this, we shall use the Riemannian geometry approach to study our problem.

Since the dynamics of system (1.1) are time-reversible and it is well known that exact controllability is equivalent to null controllability in that case, we attempt to prove the following property: Given any $(u_0, u_1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$, there exist some $T > 0$ and controls $(\varphi, \psi) \in H_0^1(0, T; L^2(\Gamma_1)) \times L^2(\Sigma_1)$ such that the corresponding solution of problem (1.1) satisfies

$$u(T) \equiv u_t(T) \equiv 0.$$

Remark 1.1. The above corresponding regularity results for problem (1.1) can be obtained by the cosine operator theory in a similar argument as in the case of constant coefficients (see [9]), during which, however, some computations on Riemannian manifold are needed to deal with the variable coefficients. Besides, it is worth noting that the recent work by Wen et al. [19] gave the well-posedness and regularity of two types of Euler–Bernoulli equations with variable coefficients and Dirichlet boundary control, in which semigroup theory and the multiplier technique with Riemannian geometry are utilized. This method can also apply to the same question for our problem (1.1), because the operator $\mathbf{A}$ we define below is quite similar to the operator A which is fundamentally used in [19].

This article is organized as follows: In Section 2, we will introduce the escape vector field and state our primary results. In Section 3, we use the duality method to find the observability inequality. The proofs of the results are given in the last section.

2. Main results

We denote the Levi-Civita connection in the metric g by D. Let X be a vector field on $(\mathbb{R}^n, g)$. The covariant differential DX of X determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$ for each $x \in \mathbb{R}^n$ by

$$DX(Y, Z) = \langle D_Z X, Y\rangle_g, \forall Y, Z \in \mathbb{R}^n_x,$$

where $D_Z X$ is the covariant derivative of X with respect to Z.

Definition 2.1. A vector field H is said to be an escape vector field for the metric g on $\overline{\Omega}$ if there exists a constant $\rho_0 > 0$ such that

$$DH(x) \geq \rho_0 g(x) \quad \text{for all } x \in \overline{\Omega}. \tag{2.1}$$

Remark 2.2. Escape vector field was introduced by Yao [21] as a checkable assumption for the exact controllability of the wave equation with variable coefficients. Actually, the existence of such a vector field can also guarantee the exact controllability of an Euler-Bernoulli plate equation with variable coefficients and the simply supported boundary condition (see our results below).

If h is a strictly convex function in the metric g on $\overline{\Omega}$, then $H = Dh$ is such an escape vector field owing to D^2h, i.e., the Hessian of h, is positive. It is well known that the square of the distance function initiating from a given point $x_0 \in \Omega$ in the metric g is strictly convex in a neighborhood of x_0 (see, e.g.,[20]), then the escape vector field certainly exists locally. Fortunately, the sectional curvature of the Riemannian metric g can provide the global information on its existence. Here are some relevant results from [21] and [23]:

Proposition 2.3. *Let $x_0 \in \mathbb{R}^n$ be given. For any $x \in \mathbb{R}^n$, $\kappa(x, \Pi)$ denotes the sectional curvature of a two-dimensional subspace $\Pi \subset \mathbb{R}^n_x$ in the metric g, set*

$$\kappa(\Omega) = \sup_{x\in\Omega,\, \Pi\subset\mathbb{R}^n_x} \kappa(x, \Pi).$$

Let $B_g(x_0, \gamma)$ be a geodesic ball in $(\mathbb{R}^n, g)$ centered at x_0 with radius γ. Denote by $\rho(x) = d_g(x, x_0)$ the distance function of the metric g from x to x_0. If $\gamma > 0$ satisfies $4\gamma^2\kappa(\Omega) < \pi^2$ and $\overline{\Omega} \subset B_g(x_0, \gamma)$, then $H = \rho D\rho$ is an escape vector field for the metric g on $\overline{\Omega}$.

Proposition 2.4. *Suppose $(\mathbb{R}^n, g)$ is a Riemannian manifold, then*

(a) If $(\mathbb{R}^n, g)$ has non-positive sectional curvature, then there exists an escape vector field for the metric g on the whole space $\mathbb{R}^n$.

(b) If $(\mathbb{R}^n, g)$ is noncompact, complete, and its sectional curvature is positive everywhere on $\mathbb{R}^n$, then there exists an escape vector field in the metric g on the whole space $\mathbb{R}^n$.

Now we present the main results.

Theorem 2.5. *Let H be an escape vector field for the metric g on $\overline{\Omega}$ and let $T > 0$ be given. Let $\|k\|^2_{L^\infty(\Gamma)} < k_0$, which will be given concretely in Section 4. Then system (1.1) is exactly controllable on the space $H^1_0(\Omega) \times H^{-1}(\Omega)$ with controls $(\varphi, \psi) \in H^1_0(0, T; L^2(\Gamma_1)) \times L^2(\Sigma_1)$, where*

$$\Gamma_1 = \{x | \langle H, \nu\rangle > 0, x \in \Gamma\}.$$

3. Observability inequality

The dual problem of system (1.1) can be readily derived as follows

$$\begin{gathered}
w_{tt} + \mathscr{A}^2 w = 0 \quad \text{in } Q,\\
w = 0 \quad \text{on } \Sigma,\\
\mathscr{A} w + a(x)Bw = 0 \quad \text{on } \Sigma,\\
w(0) = w_0, w_t(0) = w_1 \quad \text{on } \Omega.
\end{gathered} \tag{3.1}$$

Let $\mathbf{A} : L^2(\Omega) \to L^2(\Omega)$ be a linear operator defined by

$$\mathbf{A}f = \mathscr{A}^2 f, D(\mathbf{A}) = \{f \in H^4(\Omega) : f|_\Gamma = 0, \mathscr{A} f + a(x)Bf|_\Gamma = 0\}.$$

It is easy to check that $\mathbf{A}$ is a positive, self-adjoint operator. According to the interpolation results in [15], we have the following space identifications:

$$\begin{gathered}
D(\mathbf{A}^\theta) = H^{4\theta}(\Omega), \quad 0 < \theta < \frac{1}{8},\\
D(\mathbf{A}^\theta) = \{f \in H^{4\theta}(\Omega) : f|_\Gamma = 0\}, \frac{1}{8} < \theta < \frac{5}{8}.
\end{gathered} \tag{3.2}$$

In particular, $\mathbf{A}^{1/2}f = -\mathscr{A}f$ and $D(\mathbf{A}^{1/2}) = H^2(\Omega)\cap H_0^1(\Omega)$.

We introduce the energy of system (3.1) by

$$2E(t) = \int_\Omega [(\mathbf{A}^{1/4}w)^2 + (\mathbf{A}^{-1/4}w_t)^2]\mathrm{d}x.$$

Differentiating the above identity with respect to t, we have

$$\begin{aligned}
E'(t) &= (A^{1/4}w_t, A^{1/4}w) + (A^{-1/4}w_{tt}, A^{-1/4}w_t)\\
&= (A^{1/4}w_t, A^{1/4}w) - (A^{3/4}w, A^{-1/4}w_t) = 0,
\end{aligned}$$

then $E(t)\equiv E(0)$ for all $t>0$.

