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INTERACTING RAREFACTION WAVES FOR THE UNSTEADY TRANSONIC SMALL DISTURBANCE EQUATION

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Abstract. We consider a Riemann problem for the unsteady transonic small disturbance equation that results in two rarefaction waves. We write the problem in self-similar and parabolic coordinates and we obtain a system that changes type from hyperbolic to elliptic. We use the characteristic decomposition equations to study the complicated interaction of these two rarefaction waves in the hyperbolic region. We obtain local existence of the solution and we derive various properties of the solution and of the characteristic curves.

1. Introduction and summary of this article

To better understand the structure of the solutions to the Cauchy problem for the two-dimensional Euler gas dynamics equations, scientists have considered simpler models with sectorially constant initial data known as Riemann initial data. There has been a lot of progress in the past several decades in this research area using numerical and analytical approaches. Some of the simplified models that have been considered are the isentropic gas dynamics equations, the pressure-gradient system, the nonlinear wave system, the potential flow and the unsteady transonic small disturbance (UTSD) equation (see Keyfitz [12]).

Conjectures on the structure of the solutions to isentropic and full gas dynamics equations, where the Riemann initial data is constant in four quadrants and each discontinuity results in a shock, rarefaction wave or a slip line, are given by Zhang and Zheng [18], revealing very complicated interactions using the theory of generalized characteristics. The numerical simulations are given by Schulz-Rinne et al. [14]. For the analytical and numerical studies of solutions to the pressure-gradient system, with Riemann data posed in four quadrants, see [17, 19].

A Riemann problem with data constant in four quadrants that results in interacting rarefaction waves is considered numerically for the full gas dynamics equations by Glimm et al. in [6]. They observed reflection of characteristics from the sonic curves and formation of shocks. These regions where one family of characteristics starts on a sonic curve and ends on a transonic shock are known in the literature as semi-hyperbolic patches and have been studied using the characteristic decomposition equations. For the study of semi-hyperbolic patches for the pressure-gradient system, see Song and Zheng [15], for the Euler equations, see Li and Zheng [13], for

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the nonlinear wave system, see [7] by Hu and Wang, and for the UTSD equation, see Jegdić and Jegdić [10] and Tesdall and Keyfitz [16]. Conjecture on the structure of the solution for the nonlinear wave system with diverging rarefaction waves was given by Jegdić and Jegdić [9]. A Riemann problem for the pressure-gradient system with data constant in four quadrants resulting in interacting rarefaction waves without the reflection of waves from the sonic curves was studied by Bang [1].

Shock reflection phenomena for the UTSD equation have been studied in [2, 3, 4, 11]. In this article we are interested in interaction of rarefaction waves and in §2 we formulate the initial value problem we consider. As in [3, 4, 10, 11], we write the problem in self-similar coordinates and obtain a reduced system that changes type from hyperbolic to elliptic with boundary data given in the far field. Using the generalized characteristics as in [17, 18, 19], we derive conjectures on two configurations of the solution. We notice that the hyperbolic region consists of fully-hyperbolic, semi-hyperbolic and, in the case of the second configuration, impracticably hyperbolic subregions (see Zheng [19] for this terminology for the pressure-gradient system). In §3 we recall the characteristic decomposition equations from [10], which we use to study solution locally in the fully-hyperbolic and semi-hyperbolic subdomains. We prove local existence of smooth solution in these subdomains and we derive statements on the monotonicity of solution along characteristics and various properties of characteristics curves.

2. Formulation of the problem and structure of the solution

In this section we formulate the initial value problem we consider and we rewrite it using self-similar coordinates. We use the generalized characteristics to pose conjectures on the structure of two possible solutions.

2.1. **Formulation of the initial value problem.** We consider the unsteady transonic small disturbance equation

$$\begin{aligned} u_t + uu_x + v_y &= 0,\\ -v_x + u_y &= 0, \end{aligned} \tag{2.1}$$

where $(x,y) \in \mathbb{R}^2$, $t \in [0,\infty)$, and $(u,v) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are velocities in x- and y-directions, respectively. We use the notation $U = (u,v)$. Given a parameter $a > 0$, we consider initial data consisting of three constant states posed in three sectors (see Figure 1)

$$U(0,x,y) = \begin{cases} U_2 = (-1,a), & y > \max\{x/a, 0\},\\ U_3 = (-1,-a), & y < \min\{-x/a, 0\},\\ U_1 = (0,0), & \text{otherwise.} \end{cases} \tag{2.2}$$

2.2. **Formulation of the problem in self-similar coordinates.** We write system (2.1) in self-similar coordinates $\xi = x/t$ and $\eta = y/t$

$$\begin{aligned} (u-\xi)u_\xi - \eta u_\eta + v_\eta &= 0,\\ -v_\xi + u_\eta &= 0. \end{aligned} \tag{2.3}$$

The eigenvalues of this system are

$$\Lambda_\pm = \frac{\eta \pm \sqrt{\eta^2 + 4(\xi - u)}}{2(\xi - u)}, \tag{2.4}$$

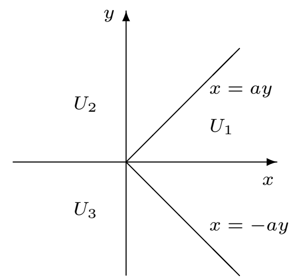

Figure 1. Initial data

implying that (2.3) changes type across the sonic parabola

$$P_u : \xi + \frac{\eta^2}{4} = u,$$

and (2.3) is hyperbolic outside of P_u and elliptic inside P_u. Hence, under this change of variables, the initial data (2.2) becomes boundary data given on the "parabola in the far field"

$$U(\xi,\eta) = \begin{cases} U_2, & \eta > \max\{\xi/a, 0\},\ \xi + \eta^2/4 \to \infty \\ U_3, & \eta < \min\{-\xi/a, 0\},\ \xi + \eta^2/4 \to \infty \\ U_1, & \xi > 0,\ -\xi/a < \eta < \xi/a,\ \xi + \eta^2/4 \to \infty. \end{cases}$$

