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RIEMANN-LIOUVILLE FRACTIONAL COSINE FUNCTIONS

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Abstract. In this article, we present the notion of Riemann-Liouville fractional cosine function. we prove that a Riemann-Liouville α-order fractional cosine function is equivalent to the Riemann-Liouville α-order fractional resolvent introduced in [15].

1. Introduction

Let X be a Banach space, and $A : D(A) \subset X \to X$, $B : D(B) \subset X \to X$ be closed linear operators. It is well-known that C_0-semigroups are important tools to study the abstract Cauchy problem of first order

$$\begin{gathered} \frac{du(t)}{dt} = Au(t), \quad t > 0 \\ u(0) = x, \end{gathered} \tag{1.1}$$

and that the cosine function essentially characterizes the abstract Cauchy problem of second order

$$\begin{gathered} \frac{d^2u(t)}{dt^2} = Bu(t), \quad t > 0 \\ u(0) = x, u'(0) = 0. \end{gathered} \tag{1.2}$$

Here a C_0-semigroup is a family $\{T(t)\}_{t\geq 0}$ of strongly continuous and bounded linear operators defined on X satisfying $T(0) = I$ and $T(t+s) = T(t)T(s)$, $t, s \geq 0$; a cosine function is a family $\{S(t)\}_{t\geq 0}$ of strongly continuous and bounded linear operators defined on X satisfying $S(0) = I$ and $2S(t)S(s) = S(t) + S(s)$, $t \geq s \geq 0$.

Concretely, system (1.1) is well-posed if and only if A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$, namely, $Ax = \lim_{t\to 0^+} t^{-1}(T(t)x - x)$ with domain $D(A) = \{x \in D(A) : \lim_{t\to 0^+} t^{-1}(T(t)x - x)$ exists $\}$; system (1.2) is well-posed if and only if B generates a cosine function $\{S(t)\}_{t\geq 0}$, namely, $Bx = 2\lim_{t\to 0^+} t^{-2}(S(t)x - x)$ with domain $D(B) = \{x \in D(B) : \lim_{t\to 0^+} t^{-2}(S(t)x - x)$ exists $\}$. Therefore, pure algebraic methods can be used to study abstract Cauchy problems of first and second orders. For details, we refer to [5, 7].

However, equations of integer order such as (1.1) and (1.2) cannot exactly describe the behavior of many physical systems; fractional differential equations maybe more suitable for describing anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous

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materials; see [1, 16] and the references therein), fractional random walk [6, 18], etc. Fractional derivatives appear in the theory of fractional differential equations; they describe the property of memory and heredity of materials, and it is the major advantage of fractional derivatives compared with integer order derivatives. Let $\alpha > 0$ and $m = [\alpha]$, The smallest integer larger than or equal to α. There are mainly two types of α-order fractional differential equations, which are most used in the real problems.

(1) Caputo fractional abstract Cauchy problem

$$\begin{gathered} {}^C D_t^\alpha u(t) = Au(t), \quad t > 0, \\ u(0) = x, u^{(k)}(0) = 0, \quad k = 1, 2, \dots, m-1. \end{gathered} \tag{1.3}$$

where ${}^C D_t^\alpha$ is the Caputo fractional differential operator defined as follows:

$$ {}^C D_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\sigma)^{-\alpha} u^{(m)}(\sigma)\, d\sigma;$$

(2) Riemann-Liouville fractional abstract Cauchy problem

$$\begin{gathered} D_t^\alpha u(t) = Au(t), \\ (g_{2-\alpha} * u)(0) = \lim_{s\to 0^+} \int_0^s \frac{(s-\sigma)^{m-1-\alpha}}{\Gamma(2-\alpha)} u(\sigma)\, d\sigma = x, \\ (g_{2-\alpha} * u)^{(k)}(0) = \lim_{s\to 0^+} \int_0^s \frac{d^k}{dt^k} \frac{(s-\sigma)^{m-1-\alpha}}{\Gamma(m-\alpha)} u(\sigma)\, d\sigma = 0, \\ k = 1, 2, \dots, m-1. \end{gathered} \tag{1.4}$$

where the Riemann-Liouville fractional differential operator is

$$D_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt} \int_0^t (t-\sigma)^{m-1-\alpha} u(\sigma)\, d\sigma.$$

Obviously, (1.1) is just the limit state of equations (1.3) and (1.4) as $\alpha \to 1$, and (1.2) is just the limit state of equations (1.3) and (1.4) as $\alpha \to 2$. Initial conditions for the Caputo fractional derivatives are expressed in terms of initials of integer order derivatives [4, 14, 17]. For some real materials, initial conditions should be expressed in terms of Riemann-Liouville fractional derivatives, and it is possible to obtain initial values for such initial conditions by appropriate measurements [8, 9].

To study Caputo fractional abstract Cauchy problem (1.3), Bajlekova [2] introduced the important notion of solution operator for equations (1.3) as follows.

