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REGULARIZATION AND ERROR ESTIMATES FOR ASYMMETRIC BACKWARD NONHOMOGENEOUS HEAT EQUATIONS IN A BALL

LE MINH TRIET, LUU HONG PHONG

Abstract. The backward heat problem (BHP) has been researched by many authors in the last five decades; it consists in recovering the initial distribution from the final temperature data. There are some articles [1, 2, 3] related the axi-symmetric BHP in a disk but the study in spherical coordinates is rare. Therefore, we wish to study a backward problem for nonhomogenous heat equation associated with asymmetric final data in a ball. In this article, we modify the quasi-boundary value method to construct a stable approximate solution for this problem. As a result, we obtain regularized solution and a sharp estimates for its error. At the end, a numerical experiment is provided to illustrate our method.

1. Introduction

Inverse problems for partial differential equations play a vital role in many physical areas. A typical example of these problems is the backward heat problem (BHP) which is also known as the final value problem. The purpose of the BHP is to retrieve the temperature distribution at a particular time $t < T$ from the final temperature data. As we known, the BHP is severely ill-posed in Hadamard's sense, i.e., the solution does not always exist. Even if the solution exists, it may not depend continuously on the given data. Therefore, an appropriate regularization is required so as to get a stable solution.

There have been a lot of research related to the BHP in different kinds of domains. For instance, the BHP has been investigated in rectangular coordinates by many authors [6, 11, 13, 15, 16], to list just a few of them. Recently, some works have considered polar coordinates and cylindrical coordinates. In particular, Cheng and Fu [1, 2, 3] studied the axisymmetric backward heat problem in a disk. Cheng and Fu [1, 3] used the modified Tikhonov method for regularizing the problem

$$
\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad 0 < r \le r_0,\ 0 < t < T, \\
u(r,T) = \varphi(r), \quad 0 \le r \le r_0, \\
u(r_0,t) = 0, \quad 0 \le t \le T, \\
|u(0,t)| < \infty, \quad 0 \le t \le T,
\end{gathered}
\tag{1.1}
$$

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where the function $\varphi(\cdot)$ in the problem (1.1) is radially symmetric or axisymmetric, i.e. it depends only on the radius r and not on θ.

Cheng W. et al. [2] considered a problem which is similar to (1.1). However, there are some differences in initial condition which is expressed as follows

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad 0 < r \le R,\ 0 < t, \\
u(r,0) = 0, \quad 0 \le r \le R, \\
u(r_1,t) = g(t), \quad 0 \le t, \\
|u(0,t)| < \infty, \quad 0 \le t,
\end{gathered} \tag{1.2}$$

in which r is the radius coordinate and $g(\cdot)$ is the temperature distribution at one fixed radius $r_1 \le R$ of a cylinder. By applying the Fourier transform, the authors found the exact solution of the problem (1.2) and used the modified Tikhonov method to construct the regularized solutions. In the above papers [1, 2, 3], although the authors suggested some methods to regularize (1.1) and (1.2), they still did not give any numerical test to prove the effectiveness of their regularization.

From the above problems, we see that BHP was considered in a rectangular domain or a disk. In our knowledge, the works for BHP in a ball are rarely studied and even we have not ever seen any results dealt with the asymmetric case. Motivated by this reason, we focus on the problem of determining the temperature distribution $u(r,\theta,\phi,t)$, for $(r,\theta,\phi,t) \in (0,a)\times(0,\pi)\times(0,2\pi)\times(0,T)$, satisfying

$$u_t = c^2\Big\{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta} + \csc^2\theta\frac{\partial^2 u}{\partial \phi^2}\Big)\Big\} + q(r,\theta,\phi), \tag{1.3}$$

$$u(a,\theta,\phi,t) = 0, \tag{1.4}$$

$$u(r,\theta,\phi,T) = f(r,\theta,\phi), \tag{1.5}$$

$$|u(0,\theta,\phi,t)| < \infty, \tag{1.6}$$

where a is the radius coordinate and $f(\cdot,\theta,\phi) \in L^2[[0;a];r]$ is the final temperature. In practice, we cannot always obtain radially symmetric or axisymmetric form of the data function f. Additionally, in physical applications, not only does the initial temperature depend on the final data but it also depends on the heat source. Hence, the heat source q is not often homogeneous. Thus, problem (1.3)-(1.6) is more general than problem (1.1) and (1.2). From that, problem (1.3)-(1.6) is more practical and applicable than (1.1) and (1.2). In this paper, we apply the modified quasi-boundary value method (MQBV) to formulate the approximate solution for (1.3)-(1.6). As we known, the quasi-boundary value (QBV) method which was given by Showalter in 1983 is one of effective regularization methods. In [12], the main idea of the QBV method is to add an appropriate "corrector term" into the boundary condition. Based on this idea, in [11] we have modified the "corrector term" to get a stable error estimations so we called it the modified quasi-boundary value method. By using the MQBV method, we can obtain the Hölder type estimate for the error between the regularized solution and the exact solution. Furthermore, one advantage of the MQBV method is easier to make numerical experiment for testing the feasibility of the method. Thus, we can make an example to illustrate our results in this paper and it is a better point of our paper when we compare with some previous papers [1, 2, 3].