Using the theory of one-dimensional hyperbolic conservation laws, we obtain that each of the discontinuities in the far field resolves in a rarefaction wave. More precisely, we obtain a rarefaction wave

$$R_{12} \begin{cases} U = U(\xi - a\eta) \\ u = \xi - a\eta - a^2 \\ v = -au \end{cases}$$

connecting the states U_1 on the left and U_2 on the right, and a rarefaction wave

$$R_{13} \begin{cases} U = U(\xi + a\eta) \\ u = \xi + a\eta - a^2 \\ v = au \end{cases}$$

connecting the states U_3 on the left and U_1 on the right. Therefore, the left and the right borders of R_{12} are given by $\xi = a\eta + a^2$ and $\xi = a\eta + a^2 - 1$, respectively, and the left and the right borders of R_{13} are given by $\xi = -a\eta + a^2 - 1$ and $\xi = -a\eta + a^2$, respectively (see Figure 2).

2.3. **Two possible structures of the solution.** We study the interaction of rarefaction waves R_{12} and R_{13} coming from the far field with the sonic parabola $P_0 : \xi + \eta^2/4 = 0$ for the state U_1 and the sonic parabola $P_{-1} : \xi + \eta^2/4 = -1$ for the states U_2 and U_3.

The rarefaction waves R_{12} and R_{13} start to interact at the point $A(a^2, 1)$. The problem is symmetric about the ξ-axis and we focus our attention to the upper half-plane. We find the $\Lambda_\pm$-characteristics at A by integrating

$$\frac{d\eta}{d\xi} = \Lambda_\pm$$

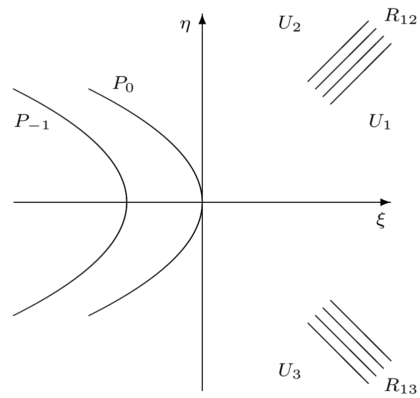

FIGURE 2. Solution in the far field and the sonic parabolas P_0 and P_{-1}

and using that $\xi - u = a\eta + a^2$ across R_{12}. We compute that Λ_+ characteristics at A is

$$\eta = \frac{\xi}{a} - a,$$

which is the left border of R_{12}, and that Λ_- characteristics at A is

$$\frac{(a+\eta)^2}{2} = -\xi + \frac{3a^2}{2},$$

which penetrates R_{12} and intersects the right border of R_{12} at the point

$$B(-a^2 + a\sqrt{4a^2+2} - 1, -2a + \sqrt{4a^2+2})$$

(see Figure 3).

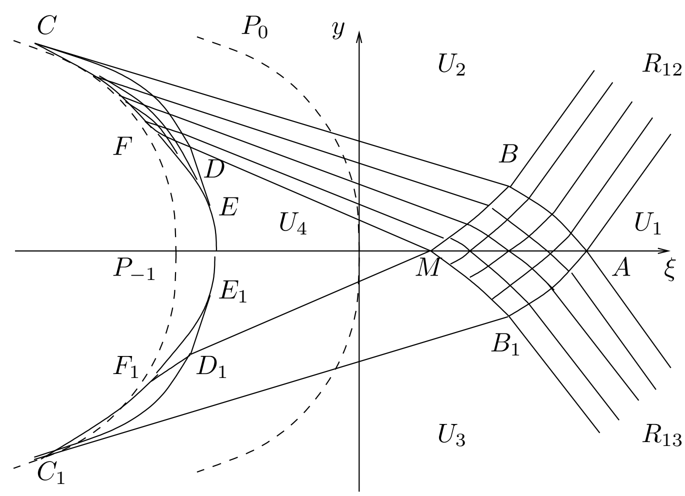

FIGURE 3. Configuration I

It is easy to check that the point B is outside of the parabola P_{-1}, implying that the point B is hyperbolic with respect to the state U_2. The straight Λ_+

characteristics of the rarefaction wave R_{12} curve bellow the parabolic arc $\widehat{AB}$. Since the problem is symmetric with respect to ξ-axis, after meeting at the point A, we conclude that the rarefaction waves R_{12} and R_{13} interact to form the penetration region $ABMB_1A$. The problem in this region is a Goursat type problem with data given on characteristic boundaries $\widehat{AB}$ and $\widehat{AB_1}$.

Upon exiting the boundaries $\widehat{BM}$ and $\widehat{B_1M}$, the curved characteristics become again straight forming two simple waves. We again focus on the upper half-plane and we find the Λ_- characteristics at B which is a line tangent to the sonic parabola P_{-1} at the point

$$C(-5a^2 + 2a\sqrt{4a^2+2} - 3, -2a + 2\sqrt{4a^2+2}). \tag{2.5}$$

It is easy to show that the characteristics $\widehat{ABC}$ has continuous first derivatives at A and B.