For $(w_0,w_1)\in H_0^1(\Omega)\times H^{-1}(\Omega)$, we solve problem (3.1) to obtain the solution w. Then we solve the terminal value problem

$$\begin{gathered}
u_{tt} + \mathscr{A}^2u = 0 \quad \text{in } Q,\\
u(T) = u_t(T) = 0 \quad \text{on } \Omega,\\
u|_{\Sigma_0} = 0, u|_{\Sigma_1} = -(\mathscr{A}w)_{\nu_{\mathscr{A}}},\\
\mathscr{A}u + a(x)Bu = 0 \quad \text{on } \Sigma_0,\\
\mathscr{A}u + a(x)Bu = -a(x)\sum_{i=2}^n e_i\langle e_i, \nabla_{\Gamma_g}u\rangle_g - w_{\nu_{\mathscr{A}}} \quad \text{on } \Sigma_1.
\end{gathered}\tag{3.3}$$

Further, we define an operator $\Lambda: H_0^1(\Omega)\times H^{-1}(\Omega)\to H^{-1}(\Omega)\times H_0^1(\Omega)$ by

$$\Lambda(w_0,w_1) = (u_t(0), -u(0)) \quad \text{on } \Omega.$$

Using equations (3.1) and (3.3), we obtain

$$\begin{aligned}
&(\Lambda(w_0,w_1),(w_0,w_1))_{L^2(\Omega)\times L^2(\Omega)}\\
&= (u_t(0),w_0) - (u(0),w_1) = [(u,w_t)-(u_t,w)]|_0^T\\
&= \int_Q (w_{tt}u - u_{tt}w)\mathrm{d}Q = \int_Q (w\mathscr{A}^2u - u\mathscr{A}^2w)\mathrm{d}Q\\
&= \int_\Sigma [w(\mathscr{A}u)_{\nu_{\mathscr{A}}} - w_{\nu_{\mathscr{A}}}\mathscr{A}u - u(\mathscr{A}w)_{\nu_{\mathscr{A}}} + u_{\nu_{\mathscr{A}}}\mathscr{A}w]\mathrm{d}\Sigma\\
&= \int_{\Sigma_1} [w_{\nu_{\mathscr{A}}}^2 + (\mathscr{A}w)_{\nu_{\mathscr{A}}}^2]\mathrm{d}\Sigma.
\end{aligned}$$

By the duality method given by Lions [14], the exact controllability of problem (1.1) on the space $H_0^1(\Omega)\times H^{-1}(\Omega)$ is equivalent to the following statement:

There is a $C_T>0$ such that

$$\int_{\Sigma_1} [w_{\nu_{\mathscr{A}}}^2 + (\mathscr{A}w)_{\nu_{\mathscr{A}}}^2]\mathrm{d}\Sigma \ge C_T\|(w_0,w_1)\|^2_{H_0^1(\Omega)\times H^{-1}(\Omega)}.\tag{3.4}$$

Using a result in [23], the norm

$$\|(w_0,w_1)\|_\star^2 = \||\nabla_g(\mathscr{A}(\mathscr{A}^{-1}w_0))|_g\|^2_{L^2(\Omega)} + \||\nabla_g(\mathscr{A}^{-1}w_1)|_g\|^2_{L^2(\Omega)}$$

is equivalent norm on $H_0^1(\Omega)\times H^{-1}(\Omega)$. Then inequality (3.4) becomes

$$\int_{\Sigma_1} [w_{\nu_{\mathscr{A}}}^2 + (\mathscr{A}w)_{\nu_{\mathscr{A}}}^2]\mathrm{d}\Sigma \ge C_TE(0).$$

Let $z = \mathbf{A}^{-1/2}w$ and define

$$\mathbf{D}\xi = \zeta \quad \text{if } \mathscr{A}\zeta = 0 \text{ in } \Omega, \text{ and } \zeta|_\Gamma = \xi.\tag{3.5}$$

Elliptic regularity theory (see [15]) gives

$$\mathbf{D} \in \mathscr{L}(L^2(\Gamma) \to H^{1/2}(\Omega)). \tag{3.6}$$

Clearly, z satisfies the boundary conditions

$$z|_\Gamma = \mathscr{A} z|_\Gamma = 0.$$

Moreover, we find that

$$\begin{aligned} z_{tt} &= \mathbf{A}^{-1/2} w_{tt} = -\mathbf{A}^{-1/2} \mathscr{A}^2 w \\ &= -\mathbf{A}^{-1/2} \mathbf{A}^{1/2} (\mathscr{A}^2 z - \mathbf{D}(\mathscr{A}^2 z|_\Gamma)) \\ &= -\mathscr{A}^2 z + \mathbf{D}(\mathscr{A}^2 z|_\Gamma). \end{aligned}$$

Since $w|_\Gamma = 0$, we obtain

$$\mathscr{A}^2 z|_\Gamma = -\mathscr{A} w|_\Gamma = a(x) B w = -k a(x) (\mathscr{A} z)_{\nu_\mathscr{A}}. \tag{3.7}$$

Consequently, z satisfies the equation

$$\begin{gathered} z_{tt} + \mathscr{A}^2 z = -\mathbf{D}(k a(x) (\mathscr{A} z)_{\nu_\mathscr{A}}), \\ z|_\Gamma = \mathscr{A} z|_\Gamma = 0, \\ z(0) = z_0, z_t(0) = z_1. \end{gathered} \tag{3.8}$$

Then the observability inequality becomes

$$\int_{\Sigma_1} [(\mathscr{A} z)^2_{\nu_\mathscr{A}} + (\mathscr{A}^2 z)^2_{\nu_\mathscr{A}}] \mathrm{d}\Sigma \ge C_T E(0), \tag{3.9}$$

where the energy is now represented as

$$2E(t) = \int_\Omega [(\mathbf{A}^{3/4} z)^2 + (\mathbf{A}^{1/4} z_t)^2] \mathrm{d}x.$$

4. Proofs of the results

We consider u as a regular solution to the problem

$$u_{tt} + \mathscr{A}^2 u = f \quad \text{in } (0, \infty) \times \Omega, \tag{4.1}$$

where f is a given function.

The following lemma from [23] will play an important role in establishing our multiplier identities.

Lemma 4.1. *Let f, h be functions on $\mathbb{R}^n$ and let H be a vector field on $\mathbb{R}^n$. Then*

$$\begin{aligned} &\langle \nabla_g f, \nabla_g (H(h)) \rangle_g + \langle \nabla_g h, \nabla_g (H(f)) \rangle_g \\ &= \operatorname{div}(\langle \nabla_g f, \nabla_g h \rangle_g H) - \langle \nabla_g f, \nabla_g h \rangle_g \operatorname{div} H + DH(\nabla_g h, \nabla_g f) + DH(\nabla_g f, \nabla_g h), \end{aligned}$$

where $\operatorname{div} H$ *is the divergence of the vector field* H *in the Euclidean metric.*

Next are our main geometric multiplier identities.

Lemma 4.2. *Let H be a vector field on $\overline{\Omega}$ and let p be a function on $\overline{\Omega}$, set $q = \operatorname{div} H$. Suppose that u is a solution to problem* (4.1). *Then (1)*

$$\begin{aligned}
&\int_{\Sigma}\{2[qu_t + H(u_t)](u_t)_{\nu_{\mathscr{A}}} + 2H(\mathscr{A}u)(\mathscr{A}u)_{\nu_{\mathscr{A}}} - u_t^2\langle\nabla_g q, \nu\rangle \\
&- (2u_t\mathscr{A}u_t + |\nabla_g u_t|_g^2 + |\nabla_g(\mathscr{A}u)|_g^2)\langle H, \nu\rangle\}\mathrm{d}\Sigma \\
&= \int_Q\{2DH(\nabla_g u_t, \nabla_g u_t) + 2DH(\nabla_g(\mathscr{A}u), \nabla_g(\mathscr{A}u)) \\
&\quad + [|\nabla_g u_t|_g^2 - |\nabla_g(\mathscr{A}u)|_g^2]q - u_t^2\mathscr{A}q + 2fH(\mathscr{A}u)\}\mathrm{d}Q - 2(u_t, H(\mathscr{A}u))\big|_0^T.
\end{aligned} \tag{4.2}$$

and (2)

$$\begin{aligned}
&\int_{\Sigma}\big\{2p[u_t(u_t)_{\nu_{\mathscr{A}}} - \mathscr{A}u(\mathscr{A}u)_{\nu_{\mathscr{A}}}] + [(\mathscr{A}u)^2 - u_t^2]p_{\nu_{\mathscr{A}}}\big\}\mathrm{d}\Sigma \\
&= 2(u_t, p\mathscr{A}u)\big|_0^T + \int_Q\big\{\mathscr{A}p[(\mathscr{A}u)^2 - u_t^2] + 2p[|\nabla_g u_t|_g^2 \\
&\quad - |\nabla_g(\mathscr{A}u)|_g^2 - f\mathscr{A}u]\big\}\mathrm{d}Q.
\end{aligned} \tag{4.3}$$

Proof. We multiply equation (4.1) by $2H(\mathscr{A}u)$ and $2p\mathscr{A}u$, respectively. Then integrating over Q by parts with Lemma 4.1 yields these identities. □

Using these multiplier identities, we can derive the following estimates.