Definition 1.1. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators of X is called a solution operator for (1.3) if the following three conditions are satisfied:

(a) $T(t)$ is strongly continuous for $t \geq 0$ and $T(0) = I$,
(b) $T(t)D(A) \subset D(A)$ and $AT(t)x = T(t)Ax$ for all $x \in D(A)$ and $t \geq 0$,
(c) for any $x \in D(A)$, it holds

$$T(t)x = x + J_t^\alpha T(t)Ax, \quad t \geq 0,$$

where

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(t)dt.$$

Chen and Li [3] developed a notion of α-resolvent operator function, which was proved to be a new characteristic of solution operator. Hence, Caputo fractional abstract Cauchy problem can be studied by pure algebraic methods. The definition of α-resolvent operator function is as follows.

Definition 1.2. Let $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on X. Then $\{S(t)\}_{t\geq 0}$ is called to be an α-resolvent operator function, if the following assumptions are satisfied:

(1) $S(t)$ is strongly continuous and $S(0)=I$.
(2) $S(s)S(t)=S(t)S(s)$ for all $t,s\geq 0$.
(3) $S(s)J_t^\alpha S(t)-J_s^\alpha S(s)S(t)=J_t^\alpha S(t)-J_s^\alpha S(s)$ for all $t,s\geq 0$.

Li and Peng [12] proposed the following notion of fractional resolvent to study Riemann-Liouville α-order fractional abstract Cauchy problem (1.4) with $\alpha\in(0,1)$.

Definition 1.3 ([12])**.** Let $0<\alpha<1$. A family $\{T(t)\}_{t>0}$ of bounded linear operators on Banach space X is called an α-order fractional resolvent if it satisfies the following assumptions:

(1) for any $x\in X$, $T(\cdot)x\in C((0,\infty),X)$, and
$$\lim_{t\to 0+}\Gamma(\alpha)t^{1-\alpha}T(t)x=x \quad \text{for all } x\in X; \tag{1.5}$$
(2) $T(s)T(t)=T(t)T(s)$ for all $t,s>0$;
(3) for all $t,s>0$, it holds
$$T(t)J_s^\alpha T(s)-J_t^\alpha T(t)T(s)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}J_s^\alpha T(s)-\frac{s^{\alpha-1}}{\Gamma(\alpha)}J_t^\alpha T(t). \tag{1.6}$$

In [15], we studied the Riemann-Liouville α-order fractional Cauchy problem (1.4) with order $\alpha\in(1,2)$. There Riemann-Liouville α-order fractional resolvent defined as follows.

Definition 1.4. A family $\{T(t)\}_{t>0}$ of bounded linear operators is called Riemann-Liouviiille α-order fractional resolvent if it satisfies the following assumptions:

(a) For any $x\in X$, $T_\alpha(\cdot)x\in C((0,\infty),X)$, and
$$\lim_{t\to 0^+}\Gamma(\alpha-1)t^{2-\alpha}T(t)x=x \quad \text{for all } x\in X; \tag{1.7}$$
(b) $T(s)T_\alpha(t)=T(t)T_\alpha(s)$ for all $t,s>0$;
(c) for all $t,s>0$, it holds
$$T(s)J_t^\alpha T(t)-J_s^\alpha T(s)T(t)=\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}J_t^\alpha T(t)-\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}J_s^\alpha T(s). \tag{1.8}$$

The linear operator A defined by
$$Ax=\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^{2\alpha}},$$
$$\text{for } x\in D(A)=\Big\{x\in X:\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^{2\alpha}}\text{ exists}\Big\}.$$
Operator A generates a Riemann-Liouville α-order fractional resolvent $\{T(t)\}_{t>0}$ in Definition 1.3.

Also, we proved that $\{T(t)\}_{t>0}$ is a Riemann-Liouville α-order fractional resolvent if and only if it is a solution operator defined as follows.

Definition 1.5. *A family $\{T(t)\}_{t>0}$ of bounded linear operators of X is called a solution operator for* (1.4) *if the following three conditions are satisfied:*

(a) *$T(t)$ is strongly continuous for $t>0$ and $\lim_{t\to 0^+}\Gamma(\alpha-1)t^{2-\alpha}T(t)x=x$, $x\in X$,*
(b) *$T(t)D(A)\subset D(A)$ and $AT(t)x=T(t)Ax$ for all $x\in D(A)$ and $t>0$,*
(c) *for any $x\in D(A)$, it holds*

$$T(t)x=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}x+J_t^{\alpha}T(t)Ax,\quad t>0.$$

However, the above functional equations for fractional differential equations are not expressed in terms of the sum of time variables: $s+t$. This is very important in concrete applications of the functional equation, just like C_0-semigroups, cosine functions. Motivated by this, Peng and Li [17] established the characterization of α-order fractional semigroup with $\alpha\in(0,1)$:

$$\begin{aligned}&\int_0^{t+s}\frac{T(\tau)}{(t+s-\tau)^{\alpha}}\,d\tau-\int_0^{t}\frac{T(\tau)}{(t+s-\tau)^{\alpha}}\,d\tau-\int_0^{s}\frac{T(\tau)}{(t+s-\tau)^{\alpha}}\,d\tau\\&=\alpha\int_0^t\int_0^s\frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2,\quad t,s\ge 0,\end{aligned}$$

where the integrals are in the sense of strong operator topology. Concretely, they proved that α-order fractional semigroup is closely related to the solution operator of Caputo fractional abstract Cauchy problem (1.3).