Next, we follow Λ_+ characteristics at C. This characteristics goes backward through the simple wave in the upper half-plane and exists the simple wave at the point D. Afterwords, this characteristics becomes straight ending tangentially at the point E on the sonic parabola for the constant state U_4 which is a solution to the Goursat problem in the region $MDEE_1D_1M$ with data given on characteristics boundaries $\overline{MD}$ and $\overline{MD_1}$. The straight Λ_- characteristics of the simple wave in the upper half-plane become curved to the right of $\widehat{CD}$ until they become sonic along $\widehat{CF}$. We can think of these Λ_- characteristics reflecting of the sonic boundary $\widehat{FC}$ as Λ_+ characteristics. After passing through $\widehat{DF}$, these Λ_+ characteristics become again straight until they reach the sonic boundary $\widehat{FE}$.

Therefore, the boundary of the interaction domain consists of the Λ_- characteristic curve $\widehat{ABC}$, Λ_+ characteristic curve $\widehat{AB_1C_1}$ and the part of the sonic parabola P_{-1} outside of the arc $\widehat{CC_1}$. The exterior of this domain consists of two rarefaction waves and three constant states. The interaction domain consists of the elliptic and hyperbolic subdomains.

The problem in the unbounded region with boundary

$$C\widehat{FEE_1}C_1 \cup \{(\xi,\eta) \in P_{-1} : \eta > \eta_C \text{ or } \eta < \eta_{C_1}\}$$

is *elliptic*, while the problem in the bounded region $ABCFEE_1F_1C_1B_1A$ is *hyperbolic*. Moreover, the problem in the region $ABCDEE_1D_1C_1B_1A$ is *fully-hyperbolic* since both sets of characteristics can be traced back to the parabola in the far-field. The regions $CDEFC$ and $C_1D_1E_1F_1C_1$ are *partially hyperbolic* and in the literature known as *semi-hyperbolic patches* as one set of characteristics can be traced to the far-field parabola and the other set of characteristics starts and ends on sonic curves. More on this terminology for the pressure-gradient system could be found in Zheng [19]. In particular, the semi-hyperbolic patches for the unsteady transonic small disturbance equation were studied in [10] where it is shown that the characteristics which start on a sonic curve form an envelope that is overtaken by a transonic shock. Hence, we conjecture that curved Λ_+ characteristics exiting the boundary $\widehat{DF}$ as straight characteristics end on a transonic shock. Numerical examples of interacting rarefaction waves for the full Euler gas dynamics equations are given by Glimm et al. [6] showing a complicated structure involving semi-hyperbolic patches and reflection of characteristics which start on a sonic curve and end on a transonic shock. Tesdall and Keyfitz [16] study the interaction of the rarefaction wave with the sonic boundary and show numerically that the the resulting shock starts (or ends) in the supersonic region.

The second possible structure of the solution is illustrated in Figure 4. In this case, the Λ_+ characteristics DE and the Λ_- characteristics D_1E_1 intersect each other before becoming sonic. Let X denote their intersection. The region EXE_1E is known in the literature (see [19]) as *impracticably hyperbolic* since at each point in this region neither set of characteristics could be traced back to the far field parabola.

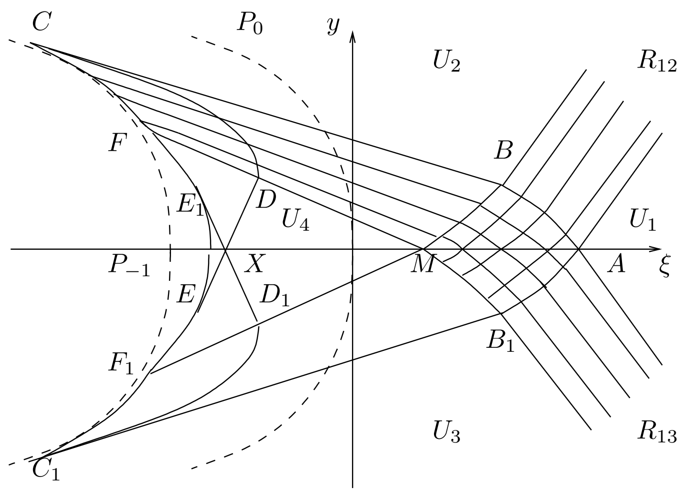

Figure 4. Configuration II

3. Analysis in the hyperbolic part of the interaction region

The rest of the paper is about the study of the solution structure in Configuration I (Figure 3) and we focus our attention to the hyperbolic part of the interaction region. We study the solution in three subdomains: penetration region $ABMB_1A$ between R_{12} and R_{13}, simple domain region $BCDMB$ and the semi-hyperbolic patch $CDEFC$. In §3.1 we write the problem in parabolic coordinates and we recall the characteristic decomposition equations derived in [10]. We use these equations to study Goursat problems in domains $ABMB_1A$ and $BCDMB$, where the data is known along two characteristic boundaries.

In the semi-hyperbolic patch $CDEFC$, positive characteristics reflect of the sonic curve $\widehat{FC}$ and the solution in $CDFC$ depends on the solution in the elliptic region with $\widehat{FC}$ being a free sonic boundary. To the best of our knowledge, understanding of this dependence is still an open problem and our future work consists of computing detailed numerical simulations using the numerical method from [8] by Jegdić to better understand this dependence. As in earlier studies of semi-hyperbolic patches [7, 10, 13, 15], to study the solution in the region $CDEFC$, we assume that the location of the characteristic curve $\widehat{DF}$ is known.

