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POSITIVE SOLUTIONS FOR A SECOND-ORDER Φ-LAPLACIAN EQUATIONS WITH LIMITING NONLOCAL BOUNDARY CONDITIONS

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Abstract. Motivated, mainly, by the works of Fewster-Young and Tisdell [9, 10] and Orpel [30], as well as the papers by Karakostas [21, 22, 23], we give sufficient conditions to guarantee the existence of (nontrivial) solutions of the second-order Φ-Laplacian equation

$$\frac{1}{p(t)}\frac{d}{dt}[p(t)\Phi(u'(t))]+(Fu)(t)=0,\quad \text{a.e. } t\in[0,1]=:I,$$

which satisfy the nonlocal boundary value conditions of the limiting Sturm-Liouville form

$$\lim_{t\to 0}[p(t)\Phi(u'(t))]=\int_0^1 u(s)d\eta(s),\quad \lim_{t\to 1}[p(t)\Phi(u'(t))]=-\int_0^1 u(s)d\zeta(s).$$

Here Φ is an increasing homeomorphism of the real line onto itself and F is an operator acting on the function u and on its first derivative with the characteristic property that $u\to p(Fu)$ is a C^0-type, or C^1-type Caratheodory operator, a meaning introduced here. Examples are given to illustrate both cases.

1. Introduction

We study the existence of positive solutions of the second-order Φ-Laplacian equation

$$\frac{1}{p(t)}\frac{d}{dt}[p(t)\Phi(u'(t))]+(Fu)(t)=0,\quad \text{a.e. } t\in[0,1]=:I, \tag{1.1}$$

associated with the nonlocal limiting boundary value conditions of the Sturm-Liouville form

$$\lim_{t\to 0}[p(t)\Phi(u'(t))]=\int_0^1 u(s)d\eta(s),\quad \lim_{t\to 1}[p(t)\Phi(u'(t))]=-\int_0^1 u(s)d\zeta(s), \tag{1.2}$$

where Φ is an increasing homeomorphism, while, conditions for p, η and ζ will be given in the text. If $p(t)=1$ and Φ is the identity, then the boundary value conditions (1.2) take a form related to that one considered in [18] and [19].

In the sequel we shall use the Banach space $C^0:=C^0(I,\mathbb{R})$ of continuous functions $u:I\to\mathbb{R}$ endowed with the sup-norm $\|\cdot\|_0$ and the Banach space $C^1:=$

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$C^1(I,\mathbb{R})$ (of all differentiable functions $u : I \to \mathbb{R}$ having derivative $u' \in C^0(I,\mathbb{R})$) endowed with the norm $\|u\|_1 := \max\{\|u\|_0, \|u'\|_0\}$. Notice that the natural imbedding $C^1 \hookrightarrow C^0$ furnishes the space C^1 with both norms, $\|\cdot\|_0$ and $\|\cdot\|_1$. Then, given a set $A \subseteq C^1$, we let cl_0A and cl_1A be the closures of A with respect to the norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. We shall denote by C^0_+ the set of all nonnegative functions of C^0. Also, let $\mathcal{M}$ be the linear space of all Lebesgue measurable functions $u : I \to \mathbb{R}$ and $\mathcal{L}^1$ the Banach space of all Lebesgue integrable functions $u \in \mathcal{M}$ with norm $\|u\|_{\mathcal{L}^1}$.

In the sequel, given two topological spaces X_1, X_2, to emphasize that an operator $V : X_1 \to X_2$ is continuous, we shall say that V is (X_1, X_2)-continuous. We introduce the following definition.

Definition 1.1. Let $i = 0, 1$. An operator $G : C^i \to \mathcal{M}$ will be said to be a C^i-type Caratheodory operator, if it is $(\|\cdot\|_i, \mathcal{L}^1)$-continuous, and for each $c > 0$, there is a function $m_c \in \mathcal{L}^1$, such that

$$\|u\|_i \le c \implies |(Gu)(t)| \le m_c(t), \quad \text{for a.a. } t \in I$$

and $u \in C^i$.

Hence, if $f(t, x, y)$ is a Caratheodory function (in the usual sense) with respect to the pair-variable (x, y), then the types

$$(F_1u)(t) := f(t, u(t), u(\frac{t}{2})),$$

$$(F_2u)(t) := f(t, u(t), \int_0^1 a(s)u^2(s)ds), \quad (a \in \mathcal{L}^1),$$

$$(F_3u)(t) := f(t, u(t), u'(\frac{t}{3}))$$

define, respectively, C^0, C^0, C^1-type Caratheodory operators.

Because of a great number of physical applications, the study of ϕ-Laplacian differential equations of second order of the form

$$(a(t)\Phi(u'(t)))' + b(t)f(t, u(t)) = 0, \quad t \in (0, 1),$$

associated with various boundary value conditions have received the attention of many authors; see, e.g., [1]-[5], [8], [11]-[15], [23], [26]-[28], [32], and the references therein. In most of these papers the response function is continuous in its arguments, thus it is a Caratheodory function. Also, many two- (or multi)-point boundary value problems involving the well-known p-Laplacian equation

$$(\phi_p(u'(t)))' = f(t, u(t), u'(t)), \quad t \in [0, 1],$$

where $\phi_p(s) := |s|^{p-2}s, \quad (p > 1)$, have received a lot of attention, see for instance, [6, 16, 17, 31] and the references therein. The key condition in these works is a growth restriction imposed on the response f.

When seeking a positive solution of the problem and the positivity of the nonlinearity is guaranteed, most of the authors mentioned above, are frequently led to use the Krasnoselskii's fixed point theorem applying to a completely continuous operator T which is defined on an appropriate cone in some Banach space. A type of a barrier strip on the function f is used elsewhere (see, e.g., [24, 25, 32] and the references therein) and then apply the Topological Transversality Theorem. The disadvantage in the latter situations is that no sign of the solutions is known.