The rest of this article is organized as follows. In Section 2, some definitions and propositions are given. In Section 3, we propose the regularized solutions for problem (1.3)-(1.6) and estimate the error between the regularized solutions and the exact solution. Then, the proof of our results is given in Section 4. Finally, we present a numerical experiment to illustrate the main results in Section 5.

2. Some definitions and propositions

Definition 2.1. Let $a > 0$ and $L^2[[0;a];r] = \{f : [0;a] \to \mathbb{R} : f$ is Lebesgue measurable with weigh r on $[0;a]\}$. The above space is equipped with norm

$$\|f\|_2 = \Big(\int_0^a r|f(r)|^2 dr \Big)^{1/2}.$$

Next some definitions and propositions, presented in [5, 9, 18], are restated.

Proposition 2.2. *Let n be a non-negative integer. Then, the spherical Bessel functions of the 1^{st} kind of order n are defined as*

$$j_n(x) = (\frac{\pi}{2x})^{1/2} J_{n+\frac{1}{2}}(x),$$

where $J_{n+\frac{1}{2}}$ is the Bessel function of the 1^{st}-kind of order $n + \frac{1}{2}$.

Proposition 2.3. *Let n be a non-negative integer and the spherical Bessel's equation of order n be defined by*

$$x^2 y'' + 2xy' + (\lambda^2 x^2 - n(n+1))y = 0, \quad 0 < x < a, \ y(a) = 0. \tag{2.1}$$

Then, we obtain the following solutions for equation (2.1),

$$y_{n,j}(x) = j_n(\lambda_{n,j} x), \quad n = 0, 1, 2, \dots, j = 1, 2, \dots,$$

where $\lambda = \lambda_{n,j} = \frac{\alpha_{n+1/2,j}}{a}$, for $\alpha_{n+1/2,j}$ denotes the jth positive zero of $J_{n+\frac{1}{2}}$.

Proposition 2.4. *Let n be a non-negative integer. Then, we have the Legendre polynomial of the 1^{st} kind of degree n,*

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m)!} x^{n-2m}, \tag{2.2}$$

in which $M = n/2$ if n is even or $M = (n-1)/2$ if n is odd. Moreover, we have the Legendre function of the 2^{nd} kind of degree n,

$$Q_n(x) = P_n(x) \int \frac{1}{[P_n(x)]^2(1-x^2)} dx, \quad (n = 0, 1, 2, \dots). \tag{2.3}$$

Proposition 2.5. *For $n = 0, 1, 2, \dots$, Legendre's equation of degree n,*

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad -1 < x < 1. \tag{2.4}$$

From which, the general solution of (2.4) is

$$y(x) = c_1 P_n(x) + c_2 Q_n(x),$$

where $P_n(x), Q_n(x)$ are defined by (2.2) and (2.3), respectively, and c_1, c_2 are arbitrary constants.

Remark 2.6. (i) For $n = 0, 1, 2, \dots$ and $m = 0, 1, 2, \dots$, the associated Legendre function $P_n^m(x)$ is defined in terms of the $m-th$ derivative of the Legendre polynomial of degree n by

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}. \tag{2.5}$$

Since P_n is a polynomial of degree n, for P_n^m to be nonzero, we must take $0 \le m \le n$. Moreover, if m is negative integer, we defined P_n^m by

$$P_n^m(x) = (-1)^m \frac{(n+m)!}{(n-m)!} P_n^{-m}(x).$$

This extends the definition of the associated Legendre function for $n = 0, 1, 2, \dots$ and $m = -n, -(n-1), \dots, n-1, n$.

(ii) After that, we define the spherical harmonics $Y_{n,m}(\theta, \phi)$ by

$$Y_{n,m}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi}, \tag{2.6}$$

where $n = 0, 1, 2, \dots$ and $m = -n, -(n-1), \dots, n-1, n$.