In §3.2 and 3.4 we prove existence of local smooth solutions in regions $ABMB_1A$ and $CDFC$, and we obtain various properties of global smooth solutions in these regions. Our future work consists of using these properties and ideas from [5] by Dai to extend these local solutions to the entire regions $ABMB_1A$ and $CDFC$.

3.1. **Formulation of the problem in parabolic coordinates and characteristic decomposition.** We consider system (2.3) in parabolic coordinates (ρ, η) where $\rho = \xi + \eta^2/4$

$$\begin{aligned} (u-\rho)u_\rho - \frac{\eta}{2}u_\eta + v_\eta = 0, \\ \frac{\eta}{2}u_\rho - v_\rho + u_\eta = 0, \end{aligned} \tag{3.1}$$

with eigenvalues

$$\lambda_\pm = \pm\frac{1}{\sqrt{\rho - u}}.$$

From [10] we recall the definitions of the directional derivatives along $\Lambda_\pm$ and $\lambda_\pm$ characteristic curves

$$\partial^\pm = \partial_\xi + \Lambda_\pm \partial_\eta \quad \text{and} \quad \partial_\pm = \partial_\eta + \lambda_\pm^{-1}\partial_\rho = \partial_\eta \pm \sqrt{\rho - u}\partial_\rho, \tag{3.2}$$

the following relationships

$$\partial^\pm \Lambda_\pm = \frac{\Lambda_\pm^3}{\eta\Lambda_\pm + 2}\partial^\pm u \quad \text{and} \quad \partial_\pm u = \frac{\eta\Lambda_\pm + 2}{\Lambda_\pm^4}\partial^\pm \Lambda_\pm, \tag{3.3}$$

and the characteristic decomposition equations

$$\partial_\pm \partial_\mp u = Q(\partial_\pm u - \partial_\mp u)\partial_\mp u, \tag{3.4}$$

where

$$Q = \frac{1}{4(\rho - u)} = -\frac{\lambda_+\lambda_-}{4}. \tag{3.5}$$

Notice that equations (3.4) could be rewritten as

$$\partial_\pm\Big(\frac{1}{\partial_\mp u}\Big) + Q\partial_\pm u\frac{1}{\partial_\mp u} = Q. \tag{3.6}$$

As in [10], we eliminate v from the system (3.1) and we obtain the second order equation

$$(u-\rho)u_{\rho\rho} + u_{\eta\eta} - \frac{u_\rho}{2} + u_\rho^2 = 0$$

and the system

$$\begin{bmatrix} \partial_+ u \\ \partial_- u \\ u \end{bmatrix}_\eta + \begin{bmatrix} -\sqrt{\rho - u} & 0 & 0 \\ 0 & \sqrt{\rho - u} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_+ u \\ \partial_- u \\ u \end{bmatrix}_\rho = \begin{bmatrix} Q(\partial_- u - \partial_+ u)\partial_+ u \\ Q(\partial_+ u - \partial_- u)\partial_- u \\ \frac{1}{2}(\partial_+ u + \partial_- u) \end{bmatrix}. \tag{3.7}$$

We use the notation

$$W = (\partial_+ u, \partial_- u, u)^T$$

and in sections §3.2 - §3.4, we consider system (3.7) in subregions of the hyperbolic interaction domain $ABCFEE_1F_1C_1B_1A$ with u given on characteristic boundaries. We find $\partial_\pm u$ along those boundaries by using the fact that they are characteristics and by using the characteristic decomposition equations written in the form (3.6). We obtain local existence of solution in these subregions and we obtain certain properties of the solution and of the characteristic curves. Our ideas are similar to those in [10] on the study of a semi-hyperbolic patch.

3.2. **Interactions in the region $ABMB_1A$.** We consider system (3.7) in the region $ABMB_1A$. The boundaries $\widehat{AB}$ and $\widehat{B_1A}$ are characteristics determined in §2.3 with the condition for u found in §2.2. More precisely, we have that $\widehat{AB}$ is given by

$$\frac{(a+\eta)^2}{2} = -\xi + \frac{3a^2}{2},$$

or, in parabolic coordinates,

$$\rho = a^2 - a\eta - \frac{\eta^2}{4},$$

and along $\widehat{AB}$ we have

$$u = \xi - a\eta - a^2 = \rho - \Big(\frac{\eta}{2} + a\Big)^2.$$

Also, $\widehat{B_1A}$ is given by

$$\frac{(a-\eta)^2}{2} = -\xi + \frac{3a^2}{2} \quad \text{or} \quad \rho = a^2 + a\eta - \frac{\eta^2}{4},$$

and along $\widehat{B_1A}$ we have

$$u = \xi + a\eta - a^2 = \rho - \Big(\frac{\eta}{2} - a\Big)^2.$$

We find $\partial_- u$ along $\widehat{AB}$ and $\partial_+ u$ along $\widehat{B_1A}$ by using the fact that these boundaries are characteristics and we find $\partial_+ u$ along $\widehat{AB}$ and $\partial_- u$ along $\widehat{B_1A}$ by using equations (3.6).

Since $\widehat{AB}$ is a λ_- characteristics, we have

$$\partial_- u = \partial_\eta - \sqrt{\rho - u}\partial_\rho u = -\eta - 2a < 0 \tag{3.8}$$

along $\widehat{AB}$. Similarly, since $\widehat{B_1A}$ is a λ_+ characteristics, we have

$$\partial_+ u = u_\eta + \sqrt{\rho - u}\,\partial_\rho u = -\eta + 2a > 0 \tag{3.9}$$

along $\widehat{B_1A}$.