The presence of a generally nonlinear operator Φ was introduced in [21, 22], which was assumed to be a sup-multiplicative-like function in the following sense: Φ is a *sup-multiplicative-like* function, if it is an odd homeomorphism of the real line $\mathbb{R}$ onto itself, for which there exists a homeomorphism ϕ of $[0,+\infty)$ onto itself, that *supports* Φ, in the sense that, for all $v_1, v_2 \geq 0$, it holds $\phi(v_1)\Phi(v_2) \leq \Phi(v_1 v_2)$. For instance, a function of the form $\Phi(v) := \sum_0^k c_j |v|^j v$, $v \in \mathbb{R}$ is sup-multiplicative-like, provided that $c_j \geq 0$ and $\sum c_j > 0$. Here a supporting function is defined by $\phi(u) := \min\{u^{k+1}, u\}$, $u \geq 0$. A good use of sup-multiplicative-like functions is made elsewhere, see, e.g., [4, 20].

The motivation of considering limiting boundary conditions comes, mainly, from the works of [9, 10, 30], which refer to $\mathbb{R}$, or to $\mathbb{R}^n$ and are of the form

$$-\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t)), \quad 0 < t < T,$$

with boundary conditions,

$$-\alpha y(0) + \beta \lim_{t\to 0^+} p(t)y'(t) = c, \quad \gamma y(T) + \delta \lim_{t\to T^-} p(t)y'(t) = d,$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative real numbers. In these problems the moment-function p is positive on $(0,1)$ and

$$\int_0^1 \frac{ds}{p(s)} < +\infty,$$

as well as either it is a C^1 function, or such that $\min_{t\in I} p(t) > 0$. The latter is not required in our first main result.

In this work we give two existence results covering the case where the operator F is a C^0- or a C^1-type operator. To prove our main theorems we shall apply the following well known Schauder-Tychonoff Fixed Point Theorem (see, e.g., [29] and [7, p. 26]).

Theorem 1.2. *Let* $\mathbf{C}$ *be a closed convex subset of a normed linear space and let* $f : \mathbf{C} \to \mathbf{C}$ *be a compact map (i.e. it is continuous and* $f(\mathbf{C})$ *is relatively compact), then* f *has a fixed point.*

We shall apply twice Theorem 1.2. First in the linear space $C^1(I, \mathbb{R})$, when it is furnished with the norm $\|\cdot\|_0$ and second in the same space, endowed with its natural norm $\|\cdot\|_1$. Notice that with respect to the first norm the space C^1 is not complete.

2. Some auxiliary facts

In the sequel we assume the following:

- (H1) The function Φ is an increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, with $\Phi(0) = 0$. Let Ψ be the inverse of Φ.
- (H2) The function $p : I \to (0, +\infty)$ is measurable, and the function $\Psi(\frac{k}{p})$ is Lebesgue integrable on I, for all $k \in \mathbb{R}$.
- (H3) The measure-functions $\eta, \zeta : [0,1] \to [0,+\infty)$ are nondecreasing and non constants on I, and, moreover, satisfy the conditions
$$\eta(0) = \zeta(0) = 0, \quad \eta(1)\zeta(t) \leq \zeta(1)\eta(t), \quad \text{for all} \quad t \in I.$$
- (H4) The quantity $(F0)(t)$ is not identically equal to zero for a.a. $t \in I$. (Obviously, the latter implies that the problem does not admit the zero solution.)

Lemma 2.1. *Under the conditions* (H1)–(H4) *the problem does not have a nonnegative solution u with $(Fu)(t) = 0$, a.e. on I.*

Proof. Indeed, if such a solution u exists, it satisfies $p(t)\Phi(u'(t)) = c$, a.e. on I, for some c. From (1.2) we obtain both $c \geq 0$ and $c \leq 0$. Thus $c = 0$ and so $\Phi(u'(t)) = 0$, a.e. on I, which means that u is a constant, v, say. Again, from the first of (1.2) we obtain $v\eta(1) = 0$, and so $v = 0$. Thus we have $(F0)(t) = 0$, a.a $t \in I$, which contradicts to (H4). □

Next, for $i = 0, 1$ we consider the condition:

$(\mathrm{H5})_i$ The operator F has the following properties:
- (i) It holds $(Fu)(t) \geq 0$, $t \in I$, for all $u \in C^i$, with $u(t) \geq 0$.
- (ii) The operator $u \to p(Fu) =: Gu$ maps the set C^i into $\mathcal{L}^1$ and it is a C^i-type Caratheodory operator. Recall that, by definition, the latter condition ensures that, for each $c > 0$, there is some $m_c \in \mathcal{L}^1$, satisfying the condition $0 \leq p(t)(Fu)(t) \leq m_c(t)$, for a.a. $t \in I$ and $u \in C^i$ with $0 \leq \|u\|_i \leq c$ and $0 \leq u(t)$, $t \in I$.

In the sequel, the most important role in our discussion will be played by the quantity

$$D(x, \phi) := \eta(1)(x - \phi(1)) + \zeta(1)x + \int_0^1 (\zeta(1)\eta(t) - \eta(1)\zeta(t))\Psi\big(\frac{1}{p(t)}[x - \phi(t)]\big)dt,$$

which is defined for each real number x and any continuous function $\phi : [0, 1] \to \mathbb{R}$. This functional has the following properties:

Lemma 2.2. *Given any nondecreasing continuous function $\phi : [0, 1] \to \mathbb{R}^+$, there exists a unique real number $\mathcal{X}\big(\phi\big) \in \big[0, \phi(1)\big]$, satisfying the equation*

$$D(\mathcal{X}(\phi), \phi) = 0.$$

If $\phi(0) = 0 < \phi(1)$, then $\mathcal{X}\big(\phi\big) \in \big(0, \phi(1)\big)$. Moreover the operator $\phi \to \mathcal{X}(\phi)$ is $(C^0, \mathbb{R})$-continuous.