Proposition 2.7. *Let n be a non-negative integer and the differential equation for the spherical harmonics be defined by*

$$\frac{\partial^2 Y}{\partial \theta^2} + \cot\theta \frac{\partial Y}{\partial \theta} + \csc^2\theta \frac{\partial^2 Y}{\partial \phi^2} + n(n+1)Y = 0,$$

where $0 < \theta < \pi$, $0 < \phi < 2\pi$. Then, we have $2n+1$ nontrivial solutions given by the spherical harmonics

$$Y(\theta, \phi) = Y_{n,m}(\theta, \phi), \quad |m| \le n,$$

where $Y_{n,m}(\theta, \phi)$ is defined by (2.6).

Proposition 2.8. *Let $f(r, \theta, \phi)$ be a square integrable function, defined for $0 < r < a$, $0 < \theta < \pi$, $0 < \phi < 2\pi$, and 2π-periodic in ϕ. Then, we have*

$$f(r, \theta, \phi) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{jnm} j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi),$$

where

$$A_{jnm} = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+\frac{1}{2},j})} \int_0^a \int_0^{2\pi} \int_0^{\pi} f(r, \theta, \phi) j_n(\lambda_{n,j} r) \overline{Y}_{n,m}(\theta, \phi) r^2 \sin\theta \, d\theta \, d\phi \, dr,$$

and $\overline{Y}_{n,m}$ is the complex conjugate of $Y_{n,m}$.

3. Regularization and main results

By employing the method of separation of variables, the exact solution u of the problem (1.3)-(1.5) corresponding to the exact data f can be found out as follows

$$u(r, \theta, \phi, t) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{jnm}(t) j_n(\lambda_{n,j} r) Y_{n,m}(\theta, \phi), \tag{3.1}$$

where

$$A_{jnm}(t) = \exp\{c^2 \lambda_{n,j}^2 (T-t)\} \Big(f_{jnm} - \frac{q_{jnm}}{c^2 \lambda_{n,j}^2} \Big) + \frac{q_{jnm}}{c^2 \lambda_{n,j}^2},$$

$$f_{jnm} = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+1/2,j})} \int_0^a \int_0^{2\pi} \int_0^{\pi} f(r,\theta,\phi) j_n(\lambda_{n,j} r) \overline{Y}_{n,m}(\theta,\phi) r^2 \sin\theta \, d\theta \, d\phi \, dr,$$

$$q_{jnm} = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+1/2,j})} \int_0^a \int_0^{2\pi} \int_0^{\pi} q(r,\theta,\phi) j_n(\lambda_{n,j} r) \overline{Y}_{n,m}(\theta,\phi) r^2 \sin\theta \, d\theta \, d\phi \, dr.$$

From (3.1), we can see that the term $\exp\{c^2\lambda_{n,j}^2(T-t)\}$ becomes large as n tends to infinity. This term causes the instability of problem (1.3)-(1.5) so that we replace this term by a better term. In fact, if we use the QBV method; the regularized problem shall be as follows

$$\omega_t^\varepsilon = c^2 \nabla^2 \omega^\varepsilon + q(r,\theta,\phi), \tag{3.2}$$

$$\omega^\varepsilon(a,\theta,\phi,t) = 0, \tag{3.3}$$

$$\omega^\varepsilon(r,\theta,\phi,T) + \varepsilon\omega^\varepsilon(r,\theta,\phi,0) = f^\varepsilon(r,\theta,\phi), \tag{3.4}$$

$$|\omega^\varepsilon(0,\theta,\phi,t)| < \infty, \tag{3.5}$$

where ∇^2 is the spherical form of the Laplacian, i.e,

$$\nabla^2 \omega^\varepsilon = \frac{\partial^2 \omega^\varepsilon}{\partial r^2} + \frac{2}{r}\frac{\partial \omega^\varepsilon}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 \omega^\varepsilon}{\partial \theta^2} + \cot\theta \frac{\partial \omega^\varepsilon}{\partial \theta} + \csc^2\theta \frac{\partial^{2\omega\varepsilon}}{\partial \phi^2}\Big).$$

Then, we have the following regularized solution of (3.2)-(3.5),

$$\omega^\varepsilon(r,\theta,\phi,t) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{jnm}^\varepsilon(t) j_n(\lambda_{n,j} r) Y_{n,m}(\theta,\phi),$$

in which

$$A_{jnm}^\varepsilon(t) = \frac{\exp\{-c^2\lambda_{n,j}^2 t\}}{\varepsilon + \exp\{-c^2\lambda_{n,j}^2 T\}}\Big(f_{jnm}^\varepsilon - \frac{q_{jnm}}{c^2\lambda_{n,j}^2}\Big) + \frac{q_{jnm}}{c^2\lambda_{n,j}^2},$$

$$f_{jnm}^\varepsilon = \frac{2}{a^3 j_{n+1}^2(\alpha_{n+1/2,j})} \int_0^a \int_0^{2\pi} \int_0^{\pi} f^\varepsilon(r,\theta,\phi) j_n(\lambda_{n,j} r) \overline{Y}_{n,m}(\theta,\phi) r^2 \sin\theta \, d\theta \, d\phi \, dr.$$