Lemma 4.3. *Let $T > 0$ be given and let H be an escape vector field for the metric g on $\overline{\Omega}$. Assume z is the solution to* (3.8). *Then there is a $C_{T,1} > 0$ such that*

$$\|(z_t)_{\nu_{\mathscr{A}}}\|_{L^2(\Sigma_1)}^2 + \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|_{L^2(\Sigma_1)}^2 \ge C_{T,1}E(0). \tag{4.4}$$

Lemma 4.4. *Let z be the solution to* (3.8). *Then there is a $C_{T,2} > 0$ such that*

$$\|(z_t)_{\nu_{\mathscr{A}}}\|_{L^2(\Sigma)}^2 + \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|_{L^2(\Sigma)}^2 \le C_{T,2}E(0). \tag{4.5}$$

Proof of Lemma 4.3. Since H is escaping on $\overline{\Omega}$, there is $\rho_0 > 0$ such that

$$DH(X, X) \ge \rho_0|X|_g^2 \quad \text{for } X \in \mathbb{R}_x^n, x \in \overline{\Omega}. \tag{4.6}$$

By the boundary conditions, $z = \mathscr{A}z = 0$ on Γ, we have

$$\nabla_g z_t = \sum_{i=1}^n\langle\nabla_g z_t, e_i\rangle_g e_i = \langle\nabla_g z_t, \frac{\nu_{\mathscr{A}}}{|\nu_{\mathscr{A}}|_g}\rangle_g\frac{\nu_{\mathscr{A}}}{|\nu_{\mathscr{A}}|_g} = \frac{(z_t)_{\nu_{\mathscr{A}}}}{|\nu_{\mathscr{A}}|_g^2}\nu_{\mathscr{A}}.$$

Similarly, $\nabla_g(\mathscr{A}z) = \frac{(\mathscr{A}z)_{\nu_{\mathscr{A}}}}{|\nu_{\mathscr{A}}|_g^2}\nu_{\mathscr{A}}$. Thus,

$$|\nabla_g z_t|_g^2 = \frac{(z_t)_{\nu_{\mathscr{A}}}^2}{|\nu_{\mathscr{A}}|_g^2}, \quad H(z_t) = \frac{\langle H, \nu\rangle}{|\nu_{\mathscr{A}}|_g^2}(z_t)_{\nu_{\mathscr{A}}}, \tag{4.7}$$

$$|\nabla_g(\mathscr{A}z)|_g^2 = \frac{(\mathscr{A}z)_{\nu_{\mathscr{A}}}^2}{|\nu_{\mathscr{A}}|_g^2}, \quad H(\mathscr{A}z) = \frac{\langle H, \nu\rangle}{|\nu_{\mathscr{A}}|_g^2}(\mathscr{A}z)_{\nu_{\mathscr{A}}}. \tag{4.8}$$

Using the boundary conditions of problem (3.8), the relations (4.7) and (4.8) in identity (4.2) with $f = -\mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}})$, we obtain

$$\begin{aligned} &\int_\Sigma [(z_t)^2_{\nu_{\mathscr{A}}} + (\mathscr{A}z)^2_{\nu_{\mathscr{A}}}]\langle H,\nu\rangle/|\nu_{\mathscr{A}}|^2_g \mathrm{d}\Sigma \\ &= \int_Q \Big\{[2DH(\nabla_g z_t, \nabla_g z_t) + 2DH(\nabla_g(\mathscr{A}z), \nabla_g(\mathscr{A}z))] \\ &\quad + [|\nabla_g z_t|^2_g - |\nabla_g(\mathscr{A}z)|^2_g] \operatorname{div} H - z_t^2 \mathscr{A}q - 2\mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}})H(\mathscr{A}z)\Big\}\mathrm{d}Q \\ &\quad - 2(z_t, H(\mathscr{A}z))|_0^T. \end{aligned} \tag{4.9}$$

Firstly,

$$\int_\Sigma [(z_t)^2_{\nu_{\mathscr{A}}} + (\mathscr{A}z)^2_{\nu_{\mathscr{A}}}]\langle H,\nu\rangle/|\nu_{\mathscr{A}}|^2_g \mathrm{d}\Sigma \le C\int_{\Sigma_1} [(z_t)^2_{\nu_{\mathscr{A}}} + (\mathscr{A}z)^2_{\nu_{\mathscr{A}}}]\mathrm{d}\Sigma. \tag{4.10}$$

Next, we shall estimate all terms on the right-hand side of (4.9). For the first term, by means of (4.6), we obtain

$$\begin{aligned} &\int_Q [2DH(\nabla_g z_t, \nabla_g z_t) + 2DH(\nabla_g(\mathscr{A}z), \nabla_g(\mathscr{A}z))]\mathrm{d}Q \\ &\ge 2\rho_0 \int_Q [|\nabla_g z_t|^2_g + |\nabla_g(\mathscr{A}z)|^2_g]\mathrm{d}Q = 4\rho_0 T E(0). \end{aligned} \tag{4.11}$$

For the second term, using the boundary conditions of problem (3.8) in identity (4.3) with $p = \operatorname{div} H/2$ and $f = -\mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}})$, we obtain

$$\Big|\int_Q [|\nabla_g z_t|^2_g - |\nabla_g(\mathscr{A}z)|^2_g] \operatorname{div} H \mathrm{d}Q\Big| \le \varepsilon \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} + C_\varepsilon L(z), \tag{4.12}$$

where

$$\begin{aligned} L(z) &= \|z(0)\|^2_{L^2(\Omega)} + \|z(T)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(Q)} + \|z_t(0)\|^2_{L^2(\Omega)} + \|z_t(T)\|^2_{L^2(\Omega)} \\ &\quad + \|z_t\|^2_{L^2(Q)} + \||D^2 z|_g(0)\|^2_{L^2(\Omega)} + \||D^2 z|_g(T)\|^2_{L^2(\Omega)} + \||D^2 z|_g\|^2_{L^2(Q)}, \end{aligned}$$

are the lower terms relative to the energy $E(t)$.

For the third term, we have

$$\int_Q -z_t^2 \mathscr{A}q \mathrm{d}Q \ge -\sup_{x\in\Omega} |\mathscr{A}q| \|z_t\|^2_{L^2(Q)}. \tag{4.13}$$

For the fourth term, by (3.2) and (3.6), $\mathbf{A}^\theta \mathbf{D} \in \mathcal{L}(L^2(\Gamma) \to L^2(\Omega))$ for $\theta < 1/8$, we have

$$\begin{aligned} &|(\mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}}), H(\mathscr{A}z))_{L^2(Q)}| \\ &= |(\mathbf{A}^{-\theta}\mathbf{A}^\theta \mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}}), H(\mathscr{A}z))_{L^2(Q)}| \\ &\le \epsilon \|\mathbf{A}^\theta \mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}})\|^2_{L^2(Q)} + C_\epsilon \|(\mathbf{A}^{-\theta} H(\mathscr{A}z)\|^2_{L^2(Q)} \\ &\le \epsilon \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} + C_\epsilon T \|\mathbf{A}^{-\theta} H(\mathscr{A}z)\|^2_{C[0,T;L^2(\Omega)]}. \end{aligned} \tag{4.14}$$

Applying Lemma 4.4, we obtain

$$\begin{aligned} \epsilon \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} &= \epsilon \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1)} + \epsilon \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_0)} \\ &\le \epsilon \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1)} + \epsilon C_{T,2} E(0). \end{aligned} \tag{4.15}$$

For the last term, we have

$$
\begin{aligned}
|(z_t, H(\mathscr{A}z))| &\le \sup_{x\in\Omega} |H|_g \int_\Omega |z_t||\nabla_g(\mathscr{A}z)|_g \mathrm{d}x \\
&\le \epsilon \int_\Omega |\nabla_g(\mathscr{A}z)|_g^2 \mathrm{d}x + C_\epsilon \int_\Omega z_t^2 \mathrm{d}x \\
&\le 2\epsilon E(0) + C_\epsilon \int_\Omega z_t^2 \mathrm{d}x.
\end{aligned} \tag{4.16}
$$