Proof. It is obvious that if $\phi(1) = 0$ then $D(x, \phi) = 0$, if and only if $x = 0$. Assume that it holds $\phi(1) > 0$. Then the existence of such a real number follows from the continuity of $D(\cdot, \phi)$ and the fact that the value

$$D(0, \phi) = -\eta(1)\phi(1) + \int_0^1 (\zeta(1)\eta(t) - \eta(1)\zeta(t))\Psi\big(\frac{1}{p(t)}[-\phi(t)]\big)dt,$$

is negative, while, for any fixed $v \geq \phi(1)$, the value

$$\begin{aligned} D(v, \phi) &= \eta(1)(v - \phi(1)) + \zeta(1)v \\ &\quad + \int_0^1 (\zeta(1)\eta(t) - \eta(1)\zeta(t))\Psi\big(\frac{1}{p(t)}[v - \phi(t)]\big)dt, \end{aligned} \tag{2.1}$$

is positive. The uniqueness is implied from the monotonicity of $D(x, \phi)$ with respect to x.

Finally, let (ϕ_n) be a sequence of nonnegative continuous functions defined on I. Assume that (ϕ_n) converges, in the sense of $\|\cdot\|_0$-norm, to a function ϕ. (By the uniform converge of bounded functions, it follows that there is a common upper bound B of all functions ϕ_n, $n = 1, 2, \dots$. Hence, by the first part of the lemma, we conclude that the quantities $(\mathcal{X}(\phi_n))$ belong to $[0, B]$, for all $n = 1, 2, \dots$). If there are subsequences $(\mathcal{X}(\phi_{k_n}))$ and $(\mathcal{X}(\phi_{l_n}))$ converging to v_1 and v_2 respectively, we must have

$$D(\mathcal{X}(\phi_{k_n}),\phi_{k_n})=0=D(\mathcal{X}(\phi_{l_n}),\phi_{l_n}).$$

Then, by the continuity of Ψ, we obtain

$$D(v_1,\phi)=0=D(v_2,\phi),$$

which, by the uniqueness, implies that $v_1=\mathcal{X}(\phi)=v_2$. This shows the continuity of $\mathcal{X}$. □

Next assume that F satisfies condition (H5)$_i$. Then the operator P defined by

$$P(u)(t):=\int_0^t p(s)(Fu)(s)ds,\quad u\in C^0_+\cap C^i, \tag{2.2}$$

is (C^i,C^0)-continuous and the function $P(u)$ is nonnegative and non-decreasing, with $P(u)(0)=0<P(u)(1)$. Therefore, by Lemma 2.2, the operator $u\to\mathcal{X}(P(u))$ is $(C^i,\mathbb{R})$-continuous on the set $C^0_+\cap C^i$. Moreover, the previous arguments ensure that the quantity $\mathcal{X}(P(u))$ satisfies the relation

$$\begin{aligned}\mathcal{X}(P(u))=\frac{\eta(1)}{\eta(1)+\zeta(1)}\Big[&P(u)(1)\\ &-\int_0^1\int_0^t\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big]\\ &+\int_0^1\int_0^t\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)dsd\eta(t).\end{aligned} \tag{2.3}$$

3. Reformulation of the Problem

By a positive solution of the problem (1.1)-(1.2) we mean a function $u\in C^1\cap C^0_+$, satisfying the conditions (1.2), and being such that the function $t\to p(t)\Phi(u'(t))$ is absolutely continuous on the interval I and the relation (1.1) is satisfied almost everywhere on the interval I.

To set the problem (1.1)-(1.2) in a form of seeking a fixed point of an appropriate functional operator, we shall reformulate it to an integral equation. To this end, assume that $u(t)$, $t\in I$, is a nonnegative solution. Then, for all $\tau,t\in[0,1]$, we have

$$p(t)\Phi(u'(t))=p(\tau)\Phi(u'(\tau))-\int_\tau^t p(s)(Fu)(s)ds. \tag{3.1}$$

By using the boundary conditions (1.2) we obtain

$$p(t)\Phi(u'(t))=\int_0^1 u(s)d\eta(s)-\int_0^t p(s)(Fu)(s)ds, \tag{3.2}$$

as well as

$$-\int_0^1 u(s)d\zeta(s)=\int_0^1 u(s)d\eta(s)-\int_0^1 p(s)(Fu)(s)ds. \tag{3.3}$$

From relation (3.2) it follows that

$$\Phi(u'(t))=\frac{1}{p(t)}[k_u-P(u)(t)],$$

where k_u is the non-negative real number

$$k_u := \int_0^1 u(s)d\eta(s)$$

and $P(u)$ is the function defined by (2.2). From Lemma 2.1 we have $P(u)(1) > 0$. Also, it is clear that it holds

$$P(u)(t) \le \int_0^1 m_c(s)ds =: B_c \tag{3.4}$$

where $c := \|u\|_0$. Moreover we have

$$u'(t) = \Psi\big(\frac{1}{p(t)}[k_u - P(u)(t)]\big). \tag{3.5}$$

Thus the solution u satisfies the integral equation

$$u(t) = u(0) + \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)ds. \tag{3.6}$$

The value of the function u at 0, is not known, so we shall express it by using the boundary values (1.2). In order to do it, we observe that, on one hand, we have

$$\int_0^1 u(s)d\zeta(s) = u(0)\zeta(1) + \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd\zeta(t),$$

and, on the other hand

$$\int_0^1 u(s)d\eta(s) = u(0)\eta(1) + \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd\eta(t),$$

Then, by using relation (3.3) we obtain

$$u(0) = \frac{1}{\eta(1)+\zeta(1)}\Big[P(u)(1) - \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big].$$

Replacing this value of $u(0)$ in equation (3.6) we obtain that

$$\begin{aligned} u(t) &= \frac{1}{\eta(1)+\zeta(1)}\Big[P(u)(1) - \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big] \\ &\quad + \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)ds. \end{aligned} \tag{3.7}$$