Next, since $\widehat{AB}$ is λ_- characteristic curve, (3.6) imply that along $\widehat{AB}$ we have

$$\partial_-\big(\frac{1}{\partial_+ u}\big) + \partial_- u \frac{1}{\partial_+ u} = Q,$$

i.e., by using (3.8), we have

$$\frac{d}{d\eta}\frac{1}{\partial_+ u} - \frac{1}{\eta + 2a}\frac{1}{\partial_+ u} = \frac{1}{(\eta + 2a)^2}.$$

Notice that this is a linear ordinary differential equation in $1/\partial_+ u$. We use (3.9) to impose a condition at A,

$$\frac{1}{\partial_+ u}(A) = \frac{1}{2a},$$

and as a solution of this initial value problem we obtain $\partial_+ u$ along $\widehat{AB}$. Similarly, (3.6) imply that along $\widehat{B_1A}$ we have

$$\frac{d}{d\eta}\frac{1}{\partial_- u} - \frac{1}{\eta - 2a}\frac{1}{\partial_- u} = \frac{1}{(\eta - 2a)^2},$$

which with the condition $\frac{1}{\partial_- u}(A) = -\frac{1}{2a}$ gives $\partial_- u$ along $\widehat{B_1A}$. Therefore, we have $W = (\partial_+ u, \partial_- u, u)^T$ along the boundaries $\widehat{AB}$ and $\widehat{B_1A}$.

Theorem 3.1. *The Goursat problem* (3.7) *with W given on the characteristic boundaries $\widehat{AB}$ and $\widehat{B_1A}$ as above has a smooth local solution at the point A.*

Proof. As in [10, Theorem 4.1], it suffices to check the following compatibility conditions at the point A:

$$W|_{\widehat{AB}}(A) = W|_{\widehat{B_1A}}(A)$$

and

$$\frac{\vec{l}_3 W'|_{\widehat{AB}}(A) - \frac{\partial_+ u(A)+\partial_- u(A)}{2}}{\lambda_1(A) - \lambda_3(A)} = \frac{\vec{l}_3 W'|_{\widehat{B_1A}}(A) - \frac{\partial_+ u(A)+\partial_- u(A)}{2}}{\lambda_2(A) - \lambda_3(A)},$$

where $\lambda_1 = -\sqrt{\rho - u}$, $\lambda_2 = \sqrt{\rho - u}$ and $\lambda_3 = 0$ are the eigenvalues of the system (3.7) and $\vec{l_1} = (1,0,0)$, $\vec{l_2} = (0,1,0)$ and $\vec{l_3} = (0,0,1)$ are the corresponding eigenvectors. It is easy to check that both conditions are satisfied. □

Lemma 3.2. *Let u be a smooth solution of the Goursat problem* (3.7) *in the domain $ABMB_1A$ with W given on characteristic boundaries $\widehat{AB}$ and $\widehat{B_1A}$ as above. Then*

(a) *$\partial_- u < 0$ and $\partial_+ u > 0$ in the interior of the domain $ABMB_1A$,*
(b) *$\partial_- u$ is increasing along λ_+ characteristics, and*
(c) *$\partial_+ u$ is decreasing along λ_- characteristics.*

Proof. The proof is similar to the proof of [10, Lemma 4.2] and for completeness, we outline it here. Let P be an interior point of the domain $ABMB_1A$ and let P_1 be the intersection of the λ_+ characteristics with $\widehat{AB}$. Then integrating (3.4) along $\widehat{PP_1}$ we have

$$\partial_- u(P) = \partial_- u(P_1) e^{\int_P^{P_1} Q(\partial_+ u - \partial_- u) ds}.$$

Since $\partial_- u(P_1) < 0$ according to (3.8), we have $\partial_- u(P) < 0$. Part (b) follows by noticing that $Q > 0$. Similarly, we prove the statement (c) for $\partial_+ u$. □

Lemma 3.3. *Let u be a smooth solution of the Goursat problem* (3.7) *in the domain $ABMB_1A$ with W given on characteristic boundaries $\widehat{AB}$ and $\widehat{B_1A}$ as above. Then Λ_+ characteristics are convex and Λ_- characteristics are concave.*

Proof. From (3.3) we have

$$\partial^{\pm} \Lambda_{\pm} = \frac{\Lambda_{\pm}^4}{\eta \Lambda_{\pm} + 2} \partial_{\pm} u.$$

Using the result (a) of the previous Lemma, it suffices to prove

$$\eta \Lambda_{\pm} + 2 > 0. \tag{3.10}$$

First, we note that $\xi - u > 0$ in the region $ABMB_1A$. Clearly, at the point A, we have $\xi - u = a^2 > 0$. Assume that there is a point P in the interior of $ABMB_1A$ such that $\xi - u = 0$ at P. By the definition of Λ_+ in (2.4), if $\eta \neq 0$ at P, then Λ_+ characteristics would be vertical at P, which is false. Hence, $\eta = 0$ at P, and therefore $\eta^2/4 = u - \xi$ at P, implying that P is sonic. This is also false since the region $ABMB_1A$ is hyperbolic.