Thus

$$
-2(z_t, H(\mathscr{A}z))|_0^T \ge -8\epsilon E(0) - 2C_\epsilon(\|z_t(0)\|^2_{L^2(\Omega)} + \|z_t(T)\|^2_{L^2(\Omega)}). \tag{4.17}
$$

Combining (4.9)–(4.17), we have

$$
\begin{aligned}
&C \int_{\Sigma_1} [(z_t)^2_{\nu_\mathscr{A}} + (\mathscr{A}z)^2_{\nu_\mathscr{A}}]\mathrm{d}\Sigma \\
&\ge 4(\rho_0 T - 2\epsilon - \frac{\epsilon}{2} C_{T,2})E(0) - 2C_\epsilon L(z) - \varepsilon \|(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma)} \\
&- 2\epsilon \|(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma_1)} - 2C_\epsilon T \|\mathbf{A}^{-\theta} H(\mathscr{A}z)\|^2_{C[0,T;L^2(\Omega)]}.
\end{aligned} \tag{4.18}
$$

Then for ϵ small enough, there are constants $C_i > 0$ for $1 \le i \le 3$ such that

$$
C_1 \int_{\Sigma_1} [(z_t)^2_{\nu_\mathscr{A}} + (\mathscr{A}z)^2_{\nu_\mathscr{A}}]\mathrm{d}\Sigma + C_2 L(z) \ge C_3 E(0), \tag{4.19}
$$

for all solutions z to (3.8). Then inequality (4.4) follows by Lemma 4.5 below. $\square$

Lemma 4.5. *Let inequality* (4.19) *hold for all solutions z of* (3.8). *Then there is a $C > 0$ such that*

$$
\int_{\Sigma_1} [(z_t)^2_{\nu_\mathscr{A}} + (\mathscr{A}z)^2_{\nu_\mathscr{A}}]\mathrm{d}\Sigma \ge C E(0). \tag{4.20}
$$

To prove this lemma, we need the following uniqueness result from [16].

Proposition 4.6. *Let $\hat\Gamma$ be a relatively open subset of Γ. If w solves the problem*

$$
\begin{gathered}
\mathscr{A}^2 w = F(w, Dw, D^2 w, D^3 w) \quad \text{on } \Omega, \\
w = w_{\nu_\mathscr{A}} = \mathscr{A}w = (\mathscr{A}w)_{\nu_\mathscr{A}} = 0 \quad \text{on } \hat\Gamma,
\end{gathered} \tag{4.21}
$$

then $w = 0$ on Ω .

Proof of Lemma 4.5. **Step 1.** Let $Y = \{z \in H^3(Q) : z$ is a solution to problem (3.8) satisfying $(z_t)_{\nu_\mathscr{A}}|_{\Sigma_1} = (\mathscr{A}z)_{\nu_\mathscr{A}}|_{\Sigma_1} = 0\}$. Then

$$
Y = \{0\}. \tag{4.22}
$$

Indeed, from inequality (4.19), we have

$$
C_2 L(z) \ge C_3 E(0) \quad \text{for all } z \in Y,
$$

which implies that any bounded closed set in $Y \cap H^3(Q)$ is compact in $H^3(Q)$. Then Y is a finite-dimensional linear space. For any $z \in Y$, we can readily obtain that $z_t \in Y$. Then $\partial_t : Y \to Y$ is a linear operator. Let $Y \neq \{0\}$, then ∂_t has at least one eigenvalue λ. Assume that $v \neq 0$ is one of its eigenfunctions, then $v_t = \lambda v$. Further, v is a nonzero solution to the problem

$$
\begin{gathered}
\mathscr{A}^2 v = -\lambda^2 v - \mathbf{D}(ka(x)(\mathscr{A}v)_{\nu_\mathscr{A}}) \quad \text{on } \Omega, \\
v = (v_t)_{\nu_\mathscr{A}} = \mathscr{A}v = (\mathscr{A}v)_{\nu_\mathscr{A}} = 0 \quad \text{on } \Gamma_1.
\end{gathered} \tag{4.23}
$$

However, by Proposition 4.6, problem (4.23) only has zero solution, this contradiction shows that (4.22) holds.

Step 2. Suppose that the estimate (4.20) is not true. Then there are $(z_0^k, z_1^k) \in H_0^3(\Omega) \times H_0^1(\Omega)$, whose solutions are denoted by z^k, such that

$$E(z^k, 0) = 1, \quad \int_{\Sigma_1} [(z_t^k)^2_{\nu_{\mathscr{A}}} + (\mathscr{A} z^k)^2_{\nu_{\mathscr{A}}}] \mathrm{d}\Sigma \le \frac{1}{k} \quad \text{for } k \ge 1. \tag{4.24}$$

Then $\|z^k\|^2_{H^3(Q)} = 2T$ for all $k \ge 1$. Thus there is a subsequence, still denoted by z^k, such that

$$z^k \text{ converges in } H^2(\Omega) \text{ for each } t \in [0, T], \text{ and} \tag{4.25}$$

$$z^k \text{ converges in } H^2(Q). \tag{4.26}$$

It follows from relations (4.19), (4.24), (4.25) and (4.26) that z^k converges in $H^3(Q)$. Then there exists a solution z^0 to problem (3.8) such that

$$z^k \to z^0 \quad \text{as } k \to \infty \text{ in } H^3(Q).$$

Then

$$E(z^0, 0) = 1, \int_{\Sigma_1} [(z_t^0)^2_{\nu_{\mathscr{A}}} + (\mathscr{A} z^0)^2_{\nu_{\mathscr{A}}}] \mathrm{d}\Sigma = 0.$$

Thus

$$(z_t^0)_{\nu_{\mathscr{A}}}|_{\Sigma_1} = (\mathscr{A} z^0)_{\nu_{\mathscr{A}}}|_{\Sigma_1} = 0.$$

Then $0 \ne z^0 \in Y$, it contradicts the relation (4.22). □

Proof of Lemma 4.4. We choose a vector field H on $\overline{\Omega}$ such that

$$H = A(x)\nu \quad \text{for } x \in \Gamma,$$

and let $f = -\mathbf{D}(ka(x)(\mathscr{A} z)_{\nu_{\mathscr{A}}})$. Then using the boundary conditions of problem (3.8), relations (4.7) and (4.8) in identity (4.2), it gives

$$\begin{aligned} &\int_\Sigma [(z_t)^2_{\nu_{\mathscr{A}}} + (\mathscr{A} z)^2_{\nu_{\mathscr{A}}}] \mathrm{d}\Sigma \\ &= \int_Q \{2DH(\nabla_g z_t, \nabla_g z_t) + 2DH(\nabla_g(\mathscr{A} z), \nabla_g(\mathscr{A} z)) \\ &\quad + (|\nabla_g z_t|_g^2 - |\nabla_g(\mathscr{A} z)|_g^2) \operatorname{div} H - z_t^2 \mathscr{A} q \\ &\quad - 2\mathbf{D}(ka(x)(\mathscr{A} z)_{\nu_{\mathscr{A}}}) H(\mathscr{A} z)\} \mathrm{d}Q - 2(z_t, H(\mathscr{A} z))|_0^T. \end{aligned} \tag{4.27}$$

We shall estimate all terms on the right-hand side of (4.27) separately. For the first term, we have

$$\begin{aligned} &\int_Q [2DH(\nabla_g z_t, \nabla_g z_t) + 2DH(\nabla_g(\mathscr{A} z), \nabla_g(\mathscr{A} z))] \mathrm{d}Q \\ &\le C \int_Q [|\nabla_g z_t|_g^2 + |\nabla_g(\mathscr{A} z)|_g^2] \mathrm{d}Q = 2CTE(0). \end{aligned} \tag{4.28}$$

We have already estimated the second term in the proof of Lemma 4.3.