The latter equation gives a useful expression of the quantity k_u, as follows:

$$\begin{aligned} k_u &= \int_0^1 u(s)d\eta(s) \\ &= \frac{\eta(1)}{\eta(1)+\zeta(1)}\Big[P(u)(1) - \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big] \\ &\quad + \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd\eta(t), \end{aligned} \tag{3.8}$$

which, after some obvious manipulations, becomes

$$\begin{aligned} k_u &= \frac{1}{\eta(1)+\zeta(1)}\Big[\eta(1)P(u)(1) \\ &\quad + \int_0^1 \int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]ds\big)d[\zeta(1)\eta(t) - \eta(1)\zeta(t)]\Big] \end{aligned}$$

$$= \frac{1}{\eta(1)+\zeta(1)}\Big[\eta(1)P(u)(1) - \int_0^1 (\zeta(1)\eta(t)-\eta(1)\zeta(t))\Psi\big(\frac{1}{p(t)}[k_u - P(u)(t)]\big)dt\Big].$$

This relation means that the quantity $k_u =: x$ satisfies the equation

$$D(x, P(u)) = 0.$$

Then, by Lemma 2.2, we conclude that k_u is the unique quantity which satisfies (3.8) and it is such that

$$k_u = \mathcal{X}(P(u)) = \mathcal{X}\big(\int_0^{\cdot} p(s)(Fu)(s)ds\big) \in (0, B_c). \tag{3.9}$$

Lemma 3.1. *A function $u \in C^0_+$ solves the problem* (1.1)-(1.2)*, if and only if it satisfies the integral equation*

$$\begin{aligned} u(t) = {}& \frac{1}{\eta(1)+\zeta(1)}\Big[P(u)(1) - \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big] \\ & + \int_0^t \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - P(u)(s)]\big)ds, \end{aligned} \tag{3.10}$$

where $P(u)$ is defined by (2.2)*.*

Proof. If u is such a solution, then, by using (3.9) and (3.7) we obtain the "if" part. To prove the "only if" part, assume that a nonnegative function u satisfies (3.10). Clearly, u is differentiable and its derivative satisfies (1.1). We shall show that (1.2) is satisfied, too. To do it, observe that

$$\begin{aligned} \int_0^1 u(s)d\eta(s) = {}& \frac{\eta(1)}{\eta(1)+\zeta(1)}\Big[P(u)(1) \\ & - \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big] \\ & + \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - P(u)(s)]\big)dsd\eta(t). \end{aligned}$$

which, by (2.3), shows that the quantity $\int_0^1 u(s)d\eta(s)$ is equal to $\mathcal{X}(P(u))$, namely we have

$$\mathcal{X}(P(u)) = \int_0^1 u(s)d\eta(s) = k_u.$$

Also, we have

$$\begin{aligned} \int_0^1 u(s)d\zeta(s) = {}& \frac{\zeta(1)}{\eta(1)+\zeta(1)}\Big[P(u)(1) \\ & - \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big] \\ & + \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd\zeta(t) \end{aligned}$$

and therefore

$$\begin{aligned}
&\int_0^1 u(s)d(\zeta(s)+\eta(s)) \\
&= \frac{\zeta(1)+\eta(1)}{\zeta(1)+\eta(1)}\Big[P(u)(1) - \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd(\zeta+\eta)(t)\Big] \\
&\quad + \int_0^1\int_0^t \Psi\big(\frac{1}{p(s)}[k_u - P(u)(s)]\big)dsd(\zeta(t)+\eta(t)),
\end{aligned} \tag{3.11}$$

from which it follows that

$$-\int_0^1 u(s)d\zeta(s) = k_u - P(u)(1) = k_u - \int_0^1 p(s)(Fu)(s)ds. \tag{3.12}$$

On the other hand we observe that

$$p(t)\Phi(u'(t)) = k_u - \int_0^t p(s)(Fu)(s)ds$$

and hence the function $t \to p(t)\Phi(u'(t))$ is absolutely continuous in the interval I. Moreover, from this relation and (3.12) we conclude that conditions (1.2) are satisfied. $\square$

4. Main results

By the results of the previous section, the problem of existence of solutions of the problem (1.1)-(1.2) is equivalent to the problem of existence of a fixed point of the operator defined by

$$\begin{aligned}
(\mathcal{T}u)(t) :=& \frac{1}{\eta(1)+\zeta(1)}\Big[\int_0^1 p(s)(Fu)(s)ds \\
&- \int_0^1\int_0^\theta \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - \int_0^s p(r)(Fu)(r)dr]\big)dsd(\zeta+\eta)(\theta)\Big] \\
&+ \int_0^t \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - \int_0^s p(r)(Fu)(r)dr]\big)ds,
\end{aligned} \tag{4.1}$$

where $P(u)$ is defined by (2.2). Recall that $\mathcal{X}(P(u))$ is an element of the interval $[0, B_c]$, where $c := \|u\|_0$.

Lemma 4.1. *Assume that conditions* (H1)–(H4) *and* $(\mathrm{H5})_0$ *are satisfied. Then the operator* $\mathcal{T}$ *is* (C^0, C^0)*-continuous on* C^0_+.

Proof. Let (u_n) be a sequence of functions in C^0_+ converging to a certain $u_0 \in C^0_+$ in the $\|\cdot\|_0$-sense. It is clear that there is some $b > 0$ satisfying $0 \le u_n(t) \le b$, for all $t \in I$ and $n = 1, 2, \dots$. Hence all points $\mathcal{X}(P(u_n))$ exist in $[0, B_b]$ and it holds

$$|g_n(s)| \le \max\{-\Psi(\frac{-B_b}{p(s)}), \Psi(\frac{B_b}{p(s)})\}, \quad n = 1, 2, \dots \tag{4.2}$$

where

$$g_n(s) := \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u_n)) - \int_0^s p(r)(Fu_n)(r)dr]\big), \quad n = 0, 1, 2, \dots.$$