In this article, we modify the regularized parameter of ω^ε by a different one to get a Hölder type estimate for the error between the regularized solution and the exact solution. So we call this method the modified quasi-boundary value method. In particular, we construct the regularized solutions $u^\varepsilon, v^\varepsilon$ corresponding to the measured data f^ε and the exact data f, respectively

$$u^\varepsilon(r,\theta,\phi,t) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{jnm}^\varepsilon(t) j_n(\lambda_{n,j} r) Y_{n,m}(\theta,\phi), \tag{3.6}$$

where

$$B_{jnm}^\varepsilon(t) = \frac{\exp\{-c^2\lambda_{n,j}^2 t\}}{\alpha(\varepsilon)c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2 T\}}\Big(f_{jnm}^\varepsilon - \frac{q_{jnm}}{c^2\lambda_{n,j}^2}\Big) + \frac{q_{jnm}}{c^2\lambda_{n,j}^2},$$

and

$$v^\varepsilon(r,\theta,\phi,t) = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{jnm}(t) j_n(\lambda_{n,j} r) Y_{n,m}(\theta,\phi), \tag{3.7}$$

in which

$$B_{jnm}(t) = \frac{\exp\{-c^2\lambda_{n,j}^2 t\}}{\alpha(\varepsilon)c^2\lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2 T\}}\Big(f_{jnm} - \frac{q_{jnm}}{c^2\lambda_{n,j}^2}\Big) + \frac{q_{jnm}}{c^2\lambda_{n,j}^2}.$$

and $\alpha(\varepsilon)$ is regularization parameter such that $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$. For short notation, we denote $\alpha = \alpha(\varepsilon)$.

Lemma 3.1. *For $0 < \alpha < T$, $a > 0$, we have the following inequality*

$$\frac{1}{\alpha a + \exp\{-aT\}} \le \frac{T}{\alpha}(\ln(\frac{T}{\alpha}))^{-1}.$$

Lemma 3.2. *For $0 \le t \le s \le T$, $0 < \alpha < T$, $a > 0$ and denote $\widetilde{T} = \max\{1, T\}$, we get the following inequalities*

(i)

$$\frac{\exp\{(s-t-T)a\}}{\alpha a + \exp\{-aT\}} \le \widetilde{T}\Big(\alpha \ln(\frac{T}{\alpha})\Big)^{\frac{t-s}{T}}.$$

ii) *For $s = T$, we obtain*

$$\frac{\exp\{-ta\}}{\alpha a + \exp\{-aT\}} \le \widetilde{T}\Big(\alpha \ln(\frac{T}{\alpha})\Big)^{\frac{t}{T}-1}.$$

In this article, we require some assumptions on the exact data f and the measured data f^ε as follows

(H1) Let $f(\cdot,\theta,\phi)$, $f^\varepsilon(\cdot,\theta,\phi) \in L^2[[0;a];r]$ be the exact data and the measured data such that

$$\|f^\varepsilon(\cdot,\theta,\phi) - f(\cdot,\theta,\phi)\|_2 \le \varepsilon,$$

for $(\theta,\phi) \in (0,\pi) \times (0,2\pi)$.

(H2) There exists a non-negative number A such that

$$\sup_{(\theta,\phi)\in[0;\pi]\times[0;2\pi]} \|\frac{\partial u}{\partial t}(\cdot,\theta,\phi,0)\|_2 \le A.$$

In the following theorem, we give the stability of the modified method for problem (3.6).

Theorem 3.3. *Let $\alpha \in (0;1)$, $f^\varepsilon(\cdot,\theta,\phi)$, $f(\cdot,\theta,\phi)$ satisfy* (H1) *for all $(\theta,\phi) \in (0,\pi)\times(0,2\pi)$. Assume that u^ε and v^ε are defined by* (3.6) *and* (3.7) *corresponding to the final data $f^\varepsilon(\cdot,\theta,\phi)$ and $f(\cdot,\theta,\phi)$, respectively. Then, we obtain*

$$\|u^\varepsilon(\cdot,\theta,\phi,t) - v^\varepsilon(\cdot,\theta,\phi,t)\|_2 \le \widetilde{T}\Big(\alpha \ln(\frac{T}{\alpha})\Big)^{\frac{t}{T}-1}\varepsilon,$$

for $(\theta,\phi,t) \in (0,\pi)\times(0,2\pi)\times(0,T)$.