Next, from (2.4) we have

$$\eta\Lambda_+ + 2 = \frac{\eta^2 + \eta\sqrt{\eta^2 + 4(\xi - u)} + 4(\xi - u)}{2(\xi - u)}.$$

To prove (3.10), we need to prove that the numerator in the above expression is positive, i.e., that $\eta^2 + 4(\xi - u) > -\eta\sqrt{\eta^2 + 4(\xi - u)}$. If $\eta > 0$, this is clearly satisfied, and if $\eta < 0$, by squaring both sides of the inequality, we obtain that the inequality is also satisfied. Similarly, we prove the claim (3.10) for Λ_- characteristics. □

3.3. **Interactions in the simple wave region** $BCDMB$**.** We express the left and the right border of the rarefaction wave R_{12} in (ρ, η)-coordinates and we obtain

$$\rho = \Big(\frac{\eta}{2} + a\Big)^2 \quad \text{and} \quad \rho = \Big(\frac{\eta}{2} + a\Big)^2 - 1.$$

From Lemma 3.3 we have that the Λ_+ characteristics at the point B curves up after passing through the point B. Therefore, if $\widehat{MB}$ is given by $\rho = f(\eta)$ in the (ρ, η)-coordinates, we have $f' > 0$ and $f'' < \frac{1}{2}$. Then

$$f' = \frac{d\rho}{d\eta} = \sqrt{\rho - u} = \sqrt{f - u}$$

implies $u = f - (f')^2$, and along $\widehat{MB}$ we have

$$\partial_+ u = \partial_\eta u + \sqrt{\rho - u}\partial_\rho u = f'(1 - 2f'') > 0. \tag{3.11}$$

Next, we prove that u is continuous in the simple wave region $BCDMB$.

Theorem 3.4. *Let u be a solution of the Goursat problem* (3.7) *with W given along the characteristic boundaries $\widehat{BC}$ and $\widehat{MB}$. Then λ_- characteristics do not intersect each other in the domain $BCDMB$.*

Proof. We prove the statement by contradiction. Assume that $Z \in BCDMB$ is the first intersection point of two negative characteristics $\overline{XZ}$ and $\overline{YZ}$, where $X, Y \in \widehat{MB}$. Assume that $\partial_+ u(Z) = \infty$.

Since u is constant along λ_- characteristics, we have $\partial_- u = 0$ along $\overline{XZ}$. From the characteristic decomposition equation (3.4), we have

$$\partial_-\partial_+ u = -Q(\partial_+ u)^2$$

i.e.,

$$-\frac{\partial_-\partial_+ u}{(\partial_+ u)^2} = Q$$

on $\overline{XZ}$. Integrating the last equation along the characteristic line $\overline{XZ}$, we obtain

$$\frac{1}{\partial_+ u(Z)} - \frac{1}{\partial_+ u(X)} = \int_X^Z Q ds,$$

implying

$$\partial_+ u(X) = \Big(-\int_X^Z Q ds\Big)^{-1} < 0,$$

since $Q > 0$. This is a contradiction to (3.11). □

In the next lemma we prove that u is increasing along $\widehat{DC}$ in the parabolic coordinates and decreasing in self-similar coordinates.

Lemma 3.5. *Let u be a solution of the Goursat problem (3.7) in the domain $BCDMB$ with W given along the characteristic boundaries $\widehat{BC}$ and $\widehat{MB}$. Then*

$$\partial_+ u > 0 \text{ on } \widehat{DC} \quad \text{and} \quad \partial^+ u < 0 \text{ on } \widehat{DC}.$$

Proof. Let $Y \in \widehat{DC}$ and let $\overline{XY}$ be a negative characteristics through Y, where $X \in \widehat{MB}$. We integrate the characteristic decomposition equation (3.4) along $\overline{XY}$ and obtain

$$\partial_+ u(X) = \partial_+ u(Y) e^{\int_X^Y Q(\partial_- u - \partial_+ u) ds}.$$

From (3.11) we have $\partial_+ u(X) > 0$, implying $\partial_+ u(Y) > 0$.

To prove the second part of the lemma, we consider the change of coordinates $(\xi, \eta) \mapsto (\rho, \eta)$ and we compute

$$\begin{aligned}
\partial_+ u &= \partial_\eta u + \sqrt{\rho - u}\, \partial_\rho u \\
&= -\frac{\eta}{2} \partial_\xi u + \partial_\eta u + \sqrt{\rho - u}\, \partial_\xi u \\
&= \Big(\partial^+ u - \Lambda_+ \partial_\eta u\Big) \Big(\sqrt{\rho - u} - \frac{\eta}{2}\Big) + \partial_\eta u,
\end{aligned}$$

using that $\partial_\xi u = \partial^+ u - \Lambda_+ \partial_\eta u$ from (3.2). By rewriting the last expression, we obtain

$$\partial_+ u = \partial^+ u \Big(\sqrt{\rho - u} - \frac{\eta}{2}\Big) + \partial_\eta u \Big(1 - \Lambda_+ \Big(\sqrt{\rho - u} - \frac{\eta}{2}\Big)\Big).$$

Using the definition of Λ_+ in (2.4), it is easy to show

$$1 - \Lambda_+ \Big(\sqrt{\rho - u} - \frac{\eta}{2}\Big) = 0.$$

Therefore,

$$\partial_+ u = \partial^+ u \Big(\sqrt{\rho - u} - \frac{\eta}{2}\Big),$$

and to finish the proof, we need to prove that

$$\sqrt{\rho - u} - \frac{\eta}{2} < 0 \quad \text{on } \widehat{DC}. \tag{3.12}$$

Clearly, since C is a sonic point, we have

$$\sqrt{\rho - u} - \frac{\eta}{2} = -\frac{\eta_C}{2} < 0,$$

since C is in the upper half-plane. If there exists a point $X \in \widehat{DC}$ such that

$$\sqrt{\rho - u} = \frac{\eta}{2} \quad \text{at } X,$$

then we have $\eta = \sqrt{\eta^2 + 4(\xi - u)}$ at X, implying by (2.4), that Λ_-characteristics is horizontal at X. From the geometry of the problem, as described in §2.3, this is impossible since every Λ_- characteristics in the upper half-plane in the domain $BCDMB$ is decreasing. Hence, (3.12) is proved, implying $\partial^+ u < 0$ along $\widehat{DC}$. □