Notice that the right side of (4.2) defines an integrable function and, moreover, due to $(\mathrm{H5})_0$ (ii), it holds

$$\lim g_n(s) = g_0(s), \quad s \in I.$$

Finally observe that $(\mathcal{T}u)$ takes the form

$$\begin{aligned}(\mathcal{T}u_n)(t) = \frac{1}{\eta(1)+\zeta(1)}\Big[\int_0^1 p(s)(Fu_n)(s)ds \\ - \int_0^1 (\eta(1)+\zeta(1)-\eta(s)-\zeta(s))g_n(s)ds + \int_0^t g_n(s)ds\Big]\end{aligned}$$

Now, we apply the Lebesgue Dominated Convergence Theorem and get the desired result. ☐

Lemma 4.2. *Assume that conditions* (H1), (H3) *and* $(\mathrm{H5})_1$ *are satisfied. Then the operator* $\mathcal{T}$ *is* (C^0, C^1)*-continuous, provided that the following condition holds:*

$$p : I \to (0, +\infty) \text{ is continuous.} \tag{4.3}$$

Proof. Condition (4.3) implies that there is some $\rho > 0$ such that $p(t) \geq \rho$, for $t \in I$. As in previous lemma, let (u_n) be a sequence of functions in C^0_+ converging to a certain $u \in C^0_+$ in the $\|\cdot\|_0$-sense. Since, obviously, condition (4.3) implies condition (H2), from the previous lemma we know that the sequence $(\mathcal{T}u_n)$ converges to $\mathcal{T}u$ in the $\|\cdot\|_0$-sense.

Notice, also, that the function

$$u \to \mathcal{X}(P(u)) - \int_0^{\cdot} p(r)(Fu)(r)dr$$

is (C^0, C^0)-continuous. From the form of the operator $\mathcal{T}$ we see that it holds

$$\frac{d}{dt}(\mathcal{T}u)(t) = \Psi(w(t; u)),$$

where

$$w(t; u) := \frac{1}{p(t)}[\mathcal{X}(P(u)) - \int_0^t p(r)(Fu)(r)dr].$$

To proceed, consider any $\epsilon > 0$. By the uniform continuity of the function Ψ on the interval $[-\frac{B_b}{\rho}, \frac{B_b}{\rho}]$, there is a $\delta_0 > 0$ such that for all $v_1, v_2 \in [-\frac{B_b}{\rho}, \frac{B_b}{\rho}]$, it holds

$$|v_1 - v_2| < \delta_0 \implies |\Psi(v_1) - \Psi(v_2)| < \epsilon.$$

Fix any $\delta \in (0, \delta_0\rho)$. By the previous argument it follows that there is some n_0 such that

$$|[\mathcal{X}(P(u_n)) - \int_0^t p(r)(Fu_n)(r)dr] - [\mathcal{X}(P(u)) - \int_0^t p(r)(Fu)(r)dr]| \leq \delta,$$

for all $t \in I$ and $n \geq n_0$. This and the assumption that $p(t) \geq \rho$, for all $t \in I$, imply that

$$|w(t; u_n) - w(t; u)| \leq \frac{\delta}{\rho} < \delta_0,$$

for all $t \in I$ and $n \geq n_0$. We conclude that

$$|\Psi(w(t, u_n)) - \Psi(w(t; u))| \leq \epsilon,$$

for all $t \in I$ and $n \geq n_0$. The proof is complete. ☐

4.1. **Existence in the C^0 case.** Our first main result is the following.

Theorem 4.3. *Let the conditions* (H1)–(H4) *and* $(\mathrm{H5})_0$ *be satisfied. Also, assume that there is a $c>0$ such that*

(H6)
$$\frac{B_c}{\eta(1)}+\int_0^1\frac{\eta(s)}{\eta(1)}\Psi\big(\frac{B_c}{p(s)}\big)ds-\int_0^1\Psi\big(\frac{-B_c}{p(s)}\big)ds\le c,$$
where B_c is defined by (3.4), *and*

(H7) *for all $\lambda\in[0,B_c]$, it holds*
$$\lambda\ge\max\Big\{\int_0^1[\zeta(1)-\zeta(t)+\eta(1)-\eta(t)]\Psi\big(\frac{\lambda}{p(t)}\big)dt,\quad -\int_0^1[\zeta(t)+\eta(t)]\Psi\big(\frac{-\lambda}{p(t)}\big)dt\Big\}.$$

Then the operator $\mathcal{T}$ admits a fixed point in the set
$$S_c:=\{u\in C^0_+:0<\|u\|_0\le c\}.$$

Proof. Because of $(\mathrm{H5})_0$, to prove the result it is sufficient to show that the Schauder's fixed point theorem is applicable on the $\|\cdot\|_0$-closure cl_0S_c of the set S_c.

We shall show that the operator $\mathcal{T}$ maps the set $\overline{S_c}$ into itself. Fix any $u\in cl_0S_c$ and consider the quantity $P(u)$ defined in (2.2). Then, from the relation
$$\begin{aligned}
0&=(\eta(1)+\zeta(1))\mathcal{X}(P(u))-\eta(1)P(u)(1)\\
&\quad+\int_0^1(\zeta(1)\eta(t)-\eta(1)\zeta(t))\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))-P(u)(t)]\big)dt\\
&=\eta(1)(\mathcal{X}(P(u))-P(u)(1))+\zeta(1)\mathcal{X}(P(u))\\
&\quad+\int_0^1(\zeta(1)\eta(t)-\eta(1)\zeta(t))\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))-P(u)(t)]\big)dt,
\end{aligned}\tag{4.4}$$
it follows that we must have $\mathcal{X}(P(u))-P(u)(1)<0$. Thus
$$0\le\mathcal{X}(P(u))\le P(u)(1)\le B_c.$$
Also, by using the non-negativity of F, on the set C^0_+, we have
$$0\le P(u)(t)\le P(u)(1).$$
These facts imply that there is a certain point $\xi\in(0,1)$ such that
$$(\xi-t)[\mathcal{X}(P(u))-P(u)(t)]\ge0,\quad t\in(0,1).$$
Now, from the form of the operator $\mathcal{T}$, we observe that it holds
$$\frac{d}{dt}(\mathcal{T}u)(t)=\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))-\int_0^t p(r)(Fu)(r)dr]\big)$$
and so we conclude that at the point ξ the function $\mathcal{T}u$ admits a maximum, it is increasing for $t\le\xi$ and decreasing for $t\ge\xi$.