Mei, Peng and Zhang [13] developed the notion of Riemann-Liouville fractional semigroup as follows.

Definition 1.6. A family $\{T(t)\}_{t>0}$ of bounded linear operators is called Riemann-Liouville α-order fractional semigroup on Banach space X, if the following conditions are satisfied:

(i) for any $x\in X$, $t\mapsto T(t)x$ is continuous over $(0,\infty)$ and

$$\lim_{t\to 0+}\Gamma(\alpha)t^{1-\alpha}T(t)x=x; \tag{1.9}$$

(ii) for all $t,s>0$, it holds

$$\Gamma(1-\alpha)T(t+s)=\alpha\int_0^t\int_0^s\frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2, \tag{1.10}$$

where the integrals are in the sense of strong operator topology.

It is proved in [13] that A generates a Rimann-Liouville fractional semigroup if and only if it generates a fractional resolvent developed in [11].

To study Caputo fractional Cauchy problem of order $\alpha\in(1,2)$, we recently studied in [14] the notion of fractional cosine function as follows.

Definition 1.7. A family $\{T(t)\}_{t\geq 0}$ of bounded and strongly continuous operators is called an α-fractional cosine function if $T(0)=I$ and it holds

$$\begin{aligned}
&\int_0^{t+s}\int_0^{\sigma}\frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma-\int_0^{t}\int_0^{\sigma}\frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\\
&-\int_0^{s}\int_0^{\sigma}\frac{T(\tau)}{(t+s-\sigma)^{\alpha-1}}\,d\tau\,d\sigma\\
&=\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma+\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\
&\quad-\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma,\quad t,s\geq 0,
\end{aligned}\tag{1.11}$$

where the integrals are in the sense of strong operator topology.

There, we proved that A generates a fractional cosine function $\{T(t)\}_{t\geq 0}$ if and only if it generates an α-resolvent operator function; that is, the following equalities hold:

$$T(s)J_t^{\alpha}T(t)-J_s^{\alpha}T(s)T(t)=J_t^{\alpha}T(t)-J_s^{\alpha}T(s),\quad t,s\geq 0.$$

As stated above, functional equations involving t, s and $t+s$ have been discussed for Caputo fractional differential equations (1.3) with $\alpha\in(0,1)$ and $\alpha\in(1,2)$, Riemann-Liouville fractional equation (1.4) with $\alpha\in(0,1)$. To close the gap, we will discuss the residual case, that is, functional equations involving t, s and $t+s$ for Riemann-Liouville fractional equation (1.4) with $\alpha\in(1,2)$. To this end, we first consider the special case that $T(\cdot)$ is exponentially bounded (hence it is Laplace transformable). Take laplace transform on both sides of (1.6) with respect to s and t to obtain

$$(\lambda^{-\alpha}-\mu^{-\alpha})\hat{T}(\mu)\hat{T}(\lambda)=\lambda^{1-\alpha}\mu^{1-\alpha}(\lambda^{-1}\hat{T}(\lambda)-\mu^{-1}\hat{T}(\mu)).\tag{1.12}$$

It follows from [14, (3.8)] that the Laplace transform of the right-hand side of (1.10) satisfies

$$\begin{aligned}
&\int_0^{\infty}e^{-\mu t}\int_0^{\infty}e^{-\lambda s}\Big(\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma+\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\
&-\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\Big)\,ds\,dt\\
&=\frac{\Gamma(2-\alpha)(\lambda^{\alpha}-\mu^{\alpha})}{\lambda\mu(\lambda-\mu)}\hat{T}(\mu)\hat{T}(\lambda).
\end{aligned}\tag{1.13}$$

The combination of (1.12) and (1.13) implies

$$\begin{aligned}
&\int_0^{\infty}e^{-\mu t}\int_0^{\infty}e^{-\lambda s}\Big(\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma+\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\
&-\int_0^{t}\int_0^{s}\frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\Big)\,ds\,dt\\
&=\frac{\Gamma(2-\alpha)(\lambda^{-1}\hat{T}(\lambda)-\mu^{-1}\hat{T}(\mu))}{\mu-\lambda}.
\end{aligned}$$

Let $m(t)=\int_0^t T(\sigma)\,d\sigma$, by similar proof of [10, (4.2)], it holds

$$\int_0^{\infty}e^{-\mu t}\int_0^{\infty}e^{-\lambda s}m(t+s)\,ds\,dt=\frac{\hat{m}(\mu)-\hat{m}(\lambda)}{\lambda-\mu}=\frac{\lambda^{-1}\hat{T}(\lambda)-\mu^{-1}\hat{T}(\mu)}{\mu-\lambda}.$$

By the Laplace transform, it follows that

$$\begin{aligned}
&\Gamma(2-\alpha)\int_0^{t+s} T(\sigma)\,d\sigma \\
&= \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma \\
&\quad - \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma,
\end{aligned} \tag{1.14}$$

In the following two sections, we show that (1.14) also holds without the assumption that $\{T(t)\}_{t>0}$ is exponentially bounded and it essentiality describes a Riemann-Liouville fractional resolvent.