For the third term, we have

$$-\int_Q z_t^2 \mathscr{A} q \mathrm{d}Q \le T \sup_{x \in \Omega} |\mathscr{A} q| \|z_t\|^2_{C[0,T; L^2(\Omega)]}. \tag{4.29}$$

For the fourth term, we have

$$\begin{aligned}
&\Big|\int_Q \mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}})H(\mathscr{A}z)\mathrm{d}Q\Big| \\
&\le \epsilon\|\mathbf{D}(ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}})\|^2_{L^2(Q)} + \sup_{x\in\Omega}|H|^2_g C_\epsilon \||\nabla_g(\mathscr{A}z)|_g\|^2_{L^2(Q)} \\
&\le \epsilon\|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} + 2T\sup_{x\in\Omega}|H|^2_g C_\epsilon E(0).
\end{aligned} \tag{4.30}$$

For the last term, we have

$$\begin{aligned}
&-2(z_t, H(\mathscr{A}z))|_0^T \\
&\le 2\int_\Omega [|z_t(0)||H(\mathscr{A}z)(0)| + |z_t(T)||H(\mathscr{A}z)(T)|]\mathrm{d}x \\
&\le \|z_t(0)\|^2_{L^2(\Omega)} + \|z_t(T)\|^2_{L^2(\Omega)} + \sup_{x\in\Omega}|H|^2_g\int_\Omega[|\nabla_g(\mathscr{A}z)(0)|^2_g + |\nabla_g(\mathscr{A}z)(T)|^2_g]\mathrm{d}x \\
&\le 2\|z_t\|^2_{C[0,T;L^2(\Omega)]} + 4\sup_{x\in\Omega}|H|^2_g E(0).
\end{aligned}$$

Since $z_t = \mathscr{A}z = 0$ on Γ, according to the Poincare's inequality, we have

$$\|z_t\|^2 \le C\||\nabla_g z_t|_g\|^2_{L^2(\Omega)}, \|\mathscr{A}z\|^2 \le C\||\nabla_g(\mathscr{A}z)|_g\|^2_{L^2(\Omega)}. \tag{4.31}$$

Combining (4.12), (4.27)–(4.31), we obtain the desired estimate (4.5). □

Using some ideas from [6], we can eliminate the term $\|(z_t)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1)}$ from the inequality (4.4). Firstly, we have the following lemma.

Lemma 4.7. *Let $\alpha > 0$ be a given constant and define $\Sigma^\alpha = [-\alpha, T+\alpha]\times\Gamma$. Assume z satisfies problem* (3.8). *Then for any $\epsilon > 0$, there is a $C_{T,3} > 0$ such that*

$$\begin{aligned}
&\|(z_t)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1)} \\
&\le \|(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1^\alpha)} + \epsilon C_{T,3}E(0) + (\frac{8}{\epsilon}+2)C_{K,D,T}\|ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma^\alpha)} \\
&\quad + C(\|z_0\|^2_{L^2(\Omega)} + \|z_1\|^2_{L^2(\Omega)}).
\end{aligned} \tag{4.32}$$

Proof. We shall take four steps to prove it.

Step 1. Let z be a complex solution to problem (3.8). By using the cosine operator theory (see [3]), we obtain

$$z(t) = e^{i\mathscr{A}t}\tilde{z}_0 + e^{-i\mathscr{A}t}\tilde{z}_1 + \mathscr{A}^{-1}\int_0^t \frac{1}{2i}(e^{i\mathscr{A}(t-\tau)} - e^{-i\mathscr{A}(t-\tau)})\mathbf{D}f(\tau)\mathrm{d}\tau, \tag{4.33}$$

where

$$f = -ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}}, \quad \tilde{z}_0 = \frac{z_0}{2} - \frac{i}{2}\mathscr{A}^{-1}z_1, \quad \tilde{z}_1 = \frac{z_0}{2} + \frac{i}{2}\mathscr{A}^{-1}z_1.$$

To simplify notation, we define

$$\begin{aligned}
&A_1 = (\mathscr{A}e^{i\mathscr{A}t}\tilde{z}_0)_{\nu_{\mathscr{A}}}, \quad B_1 = \frac{1}{2}(\int_0^t e^{-i\mathscr{A}(t-\tau)}\mathbf{D}f(\tau)\mathrm{d}\tau)_{\nu_{\mathscr{A}}}, \\
&A_2 = (\mathscr{A}e^{-i\mathscr{A}t}\tilde{z}_1)_{\nu_{\mathscr{A}}}, \quad B_2 = \frac{1}{2}(\int_0^t e^{i\mathscr{A}(t-\tau)}\mathbf{D}f(\tau)\mathrm{d}\tau)_{\nu_{\mathscr{A}}}.
\end{aligned} \tag{4.34}$$

Using these definitions, we have

$$(z_t)_{\nu_{\mathscr{A}}} = i(A_1 - A_2) + (B_1 + B_2), \tag{4.35}$$

$$(\mathscr{A} z)_{\nu_{\mathscr{A}}} = (A_1 + A_2) + i(B_1 - B_2). \tag{4.36}$$

Thus, we obtain

$$|(z_t)_{\nu_{\mathscr{A}}}|^2 - |(\mathscr{A} z)_{\nu_{\mathscr{A}}}|^2 = 4Re(-A_1\bar{A}_2 + iA_1\bar{B}_1 - iA_2\bar{B}_2 + \bar{B}_1 B_2). \tag{4.37}$$

Step 2. Let $\phi(t) \in C_0^\infty(\mathbb{R})$ be such that $0 \le \phi(t) \le 1, \phi(t) \equiv 1$ on $[0, T]$, and $\phi(t) \equiv 0$ on $(-\infty, -\alpha) \cup (T + \alpha, \infty)$. From (4.37), we obtain

$$\begin{aligned}
&\|(z_t)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1)} \\
&\le \|(\mathscr{A} z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma_1^\alpha)} + 4\Big|\int_{-\infty}^{\infty} \phi(t) \int_{\Gamma_1} A_1\bar{A}_2 \mathrm{d}x\mathrm{d}t\Big| \\
&\quad + 4\Big|\int_{-\infty}^{\infty} \phi(t) \int_{\Gamma_1} A_1\bar{B}_1 \mathrm{d}x\mathrm{d}t\Big| + 4\Big|\int_{-\infty}^{\infty} \phi(t) \int_{\Gamma_1} A_2\bar{B}_2 \mathrm{d}x\mathrm{d}t\Big| \\
&\quad + 4\Big|\int_{-\infty}^{\infty} \phi(t) \int_{\Gamma_1} \bar{B}_1 B_2 \mathrm{d}x\mathrm{d}t\Big|.
\end{aligned} \tag{4.38}$$

Step 3.

$$\begin{aligned}
A_1\bar{A}_2 &= (\mathscr{A} e^{i\mathscr{A} t}\tilde{z}_0)_{\nu_{\mathscr{A}}} \times (\mathscr{A} e^{i\mathscr{A} t}\bar{\tilde{z}}_1)_{\nu_{\mathscr{A}}} \\
&= (\sum_{n=1}^{\infty} \lambda_n e^{-i\lambda_n t}(\tilde{z}_0, \phi_n)(\phi_n)_{\nu_{\mathscr{A}}}) \times (\sum_{m=1}^{\infty} \lambda_m e^{-i\lambda_m t}(\bar{\tilde{z}}_1, \phi_m)(\phi_m)_{\nu_{\mathscr{A}}}),
\end{aligned} \tag{4.39}$$

where λ_i and ϕ_i denote the eigenvalues and eigenfunctions corresponding to the operator $-\mathscr{A}$ with $|\phi_i| = 1$. Since

$$|(\phi_n)_{\nu_{\mathscr{A}}}| \le C|\mathscr{A} \phi_n| \le C\lambda_n|\phi_n| = C\lambda_n,$$

we have

$$\int_{\Gamma_1} |(\phi_n)_{\nu_{\mathscr{A}}}||(\phi_m)_{\nu_{\mathscr{A}}}| \mathrm{d}\Gamma \le C\lambda_n\lambda_m. \tag{4.40}$$

Combining (4.39) and (4.40), we find

$$\begin{aligned}
&\Big|\int_{-\infty}^{\infty} \phi(t) \int_{\Gamma_1} A_1\bar{A}_2 \mathrm{d}x\mathrm{d}t\Big| \\
&\le C\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_n^2\lambda_m^2 |(\tilde{z}_0, \phi_n)||(\bar{\tilde{z}}_1, \phi_m)|\Big|\int_{-\infty}^{\infty} \phi(t)e^{-i(\lambda_n+\lambda_m)t}\mathrm{d}t\Big|.
\end{aligned}$$