From these remarks it follows that to show that $0\le(\mathcal{T}u)(t)\le c$, for all $t\in I$, it is sufficient to show that the following two inequalities hold:
$$(\mathcal{T}u)(\xi)\le c,\tag{4.5}$$
$$\min\{(\mathcal{T}u)(0),(\mathcal{T}u)(1)\}\ge0.\tag{4.6}$$
To show (4.5), we use relation (2.3). We have
$$(\mathcal{T}u)(\xi)=\frac{1}{\eta(1)}\Big[\mathcal{X}(P(u))-\int_0^1\int_0^t\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)dsd\eta(t)\Big]$$

$$
\begin{aligned}
&+\int_0^{\xi}\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)ds\\
&=\frac{\mathcal{X}(P(u))}{\eta(1)}+\int_0^1\frac{\eta(s)}{\eta(1)}\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)ds\\
&-\int_{\xi}^{1}\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)ds\\
&\le\frac{\mathcal{X}(P(u))}{\eta(1)}+\int_0^1\frac{\eta(s)}{\eta(1)}\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))]\big)ds-\int_0^1\Psi\big(\frac{1}{p(s)}[-P(u)(s)]\big)ds\\
&\le\frac{B_c}{\eta(1)}+\int_0^1\frac{\eta(s)}{\eta(1)}\Psi\big(\frac{B_c}{p(s)}\big)ds-\int_0^1\Psi\big(\frac{-B_c}{p(s)}\big)ds.
\end{aligned}
$$

Because of (H6), the latter quantity is less than or equal to c.

To show (4.6), by the contrary, we assume that $(\mathcal{T}u)(0)<0$. Then from (4.1) we must have

$$
\begin{aligned}
P(u)(1)&<\int_0^1\int_0^t\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)ds\,d(\zeta(t)+\eta(t))\\
&=\int_0^1[\zeta(1)-\zeta(t)+\eta(1)-\eta(t)]\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))-P(u)(t)]\big)dt\\
&\le\int_0^1[\zeta(1)-\zeta(t)+\eta(1)-\eta(t)]\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))]\big)dt
\end{aligned}
$$

and so

$$
\mathcal{X}(P(u))\le P(u)(1)<\int_0^1[\zeta(1)-\zeta(t)+\eta(1)-\eta(t)]\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))]\big)dt,
$$

which is impossible because of (H7).

Next, assuming that $(\mathcal{T}u)(1)<0$, we obtain

$$
\begin{aligned}
P(u)(1)&<-\int_0^1[\zeta(t)+\eta(t)]\Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u))-P(u)(t)]\big)dt\\
&\le-\int_0^1[\zeta(t)+\eta(t)]\Psi\big(\frac{1}{p(t)}[-P(u)(1)]\big)dt.
\end{aligned}
$$

This is impossible, too, because of (H7) and the fact that $0\le P(u)(1)\le B_c$.

Now we recall Lemma 4.1 which says that the operator $\mathcal{T}$ is (C^0,C^0)-continuous on C^0_+. Also, as we have shown above, the set cl_0S_c has a bounded image. The fact that the family $\{\mathcal{T}u: u\in cl_0S_c\}$ is equicontinuous, follows easily from the relation

$$
|(\mathcal{T}u)(t)-(\mathcal{T}u)(\tau)|\le\Big|\int_{\tau}^{t}\Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u))-P(u)(s)]\big)ds\Big|\le\int_{\tau}^{t}Y(s)ds,
$$

for all $\tau,t\in I$, with $\tau\le t$, where

$$
Y(s):=\max\{\Psi\big(\frac{B_c}{p(s)}\big),-\Psi\big(\frac{-B_c}{p(s)}\big)\}.
$$

Hence, by Arzelá Ascoli' s Theorem, we conclude that the set $\mathcal{T}cl_0S_c$ is relatively $\|\cdot\|_0$-compact. Now, the desired result follows by applying Schauder's Fixed Point Theorem 1.2. $\square$

4.2. **Existence in the C^1 case.** Our next result is the following.

Theorem 4.4. *Assume that conditions* (H1), (H3), $(\mathrm{H5})_1$ *and* (4.3) *are satisfied. Also, assume that there is some $c > 0$, satisfying the conditions* (H6) *and* (H7)*, as well as*

(H8)
$$\max\{\Psi(\frac{B_c}{\rho}), -\Psi(-\frac{B_c}{\rho})\} \leq c,$$
where $\rho := \min_{t\in I} p(t)$.

Then the operator $\mathcal{T}$ defined by (4.1) *admits a fixed point in the set*
$$R_c := \{u \in C^1(I, [0, +\infty)), \quad 0 < \|u\|_1 \leq c\}.$$

Proof. First we notice that $R_c \subseteq S_c$ and $cl_1 R_c = R_c \cup \{0\}$. Then, by the first part of the proof of Theorem 4.3, we conclude that the operator $\mathcal{T}$ maps the set R_c into S_c, namely, for any $u \in R_c$ it holds
$$0 \leq (\mathcal{T}u)(t) \leq c, \quad t \in I. \tag{4.7}$$
Also we have $\mathcal{T}_0 \in \mathcal{T}(cl_0 S_c) \subseteq cl_0 Sc$ and, so, $0 \leq (\mathcal{T}0)(t) \leq c$, $t \in I$. This and (4.7) imply that $0 \leq (\mathcal{T}u)(t) \leq c$, $t \in I$, $u \in cl_1 R_c$.