2. Riemann-Liouville fractional cosine function

Equality (1.6) is an important functional equation for the solution of (1.4) with $\alpha \in (1,2)$. However, as stated in the introduction, (1.6) does not write the functional equation in terms of the sum of time variables: $s+t$. This is very important in concrete applications of the algebraic functional equation. Therefore, it is very valuable to study functional equation (1.14), which appears in the following definitions.

Definition 2.1. A family $\{T(t)\}_{t>0}$ of bounded linear operators is called Riemann-Liouville α-order fractional cosine function on a Banach space X, if the following conditions are satisfied:

(i) $T(t)$ is strongly continuous, that is, for any $x \in X$, the mapping $t \mapsto T(t)x$ is continuous over $(0,\infty)$;

(ii) it holds
$$\lim_{t\to 0+} t^{2-\alpha}T(t)x = \frac{x}{\Gamma(\alpha-1)} \quad \text{for all } \ x \in X; \tag{2.1}$$

(iii) for all $t, s > 0$, it holds
$$\begin{aligned}
&\Gamma(2-\alpha)\int_0^{t+s} T(\sigma)\,d\sigma \\
&= \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma \\
&\quad - \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma,
\end{aligned} \tag{2.2}$$
where the integrals are in the sense of strong operator topology.

Lemma 2.2. *Let $\{T(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional cosine on Banach space X. Then $\{T(t)\}_{t>0}$ is commutative, i.e. $T(t)T(s) = T(s)T(t)$ for all $t, s > 0$.*

Proof. Observe that the left-hand side of (2.2) is symmetric with respect to t and s. Hence we can obtain the equality

$$\begin{aligned}
&\int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma \\
&- \int_0^t\int_0^s \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \int_0^t \frac{T(\sigma)T(\tau)}{(s-\sigma)^{\alpha-1}}\, d\tau\, d\sigma + \int_0^s \int_0^t \frac{T(\sigma)T(\tau)}{(t-\tau)^{\alpha-1}}\, d\tau\, d\sigma \\
&\quad - \int_0^s \int_0^t \frac{T(\sigma)T(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, d\sigma, \quad t,s>0.
\end{aligned}
$$

The commutative property is proved as in [13, Proposition 3.4]. $\square$

Definition 2.3. Let $\{T(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional cosine function on Banach space X. Denote by $D(A)$ the set of all $x \in X$ such that the limit

$$\lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha} J_t^{2-\alpha}\Big(T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x\Big)$$

exists. Then the operator $A : D(A) \to X$ defined by

$$Ax = \lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha} J_t^{2-\alpha}\Big(T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x\Big)$$

is called the generator of $\{T(t)\}_{t>0}$.

Proposition 2.4. *Assume $\{T(t)\}_{t>0}$ is a Riemann-Liouville α-order fractional cosine function on Banach space X. Suppose that A is the generator of $\{T(t)\}_{t>0}$. Then*

(a) *For any $x \in X$ and $t > 0$, it holds $J_t^\alpha T(t)x \in D(A)$ and*
$$T(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + AJ_t^\alpha T(t)x; \tag{2.3}$$
(b) *$T(t)D(A) \subset D(A)$ and $T(t)Ax = AT(t)x$, for all $x \in D(A)$.*
(c) *For all $x \in D(A)$, we have*
$$T(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^\alpha T(t)Ax;$$
(d) *A is equivalently defined by*
$$Ax = \Gamma(2\alpha-1) \lim_{t\to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x}{t^{2\alpha-2}} \tag{2.4}$$
and $D(A)$ is just consists of those $x \in X$ such that the above limit exists.
(e) *A is closed and densely defined.*
(f) *A admits at most one Riemann-Liouville α-order fractional cosine function.*

Proof. (a) Let $x \in X$ and $b > 0$ be fixed. Denote by $g_b(\cdot)$ the truncation of $T(\cdot)$ at b; that is,

$$g_b(\sigma) = \begin{cases} T(\sigma), & \text{if } 0 < \sigma \le b \\ 0, & \text{if } \sigma > b. \end{cases}$$

Define the function $H_b(r,s)$ for $r,s > 0$ by

$$H_b(r,s) = \Big(g_b(r) - \frac{r^{\alpha-2}}{\Gamma(\alpha-1)}I\Big) J_s^\alpha g_b(s)x. \tag{2.5}$$

Obviously, for $0 < r \le t$,

$$H_t(r,t) = \Big(T(r) - \frac{r^{\alpha-2}}{\Gamma(\alpha-1)}I\Big) J_t^\alpha T(t)x. \tag{2.6}$$

Take Laplace transform with respect to r and s successively for both sides of (2.5) to obtain

$$\hat{H}_b(\mu,\lambda)=\lambda^{-\alpha}\hat{g}_b(\mu)\hat{g}_b(\lambda)x-\lambda^{-\alpha}\mu^{1-\alpha}\hat{g}_b(\lambda)x. \tag{2.7}$$

Denote by $L(t,s)$ and $R(t,s)$ the left and right sides of (2.2), respectively. Moreover, denote by $R_b(t,s)$, and $L_b(t,s)$ the quantities resulted by replacing $T(t)$ with $g_b(t)$ in $R(t,s)$, $L(t,s)$, respectively.