Since $\phi(t) \in C_0^\infty(\mathbb{R})$, for any N, we have

$$\Big|\int_{-\infty}^{\infty} \phi(t)e^{-i(\lambda_n+\lambda_m)t}\mathrm{d}t\Big| \le \frac{c_\phi}{|\lambda_n + \lambda_m|^N}. \tag{4.41}$$

Thus,

$$\begin{aligned}
\Big|\int_{-\infty}^{\infty} \phi(t) \int_{\Gamma_1} A_1\bar{A}_2 \mathrm{d}x\mathrm{d}t\Big| &\le C_\phi(\sum_{n=1}^{\infty} \frac{|(\tilde{z}_0, \phi_n)|^2}{\lambda_n^{N-4}} + \sum_{m=1}^{\infty} \frac{|(\bar{\tilde{z}}_1, \phi_m)|^2}{\lambda_m^{N-4}}) \\
&= C_\phi(\|\mathscr{A}^{2-\frac{N}{2}}\tilde{z}_0\|^2_{L^2(\Omega)} + \|\mathscr{A}^{2-\frac{N}{2}}\bar{\tilde{z}}_1\|^2_{L^2(\Omega)}) \\
&\le C(\|z_0\|^2_{H^{4-N}(\Omega)} + \|z_1\|^2_{H^{2-N}(\Omega)}).
\end{aligned} \tag{4.42}$$

Step 4. Before we complete the proof of Lemma 4.7, we shall need the following result to estimate the remaining three terms on the right-hand side of inequality (4.38).

Proposition 4.8. *Let y be a solution of the problem*

$$\begin{gathered} y_t = i\mathbf{A}^{1/2}y + \mathbf{D}\xi, \\ y(0) = y_0 \in \mathbf{D}(\mathbf{A}^{1/4}). \end{gathered} \tag{4.43}$$

Then

$$\|y_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} \le C_{T,3}\|\mathscr{A}^{1/2}y_0\|^2_{L^2(\Omega)} + C_{K,D,T}\|\xi\|^2_{L^2(\Sigma)}. \tag{4.44}$$

The above proposition will be proven later. Applying the result of Proposition 4.8 with $\xi = 0$, we obtain

$$\begin{aligned} \|A_1\|^2_{L^2(\Sigma^\alpha)} + \|A_2\|^2_{L^2(\Sigma^\alpha)} &\le C_{T,3}(\|\mathscr{A}^{3/2}\tilde{z}_0\|^2_{L^2(\Omega)} + \|\mathscr{A}^{3/2}\tilde{z}_1\|^2_{L^2(\Omega)}) \\ &\le C_{T,3}(\|\mathscr{A}^{3/2}z_0\|^2_{L^2(\Omega)} + \|\mathscr{A}^{1/2}z_1\|^2_{L^2(\Omega)}), \end{aligned} \tag{4.45}$$

with $y_0 = 0$, we obtain

$$\|B_1\|^2_{L^2(\Sigma^\alpha)} + \|B_2\|^2_{L^2(\Sigma^\alpha)} \le C_{K,D,T}\|f\|^2_{L^2(\Sigma^\alpha)} = C_{K,D,T}\|ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma^\alpha)}.$$

Then

$$\begin{aligned} &\int_{-\alpha}^{T+\alpha}\int_{\Gamma_1}(|A_1\bar{B}_1| + |A_2\bar{B}_2| + |\bar{B}_1B_2|)\mathrm{d}x\mathrm{d}t \\ &\le \int_{\Sigma_1^\alpha}[\epsilon(|A_1|^2 + |A_2|^2) + (C_\epsilon + \frac{1}{2})(|B_1|^2 + |B_2|^2)]\mathrm{d}\Sigma \\ &\le \epsilon C_{T,3}(\|\mathscr{A}^{3/2}z_0\|^2_{L^2(\Omega)} + \|\mathscr{A}^{1/2}z_1\|^2_{L^2(\Omega)}) \\ &\quad + (C_\epsilon + \frac{1}{2})C_{K,D,T}\|ka(x)(\mathscr{A}z)_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma^\alpha)}. \end{aligned} \tag{4.46}$$

Combining (4.38), (4.42) and (4.46), we obtain the desired inequality (4.32). □

Proof of Proposition 4.8. Let

$$y(t) = y_1 + y_2 = e^{-i\mathscr{A}t}y_0 + \int_0^t e^{-i\mathscr{A}(t-\tau)}\mathbf{D}\xi(\tau)\mathrm{d}\tau. \tag{4.47}$$

Clearly, y satisfies (4.43). We shall do the proof by several steps.

Step 1. We firstly prove the estimate

$$\|y_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} \le C_{T,4}(\|\mathbf{D}\xi\|^2_{L^1[0,T;L^2(\Omega)]} + \|\mathscr{A}^{1/2}y\|^2_{L^\infty[0,T;L^2(\Omega)]}). \tag{4.48}$$

Proof. We multiply (4.43) by $h(\bar{y})$, where $h|_\Gamma = \nu_{\mathscr{A}}$, and integrate over Q by parts, with Lemma 4.1 to obtain

$$\begin{aligned}
&\operatorname{Im}\int_Q y_t h(\bar{y})\mathrm{d}Q \\
&= \operatorname{Im}\Big(-\int_Q i\mathscr{A}yh(\bar{y}) + \mathbf{D}\xi h(\bar{y})\mathrm{d}Q\Big) \\
&= \int_Q \operatorname{Re}[-\operatorname{div} h(\bar{y})A(x)\nabla y + \langle \nabla_g h(\bar{y}), \nabla_g y\rangle]\mathrm{d}Q + \operatorname{Im}\int_Q \mathbf{D}\xi h(\bar{y})\mathrm{d}Q \\
&= \int_Q \{\operatorname{Re}[-\operatorname{div} h(\bar{y})A(x)\nabla y + Dh(\nabla_g y, \nabla_g \bar{y})] \\
&\quad + \frac{1}{2}\operatorname{div}(\langle \nabla_g y, \nabla_g \bar{y}\rangle_g h) - \frac{1}{2}\langle \nabla_g y, \nabla_g \bar{y}\rangle_g \operatorname{div} h\}\mathrm{d}Q + \operatorname{Im}\int_Q \mathbf{D}\xi h(\bar{y})\mathrm{d}Q \\
&= \operatorname{Im}\int_Q \mathbf{D}\xi h(\bar{y})\mathrm{d}Q - \frac{1}{2}\int_\Sigma |y_{\nu_{\mathscr{A}}}|^2\mathrm{d}\Sigma \\
&+ \int_Q [\operatorname{Re} Dh(\nabla_g y, \nabla_g \bar{y}) - \frac{1}{2}|\nabla_g y|_g^2 \operatorname{div} h]\mathrm{d}Q,
\end{aligned} \tag{4.49}$$

where the notation "Im" and "Re" denote the imaginary part and the real part of a complex number, respectively.