Consider a function $u \in cl_1 R_c$. At first we observe that the function
$$(\mathcal{T}u)'(t) = \Psi\big(\frac{1}{p(t)}[\mathcal{X}(P(u)) - \int_0^t p(r)(Fu)(r)dr]\big), \quad t \in I$$
is continuous. Also, since, $0 \leq \|u\|_0 \leq \|u\|_1 \leq c$, it holds
$$-B_c \leq \mathcal{X}(P(u)) - \int_0^t p(r)(Fu)(r)dr \leq B_c, \quad t \in I$$
and so condition (H8) gives
$$|(\mathcal{T}u)'(t)| \leq \max\{\Psi(\frac{B_c}{\rho}), -\Psi(-\frac{B_c}{\rho})\} \leq c, \quad t \in I.$$
This and (4.7) imply that $\|\mathcal{T}u\|_1 \leq c$, hence $\mathcal{T}$ maps $cl_1 R_c$ into itself.

Now, by $(\mathrm{H5})_1$, to complete the proof of the theorem, it is sufficient to show that the Schauder's fixed point theorem is applicable on the set $cl_1 R_c$, namely on $\overline{R_c} = \{0\} \cup R_c$. This is a $\|\cdot\|_1$-closed, bounded, convex subset of the space C^1. It remains to prove that $\mathcal{T} cl_1 R_c$ is a relatively compact subset of C^1.

We shall show that the operator $\mathcal{T}$ is compact, namely it is continuous and the set $\mathcal{T} cl_1 R_c$ is a (relatively) compact subset of C^1.

First we notice that, by Lemma 4.2, the operator $\mathcal{T}$ is (C^1, C^1)-continuous. Then taking into account, also, that the set R_c is a subset of S_c, we have $\mathcal{T} cl_1 R_c \subseteq cl_1 \mathcal{T} R_c$. Therefore it is enough to show that $\mathcal{T} S_c$ is a (relatively) $\|\cdot\|_1$-compact subset of C^1.

To do that consider a sequence (U_n) in $\mathcal{T} S_c$. Then there is a sequence $u_n \in S_c$ such that $U_n = \mathcal{T} u_n$. As we proved previously, it holds
$$\|\mathcal{T}u_n\|_0 \leq \|\mathcal{T}u_n\|_1 \leq c, \quad n = 1, 2, \dots$$

and, moreover, it satisfies

$$\begin{aligned} |U_n(t)-U_n(\tau)| &= \Big|\int_\tau^t \Psi\big(\frac{1}{p(s)}[\mathcal{X}(P(u)) - \int_0^s p(r)(Fu_n)(r)dr]\big)ds\Big| \\ &\le \max\{\Psi(\frac{B_c}{\rho}), -\Psi(-\frac{B_c}{\rho})\}|t-\tau| \\ &\le c|t-\tau|. \end{aligned} \tag{4.8}$$

The latter says that the sequence (U_n) is equicontinuous. By Arzelá-Ascoli Theorem it follows that a subsequence $(U_{k_n}) = (\mathcal{T}u_{k_n})$ exists, which converges in the $\|\cdot\|_0$-norm to a function $y \in C^0(I,\mathbb{R})$.

Next, we take any $\epsilon > 0$. By the uniform continuity of the function Ψ on the interval $[-B_c/\rho, B_c/\rho]$, there is a $\delta_1 > 0$ such that, for all $v_1, v_2 \in [-B_c/\rho, B_c/\rho]$, it holds

$$|v_1 - v_2| < \delta_1 \implies |\Psi(v_1) - \Psi(v_2)| < \epsilon. \tag{4.9}$$

It is clear that the sequence of functions defined by

$$z_n(t) := \frac{1}{p(t)}[\mathcal{X}(P(u_{k_n})) - \int_0^t p(s)(Fu_{k_n})(s)ds], \quad t \in I$$

is $\|\cdot\|_0$-bounded[1] and, for all $\tau, t \in I$, it satisfies

$$\begin{aligned} |z_n(t) - z_n(\tau)| &= \Big|\Big(\frac{1}{p(t)} - \frac{1}{p(\tau)}\Big)\mathcal{X}(P(u_{k_n})) \\ &\quad - \frac{1}{p(t)}\int_0^t p(s)(Fu_{k_n})(s)ds + \frac{1}{p(\tau)}\int_0^\tau p(s)(Fu_{k_n})(s)ds\Big| \\ &\le \frac{1}{p(t)}\Big|\int_\tau^t p(s)(Fu_{k_n})(s)ds\Big| \\ &\quad + \Big|\frac{1}{p(t)} - \frac{1}{p(\tau)}\Big|\Big|\mathcal{X}(P(u_{k_n})) - \int_0^\tau p(s)(Fu_{k_n})(s)ds\Big| \\ &\le \frac{1}{p(t)}\Big|\int_\tau^t m_c(s)ds\Big| + \Big|\frac{1}{p(t)} - \frac{1}{p(\tau)}\Big|B_c. \end{aligned}$$

Therefore, the sequence (z_n) is equicontinuous and so it has a subsequence (z_{l_n}), which converges to a function $z \in C^0$ in the sense of $\|\cdot\|_0$-norm. Equivalently, the sequence $(\Phi\big((\mathcal{T}u_{k_{l_n}})'(\cdot)\big))$ converges to the function $z(\cdot)$ in the $\|\cdot\|_0$-norm and hence it is a $\|\cdot\|_0$-Cauchy sequence. Therefore, given any $\delta > 0$ with $\delta \le \delta_1$, there is some index k_0 such that

$$|\Phi\big((\mathcal{T}u_{k_{l_n}})'(t)\big) - \Phi\big((\mathcal{T}u_{k_{l_m}})'(t)\big)| < \delta, \quad t \in I,$$

for all $m, n \ge k_0$. Thus, from 4.9, we obtain

$$|(\mathcal{T}u_{k_{l_n}})'(t) - (\mathcal{T}u_{k_{l_m}})'(t)| < \epsilon, \quad t \in I,$$

for all $m, n \ge k_0$. This proposition says that the sequence $((\mathcal{T}u_{k_{l_n}})'(\cdot))$ is a Cauchy sequence and so it converges to the function $\Psi(z)$ in the $\|\cdot\|_0$-norm. By a standard criterion of the uniform converge of sequences of differentiable functions, we conclude that y' exists, it is equal to $\Psi(z)$ and, moreover it satisfies the limiting relation

$$\lim \|\mathcal{T}u_{k_{l_n}} - y\|_1 = 0.$$

[1] with a bound $\frac{B_c}{\rho}$

These arguments and the continuity of $\mathcal{T}$ imply the relative $\|\cdot\|_1$-compactness of the set $\mathcal{T}S_c$. This implies that the set $\mathcal{T}cl_1R_c$ is relatively $\|\cdot\|_1$-compact. Now, the Schauder's Fixed Point Theorem 1.2 completes the proof. $\square$