Finally, we estimate the error between the regularized solution corresponding to the measured data f^ε and the exact solution of problem (1.3)-(1.5).

Theorem 3.4. *Let f, f^ε be as in Theorem 3.3 and $0 < \alpha < \min\{1;T\}$. Suppose that u^ε is defined by* (3.6) *corresponding to the perturbed datum f^ε and u be the exact solution of* (1.3)-(1.5) *satisfying* (H2). *Then, we have*

$$\|u^\varepsilon(\cdot,\theta,\phi,t) - u(\cdot,\theta,\phi,t)\|_2 \le \widetilde{T}\varepsilon^{\frac{t}{T}}\Big(\ln(\frac{T}{\varepsilon})\Big)^{\frac{t}{T}-1}(A+1). \tag{3.8}$$

for $(\theta,\phi,t) \in (0,\pi)\times(0,2\pi)\times(0,T)$.

4. Proofs of main results

Proof of Lemma 3.1. Let $0 < \alpha < T$ and $\psi(a) = \frac{1}{\alpha a + \exp\{-aT\}}$. By simple calculations, we have

$$\psi(a) \le \frac{T}{\alpha(1+\ln(T/\alpha))} \le \frac{T}{\alpha \ln(T/\alpha)},$$

for $a > 0$. This completes the proof. $\square$

Proof of Lemma 3.2. (i) From Lemma 3.1, we have

$$\begin{aligned}
\frac{\exp\{(s-t-T)a\}}{\alpha a + \exp\{-aT\}} &\le \frac{\exp\{(s-t-T)a\}}{(\alpha a + \exp\{-aT\})^{\frac{s-t}{T}} (\alpha a + \exp\{-aT\})^{\frac{T+t-s}{T}}} \\
&\le \frac{\exp\{(s-t-T)a\}}{(\alpha a + \exp\{-aT\})^{\frac{s-t}{T}} (\exp\{-aT\})^{\frac{T+t-s}{T}}} \\
&\le \Big(\frac{T}{\alpha \ln(T/\alpha)}\Big)^{\frac{s-t}{T}} \\
&\le \widetilde{T}[\alpha \ln(T/\alpha)]^{\frac{t-s}{T}},
\end{aligned}$$

where $\widetilde{T} = \max\{1, T\}$.

(ii) Let $s = T$, we obtain

$$\frac{\exp\{-ta\}}{\alpha a + \exp\{-aT\}} \le \widetilde{T}[\alpha \ln(T/\alpha)]^{\frac{t-T}{T}}.$$

This completes the proof. $\square$

Proof of Theorem 3.3. From (3.6), (3.7) and Lemma 3.2, we have the estimate

$$\begin{aligned}
&\|u^\varepsilon(\cdot,\theta,\phi,t) - v^\varepsilon(\cdot,\theta,\phi,t)\|_2 \\
&= \|\sum_{j=1}^\infty \sum_{n=0}^\infty \sum_{m=-n}^n \frac{\exp\{-c^2\lambda_{n,j}^2 t\}}{\alpha c^2 \lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2 T\}} (f_{jnm}^\varepsilon - f_{jnm}) j_n(\lambda_{n,j}\cdot) Y_{n,m}(\theta,\phi)\|_2 \\
&\le \widetilde{T}\Big(\alpha \ln(\frac{T}{\alpha})\Big)^{\frac{t}{T}-1} \|\sum_{j=1}^\infty \sum_{n=0}^\infty \sum_{m=-n}^n (f_{jnm}^\varepsilon - f_{jnm}) j_n(\lambda_{n,j}\cdot) Y_{n,m}(\theta,\phi)\|_2 \qquad (4.1) \\
&= \widetilde{T}\Big(\alpha \ln(\frac{T}{\alpha})\Big)^{\frac{t}{T}-1} \|f^\varepsilon(\cdot,\theta,\phi) - f(\cdot,\theta,\phi)\|_2 \\
&\le \widetilde{T}\Big(\alpha \ln(\frac{T}{\alpha})\Big)^{\frac{t}{T}-1} \varepsilon.
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 3.4. Using the triangle inequality,

$$\begin{aligned}
&\|u^\varepsilon(\cdot,\theta,\phi,t) - u(\cdot,\theta,\phi,t)\|_2 \\
&\le \|u^\varepsilon(\cdot,\theta,\phi,t) - v^\varepsilon(\cdot,\theta,\phi,t)\|_2 + \|v^\varepsilon(\cdot,\theta,\phi,t) - u(\cdot,\theta,\phi,t)\|_2.
\end{aligned} \qquad (4.2)$$