It follows from [14, (3.7)] that the Laplace transform of $R_b(t,s)$ with respect to t and s is given by

$$\hat{R}_b(\mu,\lambda)=\frac{\Gamma(2-\alpha)(\lambda^\alpha-\mu^\alpha)}{\lambda\mu(\lambda-\mu)}\hat{g}_b(\mu)\hat{g}_b(\lambda). \tag{2.8}$$

For all $t>0$, the Laplace transform of $\hat{L}_b(t,s)$ with respect to s and t can be obtained as

$$\hat{L}_b(\mu,\lambda)=\Gamma(2-\alpha)\frac{\lambda^{-1}\hat{g}_b(\lambda)-\mu^{-1}\hat{g}_b(\mu)}{\mu-\lambda}. \tag{2.9}$$

Combine (2.7), (2.8) and (2.9) to derive

$$\begin{aligned}\hat{H}_b(\mu,\lambda)&=\mu^{-\alpha}\hat{g}_b(\mu)\hat{g}_b(\lambda)x-\mu^{-\alpha}\lambda^{1-\alpha}\hat{g}_b(\mu)x\\&\quad+\frac{\lambda^{1-\alpha}\mu^{1-\alpha}(\lambda-\mu)}{\Gamma(2-\alpha)}(\hat{L}_b(\mu,\lambda)-\hat{R}_b(\mu,\lambda))x.\end{aligned}$$

Take inverse Laplace transform to obtain

$$\begin{aligned}H_b(r,s)&=\Big(g_b(s)-\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}I\Big)J_r^\alpha g_b(r)x\\&\quad+\frac{[(D_s^{2-\alpha})J_r^{\alpha-1}-(D_r^{2-\alpha})J_s^{\alpha-1}]\cdot[L_b(r,s)-R_b(r,s)]x}{\Gamma(2-\alpha)}.\end{aligned}$$

Here the Laplace transform formula

$$\widehat{D_b^\beta}f(\lambda)=\lambda^\beta\hat{f}(\lambda)-\lim_{t\to0^+}J_t^{\alpha-1}f(t),\quad 0<\beta<1,\ f\in C([0,\infty),X)$$

is used.

From the definition of g_b, it follows that $L_b(r,s)=R_b(r,s)$ for $0<s,r\le b$. Then we have

$$H_b(r,s)=\Big(T(s)-\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}I\Big)J_r^\alpha T(r)x,\quad\forall\ 0<r,s\le b.$$

This implies

$$H_t(r,t)=\Big(T(t)-\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}I\Big)J_r^\alpha T(r)x,\quad\forall 0<r\le t. \tag{2.10}$$

Combining (2.6) with (2.10), we obtain

$$\begin{aligned}&\lim_{r\to0^+}\Gamma(\alpha+1)r^{-\alpha}J_r^{2-\alpha}\Big(T(r)-\frac{r^{\alpha-2}}{\Gamma(\alpha-1)}\Big)J_t^\alpha T(t)x\\&=\lim_{r\to0^+}\Gamma(\alpha+1)r^{-\alpha}\Big(T(t)-\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}I\Big)J_r^2T(r)x\\&=\Gamma(\alpha+1)\Big(T(t)-\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}I\Big)\lim_{r\to0^+}r^{-\alpha}\int_0^r(r-\sigma)T(\sigma)x\,d\sigma\end{aligned}$$

$$
\begin{aligned}
&= \Gamma(\alpha+1)\Big(T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I\Big) \\
&\quad \times \lim_{r\to 0^+} \int_0^1 (1-\sigma)\sigma^{\alpha-2}(r\sigma)^{2-\alpha} T(r\sigma)x \, d\sigma.
\end{aligned}
$$

By the dominated convergence theorem and (b) of Definition 2.1, it follows that

$$
\begin{aligned}
&\lim_{r\to 0^+} \Gamma(\alpha+1) r^{-\alpha} J_r^{2-\alpha} \Big(T(r) - \frac{r^{\alpha-2}}{\Gamma(\alpha-1)}\Big) J_t^\alpha T(t)x \\
&= \Gamma(\alpha+1)\Big(T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I\Big) \int_0^1 (1-\sigma)\sigma^{\alpha-2} \lim_{r\to 0^+} (r\sigma)^{2-\alpha} T(r\sigma)x \, d\sigma \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-1)}\Big(T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I\Big) \int_0^1 (1-\sigma)\sigma^{\alpha-2} \, d\sigma x \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-1)}\Big(T(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I\Big) \frac{\Gamma(\alpha-1)\Gamma(2)}{\Gamma(\alpha+1)} x \\
&= T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x.
\end{aligned}
$$

This implies that $J_t^\alpha T(t)x \in D(A)$ and

$$
AJ_t^\alpha T(t)x = T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x.
$$

Conditions (b) and (c) are directly obtained by Lemma 2.2 and (a).