5. An application of Theorem 4.3

We consider the equation

$$t^{1/6}\big[t^{-1/6}\Phi(u'(t))\big]' + \beta_0 t + \beta_1 u^2(t) + \beta_2 u^3(\gamma t)\|u\|_0 + \beta_3\int_0^1 u(s)ds = 0, \quad t\in(0,1],$$

with $\gamma\in(0,1)$ and $\beta_0>0$, $\beta_1,\beta_2,\beta_3\geq 0$, associated with the boundary conditions

$$\lim_{t\to 0} t^{-1/6}\Phi(u'(t)) = a_0 u(0),$$
$$\lim_{t\to 1} t^{-1/6}\Phi(u'(t)) = -a_1 u(0) - b_1 u(\frac{1}{3}),$$

with $a_0>0$, $a_1,b_1\geq 0$, where Φ is the inverse of the function

$$\Psi(v) := v + v^3,$$

namely,

$$\Phi(w) := \Big(\frac{w-(w^2+\frac{4}{27})^{1/2}}{2}\Big)^{1/3} - \frac{1}{3}\Big(\frac{w-(w^2+\frac{4}{27})^{1/2}}{2}\Big)^{-1/3}.$$

This is of the form (1.1)-(1.2), with

$$p(t) := t^{-1/6}, \quad t\in(0,1],$$
$$\eta := a_0\chi_{(0,1]}, \quad \zeta := a_1\chi_{(0,\frac{1}{3}]} + (a_1+b_1)\chi_{(\frac{1}{3},1]}.$$

We take $c=1$ and define

$$m(t) := \beta_0 t^{5/6} + (\beta_1+\beta_2+\beta_3)t^{-1/6}, \quad t\in(0,1],$$
$$B := \frac{6}{11}\beta_0 + \frac{6}{5}(\beta_1+\beta_2+\beta_3).$$

We can assume that the coefficients of the problem satisfy the following conditions (which are easily computable):

$$B(\frac{1}{a_0}+\frac{12}{7}) + \frac{4B^3}{3} \leq 1,$$
$$(a_0+a_1)(\frac{3}{7}+\frac{B^2}{3}) + b_1\frac{3}{7}\Big(1-\frac{1}{3^{7/6}}\Big) + b_1\frac{B^2}{3}\Big(1-\frac{1}{3^{3/2}}\Big) \leq \frac{1}{2}.$$

One can see that, in case $c=1$, these conditions show, respectively, that (H6) and (H7) are satisfied. Hence, Theorem 4.3 guarantees the existence of a solution $u\in C^0$ such that $0<\|u\|_0\leq 1$.

6. An application of Theorem 4.4

We consider the equation

$$\frac{d}{dt}\Phi(u'(t)) + \alpha + \frac{\beta_1}{1+|u(\frac{t}{2})|} + \frac{\beta_2}{1+(u'(t))^2} = 0, \quad t\in I,$$

with $\alpha,\beta_1,\beta_2\geq 0$, and $\alpha+\beta_1+\beta_2>0$, associated with the boundary conditions

$$\lim_{t\to 0}\Phi(u'(t)) = au(\frac{1}{2}), \quad \lim_{t\to 1}\Phi(u'(t)) = -bu(1),$$

$(a, b > 0)$ where Φ is the function defined by $\Phi(v) := \ln(1 + |v|)sign(v)$. This problem is of the form (1.1)-(1.2), with

$$p(t) := 1, \quad t \in I,$$
$$\eta := a\chi_{[\frac{1}{2},1]}, \quad \zeta := b\chi_{\{1\}}.$$

Assuming that the condition

$$b + \frac{a}{2} \leq \frac{\alpha + \beta_1 + \beta_2}{e^{\alpha+\beta_1+\beta_2} - 1} \tag{6.1}$$

is satisfied, we shall show that the problem admits a solution u with

$$0 < \|u\|_1 \leq \frac{2(\alpha + \beta_1 + \beta_2)}{a} + 3(e^{\alpha+\beta_1+\beta_2} - 1).$$

To do that it is sufficient to prove that the conditions of Theorem 4.4 are satisfied with

$$c := \frac{\alpha + \beta_1 + \beta_2}{a} + \frac{3}{2}(e^{\alpha+\beta_1+\beta_2} - 1).$$

Indeed, first of all observe that the inverse of Φ is the function $\Psi(v) := (e^{|v|} - 1)\,\mathrm{sign}(v)$. We set $m := \alpha + \beta_1 + \beta_2$ and observe that $m_c(t) := m = B_c$. Then conditions (H6) and (H8) become

$$\frac{m}{a} + \frac{3}{2}(e^m - 1) \leq c, \quad e^m - 1 \leq c.$$

Clearly, this is true because of the choice of c. Condition (H7) becomes

$$\lambda \geq (b + \frac{a}{2})(e^\lambda - 1), \quad \lambda \geq \frac{a}{2}(e^\lambda - 1).$$

Both relations are satisfied if we have

$$b + \frac{a}{2} \leq \frac{\lambda}{e^\lambda - 1},$$

for all $\lambda \in [0, m]$. Since the right side of this inequality decreases with λ, the condition is satisfied, provided that it holds for the value $\lambda = m$. But the latter is true because of (6.1). Thus, Theorem 4.4 applies and the result follows.

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