From (3.1) and (3.7), we obtain

$$\begin{aligned}
&\|v^\varepsilon(\cdot,\theta,\phi,t) - u(\cdot,\theta,\phi,t)\|_2 \\
&= \|\sum_{j=1}^\infty \sum_{n=0}^\infty \sum_{m=-n}^n \Big(\frac{\exp\{-c^2\lambda_{n,j}^2 t\}}{\alpha c^2 \lambda_{n,j}^2 + \exp\{-c^2\lambda_{n,j}^2 T\}} - \exp\{c^2\lambda_{n,j}^2 (T-t)\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \big(f_{jnm} - \frac{q_{jnm}}{c^2\lambda_{n,j}^2}\big) j_n(\lambda_{n,j}\cdot) Y_{n,m}(\theta,\phi)\|_2 \\
&\le \alpha\widetilde{T}\big(\alpha\ln(\frac{T}{\alpha})\big)^{\frac{t}{T}-1} \|\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} c^2\lambda_{n,j}^2 \exp\{c^2\lambda_{n,j}^2 T\} \\
&\quad \times \Big(f_{jnm} - \frac{q_{jnm}}{c^2\lambda_{n,j}^2}\Big) j_n(\lambda_{n,j}\cdot) Y_{n,m}(\theta,\phi)\|_2 \\
&= \alpha\widetilde{T}\Big(\alpha\ln(\frac{T}{\alpha})\Big)^{\frac{t}{T}-1} \|\frac{\partial u}{\partial t}(\cdot,\theta,\phi,0)\|_2 \\
&\le \alpha\widetilde{T}(\alpha\ln(\frac{T}{\alpha}))^{\frac{t}{T}-1} A.
\end{aligned} \tag{4.3}
$$

Combining Theorem 3.3 and (4.3), choosing $\alpha = \varepsilon$, we have the estimate

$$
\|u^\varepsilon(\cdot,\theta,\phi,t) - u(\cdot,\theta,\phi,t)\|_2 \le \widetilde{T}\varepsilon^{\frac{t}{T}}\Big(\ln(\frac{T}{\varepsilon})\Big)^{\frac{t}{T}-1}(A+1).
$$

This completes the proof. $\square$

5. Numerical experiments

In this section, we consider the backward nonhomogeneous heat equation in a ball,

$$
u_t = c^2\Big\{\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta} + \csc^2\theta\frac{\partial^2 u}{\partial \phi^2}\Big)\Big\} + q(r,\theta,\phi), \tag{5.1}
$$

$$
u(a,\theta,\phi,t) = 0, \tag{5.2}
$$

$$
u(r,\theta,\phi,T) = f(r,\theta,\phi), \tag{5.3}
$$

where $(r,\theta,\phi,t) \in (0,1)\times(0,\pi)\times(0,2\pi)\times(0,1)$, $c = 0.05$ and q, f are defined as follows

$$
f(r,\theta,\phi) = 100, \tag{5.4}
$$

$$
q(r,\theta,\phi) = j_{12}(\alpha_{25/2,1} r)[Y_{12,-12}(\theta,\phi) + Y_{12,12}(\theta,\phi)]. \tag{5.5}
$$

By simple calculations, we have

$$
\begin{gathered}
f_{jnm} = 0 \quad \text{for all } j, n \neq 0 \text{ or } m \in [-n,n]/backslash\{0\}, \\
f_{j00} = \frac{400\sqrt{2}}{\sqrt{\alpha_{1/2,j}}J_{3/2}(\alpha_{1/2,j})} \quad \text{for all } j, \\
q_{jnm} = 0, \quad \text{for all } (j,n,m) \neq (1,12,-12) \text{ and } (1,12,12), \\
q_{jnm} = 1, \quad \text{for } (j,n,m) = (1,12,-12) \text{ or } (1,12,12).
\end{gathered}
$$

We also obtain

$$
\begin{gathered}
Y_{12,12}(\theta,\phi) = \sqrt{\frac{25}{24!.4\pi}} P_{12}^{12}(\cos\theta) e^{i12\theta}, \\
P_{12}^{12}(x) = (-1)^{12}(1-x^2)^6 \frac{d^{12}P_{12}(x)}{dx^{12}}, \\
P_{12}(x) = \frac{1}{2^{12}}\sum_{m=1}^{6}(-1)^m \frac{(24-2m)!}{m!(12-m)!(12-2m)!} x^{12-2m}, \\
Y_{12,-12}(\theta,\phi) = (-1)^{12}\overline{Y}_{12,12}(\theta,\phi).
\end{gathered}
$$