On the other hand, using the divergence theorem, we find

$$\operatorname{div} \bar{y}y_t h = \bar{y}y_t \operatorname{div} h + y_t h(\bar{y}) + [\bar{y}h(y)]_t - \bar{y}_t h(y). \tag{4.50}$$

Thus,

$$\begin{aligned}
\operatorname{Im}\int_Q y_t h(\bar{y})\mathrm{d}Q &= -\frac{i}{2}\int_\Sigma \bar{y}y_t |\nu_{\mathscr{A}}|_g^2 \mathrm{d}\Sigma + \frac{i}{2}\int_Q \bar{y}y_t \operatorname{div} h\mathrm{d}Q \\
&\quad + \frac{i}{2}\int_\Omega [\bar{y}(T)h(y)(T) - \bar{y}(0)h(y)(0)]\mathrm{d}\Omega.
\end{aligned} \tag{4.51}$$

Combining (4.49) and (4.51), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma |y_{\nu_{\mathscr{A}}}|^2\mathrm{d}\Sigma \\
&= \operatorname{Im}\int_Q \mathbf{D}\xi h(\bar{y})\mathrm{d}Q - \frac{i}{2}\int_Q \bar{y}y_t \operatorname{div} h\mathrm{d}Q \\
&\quad + \frac{i}{2}\int_\Omega [\bar{y}(0)h(y)(0) - \bar{y}(T)h(y)(T)]\mathrm{d}\Omega \\
&\quad + \int_Q [\operatorname{Re} Dh(\nabla_g y, \nabla_g \bar{y}) - \frac{1}{2}|\nabla_g y|_g^2 \operatorname{div} h]\mathrm{d}Q.
\end{aligned} \tag{4.52}$$

Next, we multiply (4.43) by $\bar{y}$ and integrate over Q by parts to obtain

$$\int_Q y_t \bar{y}\mathrm{d}Q = -i\int_Q \bar{y}\mathscr{A}y\mathrm{d}Q + \int_Q \bar{y}\mathbf{D}\xi\mathrm{d}Q = i\int_Q \langle \nabla_g \bar{y}, \nabla_g y\rangle_g \mathrm{d}Q + \int_Q \bar{y}\mathbf{D}\xi\mathrm{d}Q.$$

Thus,

$$
\begin{aligned}
\big|\int_Q y_t\bar{y}\mathrm{d}Q\big| &\le \int_Q |\nabla_g y|_g^2\mathrm{d}Q + \int_Q |\bar{y}\mathbf{D}\xi|\mathrm{d}Q\\
&\le \int_Q |\nabla_g y|_g^2\mathrm{d}Q + \frac{1}{2}\int_Q (|\bar{y}|^2+|\mathbf{D}\xi|^2)\mathrm{d}Q\\
&\le C\int_Q |\nabla_g y|_g^2\mathrm{d}Q + \frac{1}{2}T\|\mathbf{D}\xi\|^2_{L^\infty[0,T;L^2(\Omega)]}\\
&\le C_T(\||\nabla_g y|_g\|^2_{L^\infty[0,T;L^2(\Omega)]} + \|\mathbf{D}\xi\|^2_{L^1[0,T;L^2(\Omega)]}).
\end{aligned}
\tag{4.53}
$$

Furthermore, we can bound all terms of the right-hand side of (4.52) as follows:

$$
\begin{aligned}
&\mathrm{Im}\int_Q \mathbf{D}\xi h(\bar{y})\mathrm{d}Q\\
&\le \big|\int_Q \mathbf{D}\xi h(\bar{y})\mathrm{d}Q\big|\\
&\le \frac{1}{2}T\|\mathbf{D}\xi\|^2_{L^\infty[0,T;L^2(\Omega)]} + \frac{1}{2}\sup_{x\in\Omega}|h|_g^2\int_0^T \||\nabla_g\bar{y}|_g\|^2_{L^2(\Omega)}\mathrm{d}t\\
&\le \frac{1}{2}T\|\mathbf{D}\xi\|^2_{L^1[0,T;L^2(\Omega)]} + \frac{1}{2}\sup_{x\in\Omega}|h|_g^2 T\||\nabla_g\bar{y}|_g\|^2_{L^\infty[0,T;L^2(\Omega)]};
\end{aligned}
\tag{4.54}
$$

$$
\big|-\frac{i}{2}\int_Q \bar{y}y_t\,\mathrm{div}\,h\mathrm{d}Q\big| \le \frac{1}{2}\sup_{x\in\Omega}|\,\mathrm{div}\,h|\big|\int_Q y_t\bar{y}\mathrm{d}Q\big|;
\tag{4.55}
$$

$$
\begin{aligned}
&\big|\frac{i}{2}\int_\Omega [\bar{y}(0)h(y)(0)-\bar{y}(T)h(y)(T)]\mathrm{d}x\big|\\
&\le \frac{1}{2}\int_\Omega [|\bar{y}(0)\langle h,\nabla_g y\rangle_g(0)| + |\bar{y}(T)\langle h,\nabla_g y\rangle_g(T)|]\mathrm{d}x\\
&\le \frac{1}{4}\sup_{x\in\Omega}|h|_g\int_\Omega [|\nabla_g y(0)|_g^2+|\bar{y}(0)|^2+|\nabla_g y(T)|_g^2+|\bar{y}(T)|^2]\mathrm{d}x\\
&\le \frac{1}{2}\sup_{x\in\Omega}|h|_g(\||\nabla_g y|_g\|^2_{L^\infty[0,T;L^2(\Omega)]}+\|\bar{y}\|^2_{L^\infty[0,T;L^2(\Omega)]})\\
&\le C\sup_{x\in\Omega}|h|_g\||\nabla_g y|_g\|^2_{L^\infty[0,T;L^2(\Omega)]};
\end{aligned}
\tag{4.56}
$$

$$
\int_Q \mathrm{Re}\,Dh(\nabla_g y,\nabla_g\bar{y})\mathrm{d}Q \le C_T\||\nabla_g y|_g\|^2_{L^\infty[0,T;L^2(\Omega)]};
\tag{4.57}
$$

$$
-\int_Q \frac{1}{2}|\nabla_g y|_g^2\,\mathrm{div}\,h\mathrm{d}Q \le C_T\sup_{x\in\Omega}|\,\mathrm{div}\,h|\||\nabla_g y|_g\|^2_{L^\infty[0,T;L^2(\Omega)]}.
\tag{4.58}
$$

Finally, by combining (4.52), (4.53) - (4.58), we obtain the desired inequality (4.48). □

Step 2. Estimates for y_1

$$
\|\mathscr{A}^{1/2}y_1(t)\|^2_{L^2(\Omega)} = \|\mathscr{A}^{1/2}e^{-i\mathscr{A}t}y_0\|^2_{L^2(\Omega)} = \|\mathscr{A}^{1/2}y_0\|^2_{L^2(\Omega)} = \text{constant}.
$$

Therefore,

$$
\|\mathscr{A}^{1/2}y_1\|^2_{L^\infty[0,T;L^2(\Omega)]} = \|\mathscr{A}^{1/2}y_0\|^2_{L^2(\Omega)}.
\tag{4.59}
$$

Step 3. Estimates for y_2. We shall prove

$$\|y_2\|^2_{L^\infty[0,T;H^1_0(\Omega)]} \le C_K\|\xi\|^2_{L^2(\Sigma)}. \tag{4.60}$$

Proof. We define a closed and dense operator $L : L^2(\Sigma) \to L^2(Q)$ by

$$(Lf)(t) = \mathscr{A}\int_0^t e^{-i\mathscr{A}(t-\tau)}\mathbf{D}f(\tau)\mathrm{d}\tau. \tag{4.61}$$

Then we can obtain

$$(L^*\Phi)(t) = \mathbf{D}^*\mathscr{A}\int_0^t e^{-i\mathscr{A}(t-\tau)}\Phi(\tau)\mathrm{d}\tau, \tag{4.62}$$

where $\Phi = \mathbf{D}f$.

Let $\eta = \int_0^t e^{-i\mathscr{A}(t-\tau)}\Phi(\tau)\mathrm{d}\tau$, then η satisfies the equation

$$\begin{gathered}\eta_t = i\mathbf{A}^{1/2}\eta + \Phi,\\ \eta(0) = 0.\end{gathered} \tag{4.63}$$

As in the proof of Step 1, we can show that

$$\|\eta_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} \le C_{T,5}(\|\Phi\|^2_{L^1[0,T;L^2(\Omega)]} + \|\mathscr{A}^{1/2}\eta\|^2_{L^\infty[0,T;L^2(\Omega)]}). \tag{4.64}$$

Moreover.

$$\|\mathscr{A}^{1/2}\eta\|_{L^2(\Omega)} \le \int_0^t \|\mathscr{A}^{1/2}\Phi(\tau)\|_{L^2(\Omega)}\mathrm{d}\tau \le \|\mathscr{A}^{1/2}\Phi\|_{L^1[0,T;L^2(\Omega)]}. \tag{4.65}$$

Combining (4.64) and (4.65) yields

$$\|\eta_{\nu_{\mathscr{A}}}\|^2_{L^2(\Sigma)} \le C_{T,5}\|\Phi\|^2_{L^1[0,T;H^1_0(\Omega)]}. \tag{4.66}$$