(d) Denote by D the set of those $x \in X$ such that the limit

$$
\lim_{t\to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}}
$$

exists. Let $x \in D(A)$. Then, by (b), we have

$$
\begin{aligned}
&\Gamma(2\alpha-1) \lim_{t\to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}} \\
&= \Gamma(2\alpha-1) \lim_{t\to 0^+} \frac{J_t^\alpha T(t)Ax}{t^{2\alpha-2}} \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \lim_{t\to 0^+} \frac{1}{t^{2\alpha-2}} \int_0^t (t-\sigma)^{\alpha-1} T(\sigma)Ax \, d\sigma \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \lim_{t\to 0^+} \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-2}(t\sigma)^{2-\alpha} T(t\sigma)Ax \, d\sigma.
\end{aligned}
$$

The dominated convergence theorem and (b) of Definition 2.1 indicate that

$$
\begin{aligned}
&\Gamma(2\alpha-1) \lim_{t\to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x}{t^{2\alpha-2}} \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-2} \lim_{t\to 0^+} (t\sigma)^{2-\alpha} T(t\sigma)Ax \, d\sigma \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\alpha-2} Ax \, d\sigma \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha-1)\Gamma(\alpha)} \frac{\Gamma(\alpha-1)\Gamma(\alpha)}{\Gamma(2\alpha-1)} Ax = Ax.
\end{aligned}
$$

This implies that $x \in D$ and then $D(A) \subset D$. Now we prove the converse inclusion. Let $x \in D$, that is, the limit

$$\lim_{t\to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1))}x}{t^{2\alpha-2}}.$$

exists. By the dominated convergence theorem, it follows that

$$\begin{aligned}
&\lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha} J_t^{2-\alpha}\Big(T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x\Big)\\
&= \lim_{t\to 0^+} \frac{\Gamma(\alpha+1)}{\Gamma(2-\alpha)} \int_0^1 (1-\sigma)^{1-\alpha}\sigma^{2\alpha-2} \frac{T(t\sigma)x - \frac{(t\sigma)^{\alpha-2}}{\Gamma(\alpha-1)}x}{(t\sigma)^{2\alpha-2}}\, d\sigma\\
&= \frac{\Gamma(\alpha+1)}{\Gamma(2-\alpha)} \int_0^1 (1-\sigma)^{1-\alpha}\sigma^{2\alpha-2} \lim_{t\to 0^+} \frac{T(t\sigma)x - \frac{(t\sigma)^{\alpha-2}}{\Gamma(\alpha-1)}x}{(t\sigma)^{2\alpha-2}}\, d\sigma\\
&= \frac{\Gamma(\alpha+1)}{\Gamma(2-\alpha)} \frac{\Gamma(2-\alpha)\Gamma(2\alpha-1)}{\Gamma(\alpha+1)} \lim_{t\to 0^+} \frac{T(t)x - \frac{(t)^{\alpha-2}}{\Gamma(\alpha-1)}x}{t^{2\alpha-2}}.
\end{aligned}$$

Hence, $x \in D(A)$ and

$$Ax = \Gamma(2\alpha-1) \lim_{t\to 0^+} \frac{T(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1))}x}{t^{2\alpha-2}}. \tag{2.11}$$

(e) The properties that A is closed and densely defined are followed directly from the combination of (d) and [12].

(f) Assume that both $\{T(t)\}_{t>0}$ and $\{S(t)\}_{t>0}$ are Riemann-Liouville α-order fractional resolvent generated by A. Then, by (c), for all $x \in D(A)$, we have

$$\begin{aligned}
\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * T(t)x &= (S(t) - J_t^\alpha AS(t)) * T(t)x\\
&= S(t) * T(t)x - (J_t^\alpha AS(t)) * T(t)x\\
&= S(t) * (T(t)x - J_t^\alpha AT(t)x)\\
&= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * S(t)x.
\end{aligned}$$

By Titchmarsh's Theorem, for any $t > 0$, $T(t) = S(t)$ on $D(A)$. The result is obtained by the density of A. □

Corollary 2.5. *Assume that A generates a Rimann-Liouville α-order fractional cosine function on Banach space X. Then $\{T(t)\}_{t>0}$ is a Riemann-Liouville α-order fractional resolvent.*

Proof. In (a) of Theorem 2.4, replacing x with $J_s^\alpha T(s)x$, and using Lemma 2.2, we obtain

$$\begin{aligned}
T(t)J_s^\alpha T(s)x &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_s^\alpha T(s)x + AJ_t^\alpha T(t)J_s^\alpha T(s)x\\
&= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_s^\alpha T(s)x + AJ_s^\alpha T(s)J_t^\alpha T(t)x\\
&= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_s^\alpha T(s)x + \Big(T(s) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\Big) J_t^\alpha T(t)x,
\end{aligned}$$

which is just (1.6). The proof is complete. □

3. Equivalence of Riemann-Liouville fractional resolvent

In this section, we prove that equality (1.6) essentially describes a Rimann-Liouville α-order fractional cosine function.