From which, we get the exact solution u corresponding to f, q which are defined by (5.4) and (5.5), respectively.

$$\begin{aligned}
&u(r,\theta,\phi,t)\\
&=\sum_{j=1}^{\infty}\exp(\alpha_{1/2,j}^2c^2(1-t))\frac{400\sqrt{2}}{\sqrt{\alpha_{1/2,j}}J_{3/2}(\alpha_{1/2,j})}j_0(\alpha_{1/2,j}r)Y_{0,0}(\theta,\phi)\\
&\quad+\big(1-\exp(\alpha_{25/2,1}^2c^2(1-t))\big)\frac{1}{c^2\alpha_{25/2,1}^2}j_{12}(\alpha_{25/2,1}r)\\
&\quad\times(Y_{12,-12}(\theta,\phi)+Y_{12,12}(\theta,\phi)) \qquad (5.6)\\
&=\sum_{j=1}^{\infty}\exp(\alpha_{1/2,j}^2c^2(1-t))\frac{200\sqrt{2}}{\sqrt{\alpha_{1/2,j}}J_{3/2}(\alpha_{1/2,j})}\Big(\frac{1}{2\alpha_{1/2,j}r}\Big)^{1/2}J_{1/2}(\alpha_{1/2,j}r)\\
&\quad+2(1-\exp(\alpha_{25/2,1}^2c^2(1-t)))\frac{1}{c^2\alpha_{25/2,1}^2}\Big(\frac{\pi}{2\alpha_{25/2,1}r}\Big)^{1/2}\\
&\quad\times J_{25/2}(\alpha_{25/2,1}r)P_{12}^{12}(\cos\theta)\cos 12\phi.
\end{aligned}$$

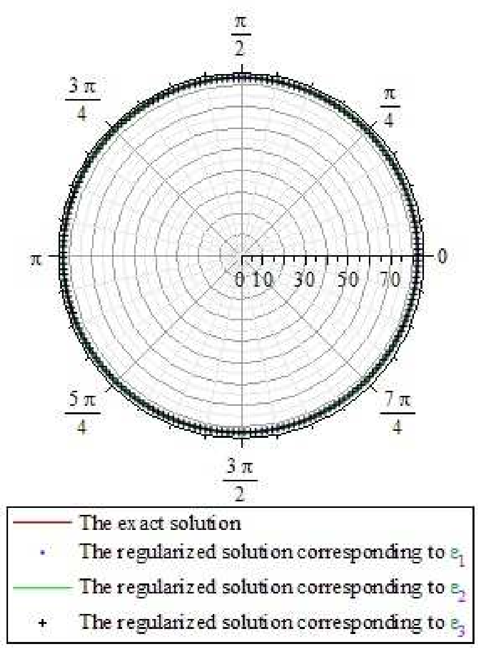

FIGURE 1. Exact and regularized solutions corresponding to ε_i, $i=1,2,3$ when $r=0.5,\theta=\frac{\pi}{6}$.

Then, we consider the measured data

$$f^{\varepsilon}(r,\theta,\phi)=100+\varepsilon. \qquad (5.7)$$

From (5.4) and (5.7), we have

$$\|f^{\varepsilon}(\cdot,\theta,\phi)-f(\cdot,\theta,\phi)\|_2=\Big(\int_0^1 r\varepsilon^2dr\Big)^{1/2}\le\varepsilon.$$

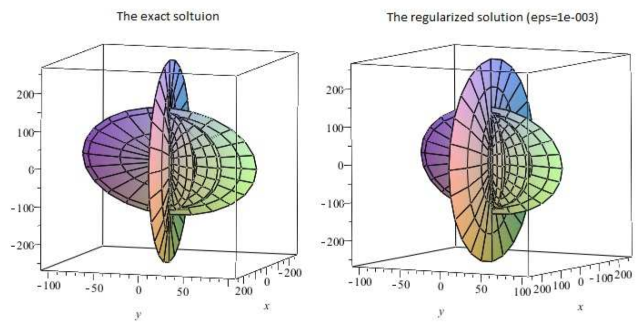

FIGURE 2. Exact and regularized solution corresponding to ε_1

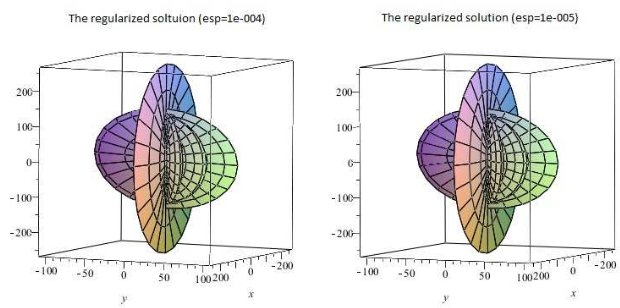

FIGURE 3. Regularized solutions corresponding to ε_i, $i = 2, 3$.