In addition,

$$\begin{aligned}(\mathbf{D}^*\mathscr{A}\eta, f)_{L^2(\Gamma)} &= (\mathscr{A}\eta, \mathbf{D}f)_{L^2(\Omega)}\\ &= \int_\Gamma [\eta_{\nu_{\mathscr{A}}}\mathbf{D}f - \eta(\mathbf{D}f)_{\nu_{\mathscr{A}}}]\mathrm{d}x + \int_\Omega \eta\mathscr{A}(\mathbf{D}f)\mathrm{d}x\\ &= (\eta_{\nu_{\mathscr{A}}}, f)_{L^2(\Gamma)},\end{aligned} \tag{4.67}$$

the last equality holds because of the definition of the operator $\mathbf{D}$ and $\eta \in D(\mathbf{A}^{1/2})$. Therefore, $\eta_{\nu_{\mathscr{A}}} = \mathbf{D}^*\mathscr{A}\eta = L^*\Phi$. It tell us that

$$L^* \in \mathscr{L}(L^1[0,T;H^1_0(\Omega)] \to L^2(\Sigma)). \tag{4.68}$$

And then, we have

$$L \in \mathscr{L}(L^2(\Sigma) \to L^\infty[0,T;H^{-1}(\Omega)]). \tag{4.69}$$

Let K be defined by

$$Kf = \mathscr{A}^{-1}Lf; \tag{4.70}$$

then $K \in \mathscr{L}(L^2(\Sigma) \to L^\infty[0,T;H^1_0(\Omega)])$. Since $K\xi = y_2$, we obtain

$$\|y_2\|^2_{L^\infty[0,T;H^1_0(\Omega)]} \le C_K\|\xi\|^2_{L^2(\Sigma)}. \tag{4.71}$$

Thus, inequality (4.60) holds. □

Step 4. Combining (4.59) and (4.71), we find

$$\begin{aligned}\|\mathscr{A}^{1/2}y\|^2_{L^\infty[0,T;L^2(\Omega)]} &= \|\mathscr{A}^{1/2}y_1+\mathscr{A}^{1/2}y_2\|^2_{L^\infty[0,T;L^2(\Omega)]}\\ &\le 2\|\mathscr{A}^{1/2}y_1\|^2_{L^\infty[0,T;L^2(\Omega)]}+2\|y_2\|^2_{L^\infty[0,T;H_0^1(\Omega)]}\\ &\le 2\|\mathscr{A}^{1/2}y_0\|^2_{L^2(\Omega)}+2C_K\|\xi\|^2_{L^2(\Sigma)}.\end{aligned}\tag{4.72}$$

By substituting inequality (4.72) into (4.48) and recalling that $\mathbf{D}\in\mathscr{L}(L^2(\Gamma)\to L^2(\Omega))$, the desired result of Proposition 4.8 is found.

Now, we are ready to complete the proof of the observability inequality (3.9). Combining the results of Lemmas 4.3 and 4.7 with $\epsilon=\frac{C_{T,1}}{2C_{T,3}}$, we obtain

$$\begin{aligned}C_{T,1}E(0) &\le 4\|(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma_1^\alpha)}+2C(\|z_0\|^2_{L^2(\Omega)}+\|z_1\|^2_{L^2(\Omega)})\\ &\quad+4(\frac{8C_{T,3}}{C_{T,1}}+1)C_{K,D,T}\|ka(x)(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma^\alpha)}.\end{aligned}\tag{4.73}$$

From Lemma 4.4, we find

$$\begin{aligned}\int_{-\alpha}^{T+\alpha}\|ka(x)(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Gamma)}\mathrm{d}t &\le a\|k\|^2_{L^\infty(\Gamma)}\|(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma^\alpha)}\\ &\le a\|k\|^2_{L^\infty(\Gamma)}C_{T,2}E(0).\end{aligned}\tag{4.74}$$

Thus, if

$$\|k\|^2_{L^\infty(\Gamma)}<k_0=\frac{C_{T,1}^2}{4aC_{T,2}C_{K,D,T}(8C_{T,3}+C_{T,1})},$$

where $a=\sup_{x\in\Gamma}|a(x)|^2$, by combining (4.73) and (4.74), we obtain

$$\begin{aligned}C_TE(0) &\le \|(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma_1^\alpha)}+C\big(\|z_0\|^2_{L^2(\Omega)}+\|z_1\|^2_{L^2(\Omega)}\big)\\ &\le \|(\mathscr{A}z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma_1^\alpha)}+\|(\mathscr{A}^2z)_{\nu_\mathscr{A}}\|^2_{L^2(\Sigma_1^\alpha)}+CL(z).\end{aligned}\tag{4.75}$$

Again, the lower terms in the right-hand side of inequality (4.75) can be absorbed by using the following compactness-uniqueness argument:

Step 1. Let $U=\{z\in H^3(Q): z$ is a solution to problem (3.8) satisfying $(\mathscr{A}z)_{\nu_\mathscr{A}}|_{\Sigma_1^\alpha}=(\mathscr{A}^2z)_{\nu_\mathscr{A}}|_{\Sigma_1^\alpha}=0\}$. Then

$$U=0.\tag{4.76}$$

Indeed, from inequality (4.75), we have

$$CL(z)\ge C_TE(0)\quad\text{for all } z\in U,$$

which implies that any bounded closed set in $U\cap H^3(Q)$ is compact in $H^3(Q)$. Then U is a finite-dimensional linear space.

For any $z\in U$, we have $z_t\in U$. Then $\partial_t: U\to U$ is a linear operator. Let $U\neq 0$, then ∂_t has at least one eigenvalue λ. Assume that $v\neq 0$ is one of its eigenfunctions, then $v_t=\lambda v$. Further, v is a nonzero solution to the problem

$$\begin{gathered}\mathscr{A}^2v=-\lambda^2v-\mathbf{D}(ka(x)(\mathscr{A}v)_{\nu_\mathscr{A}})\quad\text{on }\Omega,\\ v=\mathscr{A}v=(\mathscr{A}v)_{\nu_\mathscr{A}}=(\mathscr{A}^2v)_{\nu_\mathscr{A}}=0\quad\text{on }\Gamma_1.\end{gathered}\tag{4.77}$$

Let $\Psi=\mathbf{A}^{1/2}v$, using relations (3.5) and (4.77), it is easy to find that Ψ satisfies the problem

$$\begin{gathered}\mathscr{A}^2\Psi=-\lambda^2\Psi\quad\text{on }\Omega,\\ \Psi=\Psi_{\nu_\mathscr{A}}=\mathscr{A}\Psi=(\mathscr{A}\Psi)_{\nu_\mathscr{A}}=0\quad\text{on }\Gamma_1.\end{gathered}\tag{4.78}$$

By Proposition 4.6, the above problem only has zero solution, thus $\Psi \equiv 0$ in $\Omega \cup \Gamma_1$, i.e., $\mathscr{A} v \equiv 0$ in $\Omega \cup \Gamma_1$. Moreover, $v|_\Gamma = 0$, therefore we can obtain $v \equiv 0$, this contradiction shows that (4.76) holds.

Since the subsequent proof is basically the same as that in Step 2 of Lemma 4.5, we omit it.

Finally, we obtain

$$\int_{\Sigma_1^\alpha} [(\mathscr{A} z)^2_{\nu_{\mathscr{A}}} + (\mathscr{A}^2 z)^2_{\nu_{\mathscr{A}}}] \mathrm{d}\Sigma \geq C_T E(0). \tag{4.79}$$

Introducing the new variable $\tilde{z} = z(t - \alpha)$ into (4.79) yields

$$\int_0^{T+2\alpha} \int_{\Gamma_1} [(\mathscr{A} \tilde{z})^2_{\nu_{\mathscr{A}}} + (\mathscr{A}^2 \tilde{z})^2_{\nu_{\mathscr{A}}}] \mathrm{d}\Sigma \geq C_T E(0). \tag{4.80}$$

Since both $\tilde{z}$ and z are solutions to the same problem (3.8), the inequality (3.9) holds with T replaced by $T + 2\alpha$. $\square$

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