Lemma 3.6. *Let u be a solution of the Goursat problem (3.7) in the domain $BCDMB$ with W given along the characteristic boundaries $\widehat{BC}$ and $\widehat{MB}$. Then Λ_+ characteristics $\widehat{DC}$ is concave in (ξ, η)-plane.*

Proof. Along $\widehat{DC}$, we have

$$\frac{d^2\eta}{d\xi^2} = \partial^+\Lambda_+ = \frac{\Lambda_+^3}{\eta\Lambda_+ + 2}\partial^+ u$$

by (3.3). Recall that $\Lambda_+ < 0$ and that $\partial^+ u < 0$, by the previous Lemma.

Next, we determine the sign of

$$\eta\Lambda_+ + 2 = \frac{\eta^2 + 4(\xi - u) + \eta\sqrt{\eta^2 + 4(\xi - u)}}{2(\xi - u)} \tag{3.13}$$

along $\widehat{DC}$. We prove that

$$\eta\Lambda_+ + 2 < 0 \quad \text{on } \widehat{DC} \setminus \{C\}. \tag{3.14}$$

First, note that at the point C we have

$$\xi - u = -5a^2 + 2a\sqrt{4a^2 + 2} - 4 < 0.$$

If there is a point $P \in \widehat{DC}$ such that $\xi - u = 0$ at P, then by the definition (2.4), we would have that Λ_+ characteristics at P is vertical, which is false. Hence

$$\xi - u < 0 \quad \text{on } \widehat{DC}. \tag{3.15}$$

To prove (3.14), we need to show that the numerator in (3.13) is positive on $\widehat{DC} \setminus \{C\}$. This is clearly true as the sum of the first two terms is positive since $\widehat{DC}\setminus\{C\}$ is in the hyperbolic region, and the third term is positive since $\eta > 0$ on $\widehat{DC}$. □

3.4. **Interactions in the semi-hyperbolic patch $CDEFC$.** We follow the same idea as in [10] where the semi-hyperbolic patch resulted from the interaction of the rarefaction wave and the sonic parabola, while the semi-hyperbolic patch $CDEFC$ results from the interaction of the simple wave and the sonic parabola. We remark again that the solution in the semi-hyperbolic patch is coupled with the solution in the elliptic region through the free sonic boundary, however, this dependence is still an open problem. In this section, as in [7, 10, 13, 15], to study the solution in the region $CDEFC$, we assume that the location of the curve $\widehat{DF}$ is known.

Let us denote the Λ_+ characteristics $\widehat{DC}$ by $\xi = f(\eta)$ and Λ_- characteristics $\widehat{DF}$ by $\xi = g(\eta)$. We have $f' < 0$ and, by Lemma 3.6, we have $f'' < 0$. Also, $g' < 0$, and since the straight Λ_- characteristics of the simple wave in the region $BCDMB$ become curved to the right of $\widehat{DC}$, it is reasonable to assume that $g'' > 0$. As in §3.2, we use the fact that $\widehat{DC}$ and $\widehat{DF}$ are characteristics to derive $u, \partial_+ u$ and $\partial_- u$ along $\widehat{DC}$ and $\widehat{DF}$ and we show that the Goursat problem for the system (3.7) with data given on $\widehat{DC}$ and $\widehat{DF}$ has a smooth solution in a neighborhood of D.

In (ρ, η)-coordinates, we have that $\widehat{DC}$ is given by $\rho = f(\eta) + \eta^2/4$. Using the fact that $\widehat{DC}$ is λ_+ characteristics, we have

$$f' + \frac{\eta}{2} = \frac{d\rho}{d\eta} = \sqrt{\rho - u} = \sqrt{f + \frac{\eta^2}{4} - u},$$

implying

$$u = f + \frac{\eta^2}{4} - \big(f' + \frac{\eta}{2}\big)^2. \tag{3.16}$$

Hence, along $\widehat{DC}$

$$\partial_+ u = u_\eta + \sqrt{\rho - u}\, u_\rho = -2f''\big(f' + \frac{\eta}{2}\big). \tag{3.17}$$

We claim $f' + \frac{\eta}{2} > 0$ along $\widehat{DC}$. At the point C, given by (2.5), we have

$$f' + \frac{\eta}{2} = -\frac{2}{\eta_C} + \frac{\eta_C}{2} = \frac{1}{a - \sqrt{4a^2+2}} - (a - \sqrt{4a^2+2}) = \frac{-5a^2 - 1 + 2a\sqrt{4a^2+2}}{a - \sqrt{4a^2+2}}.$$

It is easy to show that both the numerator and the denominator in the above expression are negative, implying that the claim holds at the point C. Assume that there is a point $X \in \widehat{DC}$ such that $f' + \frac{\eta}{2} = 0$ at X. From (3.16) and definition of $\widehat{DC}$ in (ρ, η)-coordinates, we have that at the point X

$$u = f + \frac{\eta^2}{4} = \rho,$$

implying that X is sonic, which is false. Therefore, the above claim is proved and

$$\partial_+ u > 0 \quad \text{on } \widehat{DC}. \tag{3.18}$$

Similarly, λ_- characteristics $\widehat{DF}$ is given by $\rho = g(\eta) + \eta^2/4$, implying

$$g' + \frac{\eta}{2} = \frac{d\rho}{d\eta} = -\sqrt{\rho - u} = -\sqrt{g(\eta) + \frac{\eta^2}{4} - u}.$$