From (3.6) and (5.7), we have the regularized solution u^ε as follows

$$
\begin{aligned}
&u^\varepsilon(r,\theta,\phi,t)\\
&= \sum_{j=1}^{\infty} \frac{\exp(-\alpha_{1/2,j}^2 c^2 t)}{\varepsilon\alpha_{1/2,j}^2 c^2 + \exp(-\alpha_{1/2,j}^2 c^2)} \frac{4(100+\varepsilon)\sqrt{2}}{\sqrt{\alpha_{1/2,j}}J_{3/2}(\alpha_{1/2,j})} j_0(\alpha_{1/2,j} r) Y_{0,0}(\theta,\phi)\\
&\quad + \Big(1 - \frac{\exp(-\alpha_{25/2,1}^2 c^2 t)}{\varepsilon\alpha_{25/2,1}^2 c^2 + \exp(-\alpha_{25/2,1}^2 c^2)}\Big) \frac{1}{c^2\alpha_{25/2,1}^2} j_{12}(\alpha_{25/2,1} r)\\
&\quad \times (Y_{12,-12}(\theta,\phi) + Y_{12,12}(\theta,\phi))\\
&= \sum_{j=1}^{\infty} \frac{\exp(-\alpha_{1/2,j}^2 c^2 t)}{\varepsilon\alpha_{1/2,j}^2 c^2 + \exp(-\alpha_{1/2,j}^2 c^2)} \frac{2(100+\varepsilon)\sqrt{2}}{\sqrt{\alpha_{1/2,j}}J_{3/2}(\alpha_{1/2,j})} \big(\frac{1}{2\alpha_{1/2,j} r}\big)^{1/2}\\
&\quad \times J_{1/2}(\alpha_{1/2,j} r)\\
&\quad + 2\Big(1 - \frac{\exp(-\alpha_{25/2,1}^2 c^2 t)}{\varepsilon\alpha_{25/2,1}^2 c^2 + \exp(-\alpha_{25/2,1}^2 c^2)}\Big) \frac{1}{c^2\alpha_{25/2,1}^2} \big(\frac{\pi}{2\alpha_{5/2,1} r}\big)^{1/2}\\
&\quad \times J_{25/2}(\alpha_{25/2,1} r) P_{12}^{12}(\cos\theta)\cos 12\phi.
\end{aligned} \tag{5.8}
$$

Next, we calculate the first seven coefficients of (5.6) and (5.8) at various values of t. Let ε be $\varepsilon_1 = 10^{-3}$, $\varepsilon_2 = 10^{-4}$, $\varepsilon_3 = 10^{-5}$, respectively and $t \in \{0; 0.5\}$. The following table shows estimates for the error between the exact solution (5.6) and the regularized solutions (5.8).

Table 1. Error between exact and regularized solutions when $(\theta, \phi) = (\frac{\pi}{6}, \frac{\pi}{6})$.

	$\|u^\varepsilon(\cdot, \frac{\pi}{6}, \frac{\pi}{6}, t) - u(\cdot, \frac{\pi}{6}, \frac{\pi}{6}, t)\|_2$		
t	$\varepsilon_1 = 10^{-3}$	$\varepsilon_2 = 10^{-4}$	$\varepsilon_3 = 10^{-5}$
0	1.2431×10^{-1}	1.2475×10^{-2}	1.2479×10^{-3}
0.5	6.9674×10^{-2}	6.9906×10^{-3}	6.9929×10^{-4}

Figure 1 shows the exact and regularized solutions u^{ε_i}, $i = 1, 2, 3$ at the time $t = 0.5$ when $r = 0.5$ and $\theta = \frac{\pi}{6}$. Finally, we plot the graphs of the exact and regularized solutions u^{ε_i}, $i = 1, 2, 3$ at the time $t = 0.5$ corresponding to $\theta = \frac{\pi}{6}$ in Figures 2–3.

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Le Minh Triet
Division of Computational Mathematics and Engineering, Institute for Computational Science.
Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
E-mail address: `leminhtriet@tdt.edu.vn`

Luu Hong Phong
Faculty of Mathematics, University of Science, Vietnam National University, Ho chi Minh city, Vietnam
E-mail address: `luuhongphong2812@gmail.com`