Theorem 3.1. *Suppose that $\{T(t)\}_{t>0}$ is a Riemann-Liouville α-order fractional resolvent on Banach space X. Then, the family is a Riemann-Liouville α-order fractional cosine function.*

Proof. Denote by $L(t,s)$ and $R(t,s)$ the left and right sides of equality (2.2), respectively. Obviously, we need to prove that $L(t,s) = R(t,s)$ for all $t,s > 0$. For brevity, we introduce the following notation. Let

$$
\begin{gathered}
H(t,s) = T(t)J_s^\alpha T(s) - J_t^\alpha T(t)T(s), \\
K(t,s) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_s^\alpha T(s) - \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} J_t^\alpha T(t), t,s > 0.
\end{gathered}
$$

Moreover, for sufficiently large $b > 0$ denote by $g_b(t)$ the truncation of $T(t)$ at b, and by $R_b(t,s)$, $L_b(t,s)$, $H_b(t,s)$ and $K_b(t,s)$ the quantities resulted by replacing $T(t)$ with $g_b(t)$ in $R(t,s)$, $L(t,s)$, $H(t,s)$ and $K(t,s)$, respectively.

We set

$$
\begin{aligned}
P_b(t,s) = &\int_0^t \int_0^s \frac{H_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\, d\tau\, d\sigma + \int_0^t \int_0^s \frac{H_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\, d\tau\, d\sigma \\
&- \int_0^t \int_0^s \frac{H_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, d\sigma
\end{aligned}
$$

and

$$
\begin{aligned}
Q_b(t,s) = &\int_0^t \int_0^s \frac{K_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\, d\tau\, d\sigma + \int_0^t \int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\, d\tau\, d\sigma \\
&- \int_0^t \int_0^s \frac{K_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, d\sigma.
\end{aligned} \tag{3.1}
$$

Observe that the equality (1.6) implies $H(t,s) = K(t,s)$ for any $t,s > 0$. Thus, for all $t,s > 0$,

$$
\lim_{b\to\infty} P_b(t,s) = \lim_{b\to\infty} Q_b(t,s). \tag{3.2}
$$

By [14, (3.13)], it follows that

$$
P_b(t,s) = (J_s^\alpha - J_t^\alpha)R_b(t,s), \quad \forall\, t,s > 0. \tag{3.3}
$$

We now compute Laplace transform of the first term of $Q_b(t,s)$ with respect to s and t as follows,

$$
\begin{aligned}
&\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{K_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt \\
&= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{\frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} J_\tau^\alpha g_b(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} J_\sigma^\alpha g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt \\
&= \int_0^\infty e^{-\mu t} \int_0^t \int_0^\infty e^{-\lambda s} \int_0^s \frac{\frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} J_\tau^\alpha g_b(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} J_\sigma^\alpha g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\, d\tau\, ds\, d\sigma dt \\
&= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{\frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} J_\tau^\alpha g_b(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} J_\sigma^\alpha g_b(\sigma)}{(t-\sigma)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\mu t} \int_0^t \frac{\frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)}}{(t-\sigma)^{\alpha-1}} \int_0^\infty e^{-\lambda s} \int_0^s J_\tau^\alpha g_b(\tau)\, d\tau ds\, d\sigma dt \\
&- \int_0^\infty e^{-\mu t} \int_0^t \frac{J_\sigma^\alpha g_b(\sigma)}{(t-\sigma)^{\alpha-1}} \int_0^\infty e^{-\lambda s} \int_0^s \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)}\, d\tau ds\, d\sigma dt \\
&= \Gamma(2-\alpha)\mu^{-1}\lambda^{-\alpha-1}\hat{g}_b(\lambda) - \Gamma(2-\alpha)\mu^{-2}\lambda^{-\alpha}\hat{g}_b(\mu),
\end{aligned}
$$

The Laplace transform of the second term of $Q_b(t,s)$ with respect to s and t is computed as follows

$$
\begin{aligned}
&\int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt \\
&= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{\frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} J_\tau^\alpha g_b(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} J_\sigma^\alpha g_b(\sigma)}{(s-\tau)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt \\
&= \int_0^\infty e^{-\mu t} \int_0^t \frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1))} \int_0^\infty e^{-\lambda s} \int_0^s \frac{J_\tau^\alpha g_b(\tau)}{(s-\tau)^{\alpha-1}}\, d\tau\, ds\, d\sigma dt \\
&\quad - \int_0^\infty e^{-\mu t} \int_0^t J_\sigma^\alpha g_b(\sigma) \int_0^\infty e^{-\lambda s} \int_0^s \frac{\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)}}{(s-\tau)^{\alpha-1}}\, d\tau\, ds\, d\sigma dt \\
&= \Gamma(2-\alpha)\mu^{-\alpha}\lambda^{-2}\hat{g}_b(\lambda) - \Gamma(2-\alpha)\lambda^{-1}\mu^{-\alpha-1}\hat{g}_b(\mu).
\end{aligned}
$$

We compute the Laplace transform of the third term of $Q_b(t,s)$ with respect to s and t as follows.