Hence,

$$u = g(\eta) + \frac{\eta^2}{4} - \big(g' + \frac{\eta}{2}\big)^2,$$

and along $\widehat{DF}$ we have

$$\partial_- u = u_\eta - \sqrt{\rho - u} = -2g''\big(g' + \frac{\eta}{2}\big).$$

We claim that $g' + \frac{\eta}{2} < 0$ on $\widehat{DF}$. Using that at the point D we have

$$f' + \frac{\eta}{2} = \sqrt{\rho - u} = -\big(g' + \frac{\eta}{2}\big),$$

and we proved $f' + \frac{\eta}{2} > 0$ at D, we obtain $g' + \frac{\eta}{2} < 0$ at D. Similarly, as above if there is a point $X \in \widehat{DF}$ such that $g' + \frac{\eta}{2} = 0$ at X, then X would be a sonic point. Therefore, the above claim is proved and

$$\partial_- u > 0 \quad \text{on } \widehat{DF}. \tag{3.19}$$

Using (3.6), as in §3.2, we have that ∂_u along $\widehat{DC}$ is found from a solution of the linear ordinary equation

$$\frac{d}{d\eta}\frac{1}{\partial_- u} - \frac{f''}{2f' + \eta}\frac{1}{\partial_- u} = \frac{1}{(2f' + \eta)^2},$$

with the initial condition

$$\frac{1}{\partial_- u}(D) = -\frac{1}{2g''(D)(g'(D) + \eta_D/2)},$$

and $\partial_+ u$ along $\widehat{DF}$ is found from the solution of the linear ordinary equation

$$\frac{d}{d\eta}\frac{1}{\partial_+ u} - \frac{g''}{2g' + \eta}\frac{1}{\partial_+ u} = \frac{1}{(2g' + \eta)^2}$$

with the initial condition

$$\frac{1}{\partial_+ u}(D) = -\frac{1}{2f''(D)(f'(D) + \eta_D/2)}.$$

We obtain analogous results to those in §3.2.

Theorem 3.7. *The Goursat problem* (3.7) *with W given on the characteristic boundaries $\widehat{DC}$ and $\widehat{DF}$ as above has a smooth local solution at the point D.*

Lemma 3.8. *Let u be a smooth solution of the Goursat problem* (3.7) *in the domain $DCFD$ with W given on characteristic boundaries $\widehat{DC}$ and $\widehat{DF}$ as above. Then*

(a) *$\partial_- u > 0$ and $\partial_+ u > 0$ in the interior of the domain $DCFD$,*
(b) *$\partial_- u$ is increasing (or decreasing) along λ_+ characteristics if*

$$\partial_- u > \partial_+ u \quad (or \quad \partial_- u < \partial_+ u),$$

(c) *$\partial_+ u$ is increasing (or decreasing) along λ_- characteristics if*

$$\partial_+ u > \partial_- u \quad (or \quad \partial_+ u < \partial_- u).$$

Remark 3.9. We remark that from part (a) of the previous Lemma we have that the minimum of u in the domain $DCFD$ is achieved at the point D.

Lemma 3.10. *Let u be a smooth solution of the Goursat problem* (3.7) *in the domain $DCFD$ with W given on characteristic boundaries $\widehat{DC}$ and $\widehat{DF}$ as above. Then Λ_+ characteristics are concave and Λ_- characteristics are convex.*

Proof. The proof is similar to the proofs of Lemmas 3.3 and 3.6. We recall from (3.3) that

$$\partial^{\pm} \Lambda_{\pm} = \frac{\Lambda_{\pm}^4}{\eta \Lambda_{\pm} + 2} \partial_{\pm} u.$$

Using the previous Lemma, it suffices to show that

$$\eta \Lambda_+ + 2 < 0 \quad \text{and} \quad \eta \Lambda_- + 2 > 0. \tag{3.20}$$

First recall from (3.15) that $\xi - u < 0$ on $\widehat{DC}$. If there is a point P inside the domain $DCFD$ such that $\xi - u = 0$ at P, then, by (2.4), we would have that Λ_+ characteristics at P is vertical, which is false. Hence $\xi - u < 0$ in the domain $DCFD$.

Next, we have

$$\eta \Lambda_{\pm} + 2 = \frac{\eta^2 + 4(\xi - u) \pm \eta \sqrt{\eta^2 + 4(\xi - u)}}{2(\xi - u)}.$$

Clearly, $\eta^2 + 4(\xi - u) + \eta \sqrt{\eta^2 + 4(\xi - u)} > 0$ in $DCFD \setminus \{C\}$, since the sum of the first two terms is positive because the points in $DCFD \setminus \{C\}$ are hyperbolic and the third term is positive since $\eta > 0$ in $DCFD$. Hence, (3.20) holds for Λ_+.

To show that $\eta^2 + 4(\xi - u) - \eta \sqrt{\eta^2 + 4(\xi - u)} < 0$ in $DCFD \setminus \{C\}$, we prove

$$\eta^2 + 4(\xi - u) < \eta \sqrt{\eta^2 + 4(\xi - u)}.$$

We note that the left-hand side is positive, and by squaring both sides and recalling that $\xi - u < 0$, we obtain that the inequality is satisfied. □

The reflected Λ_+ characteristics pass through $\widehat{DF}$ and continue as straight characteristics forming a simple wave in the region $DFED$. As in [10, §5], these characteristics form an envelope before their sonic points indicating existence of a shock.

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