$$
\begin{aligned}
&- \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{K_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt \\
&= - \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} \int_0^t \int_0^s \frac{\frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} J_\tau^\alpha g_b(\tau) - \frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)} J_\sigma^\alpha g_b(\sigma)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, d\sigma\, ds\, dt \\
&= - \int_0^\infty e^{-\mu t} \int_0^t \frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^\infty e^{-\lambda s} \int_0^s \frac{J_\tau^\alpha g_b(\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, ds\, d\sigma dt \\
&\quad + \int_0^\infty e^{-\mu t} \int_0^t J_\sigma^\alpha g_b(\sigma) \int_0^\infty e^{-\lambda s} \int_0^s \frac{\frac{\tau^{\alpha-2}}{\Gamma(\alpha-1)}}{(t+s-\sigma-\tau)^{\alpha-1}}\, d\tau\, ds\, d\sigma dt \\
&= - \int_0^\infty e^{-\mu t} \int_0^t \frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} \int_0^\infty e^{-\lambda s} \frac{1}{(t+s-\sigma)^{\alpha-1}}\, ds\, d\sigma dt \lambda^{-\alpha} g_b(\lambda) \\
&\quad + \lambda^{1-\alpha} \int_0^\infty e^{-\mu t} \int_0^t J_\sigma^\alpha g_b(\sigma) \int_0^\infty e^{-\lambda s} \frac{1}{(t+s-\sigma)^{\alpha-1}}\, ds\, d\sigma dt \\
&= - \int_0^\infty e^{-\mu t} \int_0^t \frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)} e^{\lambda(t-\sigma)} \\
&\quad \times \Big(\int_0^\infty e^{-\lambda r} r^{1-\alpha} dr - \int_0^{t-\sigma} e^{-\lambda r} r^{1-\alpha} dr \Big)\, d\sigma dt \lambda^{-\alpha} g_b(\lambda) \\
&\quad + \lambda^{1-\alpha} \int_0^\infty e^{-\mu t} \int_0^t J_\sigma^\alpha g_b(\sigma) e^{\lambda(t-\sigma)} \\
&\quad \times \Big(\int_0^\infty e^{-\lambda r} r^{1-\alpha} dr - \int_0^{t-\sigma} e^{-\lambda r} r^{1-\alpha}\, dr \Big)\, d\sigma dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\Gamma(2-\alpha)\lambda^{\alpha-2}\int_0^\infty e^{-\mu t}\int_0^t \frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)}e^{\lambda(t-\sigma)}\,d\sigma dt\lambda^{-\alpha}g_b(\lambda)\\
&\quad + \int_0^\infty e^{-\mu t}\int_0^t \frac{\sigma^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^{t-\sigma} e^{\lambda(t-\sigma-r)}r^{1-\alpha}\,dr\,d\sigma dt\lambda^{-\alpha}g_b(\lambda)\\
&\quad + \Gamma(2-\alpha)\lambda^{\alpha-2}\lambda^{1-\alpha}\int_0^\infty e^{-\mu t}\int_0^t J_\sigma^\alpha g_b(\sigma)e^{\lambda(t-\sigma)}\,d\sigma dt\\
&\quad - \lambda^{1-\alpha}\int_0^\infty e^{-\mu t}\int_0^t J_\sigma^\alpha g_b(\sigma)\int_0^{t-\sigma} e^{\lambda(t-\sigma-r)}r^{1-\alpha}dr\,d\sigma dt\\
&= -\Gamma(2-\alpha)\lambda^{\alpha-2}\lambda^{-\alpha}\frac{\mu^{1-\alpha}}{\mu-\lambda}g_b(\lambda) + \Gamma(2-\alpha)\frac{\mu^{-1}}{\mu-\lambda}\lambda^{-\alpha}g_b(\lambda)\\
&\quad + \Gamma(2-\alpha)\lambda^{\alpha-2}\lambda^{1-\alpha}\frac{\mu^{-\alpha}}{\mu-\lambda}\hat{g}_b(\mu) - \Gamma(2-\alpha)\lambda^{1-\alpha}\mu^{\alpha-2}\frac{\mu^{-\alpha}}{\mu-\lambda}\hat{g}_b(\mu).
\end{aligned}
$$

Using (2.9), we obtain

$$
\begin{aligned}
&\hat{Q}_b(\mu,\lambda)\\
&= \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\Big(\int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t-\sigma)^{\alpha-1}}\,d\tau\,d\sigma + \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(s-\tau)^{\alpha-1}}\,d\tau\,d\sigma\\
&\quad - \int_0^t\int_0^s \frac{K_b(\sigma,\tau)}{(t+s-\sigma-\tau)^{\alpha-1}}\,d\tau\,d\sigma\Big)\,ds\,dt\\
&= (\lambda^{-\alpha}-\mu^{-\alpha})\hat{L}_b(\mu,\lambda).
\end{aligned}
\tag{3.4}
$$

Taking inverse Laplace transform on both sides of (3.4), we derive

$$
Q_b(t,s) = (J_s^\alpha - J_t^\alpha)L_b(t,s), \quad \forall t,s>0. \tag{3.5}
$$

Form (3.3) and (3.5), we have

$$
(J_s^\alpha - J_t^\alpha)L(t,s) = (J_s^\alpha - J_t^\alpha)R(t,s), \quad \forall t,s>0.
$$

Therefore, $L(t,s)=R(t,s)$. This completes the proof. □

Combining Corollary 2.5 and Theorem 3.1, we can obtain the equivalent of Riemann-Liouville α-order fractional resolvents and Riemann-Liouville α-order fractional cosine functions.

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