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OSCILLATION AND PROPERTY B FOR THIRD-ORDER DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS

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Abstract. We establish sufficient conditions for the third-order nonlinear advanced differential equation

$$\Big(a(t)[\big(b(t)y'(t)\big)']^{\gamma}\Big)' - p(t)f(y(\sigma(t))) = 0$$

to have property B or to be oscillatory. These conditions are based on monotonic properties and estimates of non-oscillatory solutions, and essentially improve known results for differential equations with deviating arguments and for ordinary differential equations.

1. Introduction

We consider the nonlinear third-order differential equation with advanced argument

$$\Big(a(t)[\big(b(t)y'(t)\big)']^{\gamma}\Big)' - p(t)f(y(\sigma(t))) = 0. \tag{1.1}$$

In the sequel we will assume:

- (H0) γ is quotient of odd positive integers.
- (H1) $a(t), b(t), p(t) \in C([t_0,\infty))$, $\sigma(t) \in C^1([t_0,\infty))$, $a(t), b(t), p(t)$ are positive, $\sigma'(t) > 0$, $\sigma(t) \ge t$.
- (H2) $f(u) \in C(\mathbb{R})$, $uf(u) > 0$ for $u \ne 0$, $f(uv) \ge f(u)f(v)$ for $uv > 0$, f is nondecreasing.
- (H3) $\int_{t_0}^{\infty} \frac{1}{a^{1/\gamma}(t)}\mathrm{d}t = \infty$, $\int_{t_0}^{\infty} \frac{1}{b(t)}\mathrm{d}t = \infty$.

By a solution of (1.1), we mean a function $y(t) \in C^1([T_y,\infty))$, $T_y \ge t_0$, that satisfies (1.1) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (1.1) that satisfy $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise, it is called to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

The study of oscillatory properties of third and higher order linear ordinary differential equations began as far back in the pioneering work of Kneser [8].

A new impetus to investigations in this direction was given by the works of Chanturia and Kiguradze [7]. Their results concern property B for the linear differential

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equation

$$y'''(t) - q(t)y(\sigma(t)) = 0 \tag{1.2}$$

with $\sigma(t) \equiv t$. By property B of (1.2) it is meant the situation when every positive solution $y(t)$ of (1.2) is strongly increasing, i.e.

$$y'(t) > 0, \quad y''(t) > 0, \quad y'''(t) > 0.$$

Over the previous few decades, oscillation theory and asymptotic behavior of differential equations related to (1.1), have drawn extensive attention and the significant body of relevant literature has been devoted to this topic (see [1]–[12])

Especially, in the earlier article [9] Koplatadze et al presented excellent criteria for the qualitative properties of solutions of binomial differential equation with deviating argument. In this article, we extend their technique that yields property B of (1.2) to (1.1).

Here, we derive new monotonic properties of nonoscillatory solutions of (1.1) that permit us to achieve new sufficient conditions for (1.1) to have property B or to be oscillatory. Our results essentially improve many known results not only for differential equations with deviating arguments but for ordinary differential equations as well.

As in oscillation theory, all functional inequalities considered are assumed to hold eventually; that is, they are satisfied for all t large enough.

2. Preliminaries

We begin with the structure of possible nonoscillatory solutions of (1.1) which follows from an analogy of Kiguradze [7] lemma and canonical form of studied equation. We introduce the following classes of nonoscillatory (let us say positive) solutions of (1.1):

$$y(t) \in \mathscr{N}_1 \iff y'(t) > 0,\ (b(t)y'(t))' < 0,\ \Big(a(t)[(b(t)y'(t))']^\gamma\Big)' > 0$$

and

$$y(t) \in \mathscr{N}_3 \iff y'(t) > 0,\ (b(t)y'(t))' > 0,\ (a(t)[(b(t)y'(t))']^\gamma)' > 0,$$

eventually.

Lemma 2.1. *Assume that $y(t)$ is an eventually positive solution of* (1.1), *then $y(t) \in \mathscr{N}_1$ or $y(t) \in \mathscr{N}_3$.*

Now, we derive some important monotonic properties and estimates of nonoscillatory solutions, that will be applied in our main results.

To simplify our notation, let us denote

$$A(t) = \int_{t_*}^t \frac{1}{a^{1/\gamma}(s)}\,\mathrm{d}s, \quad B(t) = \int_{t_*}^t \frac{1}{b(s)}\,\mathrm{d}s,$$

$$C(t) = \int_{t_*}^t \frac{1}{b(u)} \int_{t_*}^u \frac{1}{a^{1/\gamma}(s)}\,\mathrm{d}s\mathrm{d}u, \quad P(t) = \frac{1}{a^{1/\gamma}(t)}[\int_t^\infty p(s)\,\mathrm{d}s]^{1/\gamma}.$$

for t_* is large enough.

Lemma 2.2. *Let $y(t) \in \mathscr{N}_3$ be a positive solution of* (1.1) *and*

$$\int_{t_*}^\infty p(s) f(C(\sigma(s)))\mathrm{d}s = \infty. \tag{2.1}$$

Then $y(t)/C(t)$ is eventually increasing.

Proof. Assume, that $y(t)$ is a positive solution of (1.1) satisfying $y(t) \in \mathscr{N}_3$ eventually, let us say for $t \geq t_*$. We claim that (2.1) implies

$$\lim_{t\to\infty} a^{1/\gamma}(t)(b(t)y'(t))' = \infty. \tag{2.2}$$

If not, then

$$\lim_{t\to\infty} a^{1/\gamma}(t)(b(t)y'(t))' = 2\ell > 0$$

and since $a^{1/\gamma}(t)(b(t)y'(t))'$ is increasing, we have

$$a^{1/\gamma}(t)(b(t)y'(t))' > \ell,$$

eventually. An integration of the last inequality leads to

$$b(t)y'(t) \geq \ell A(t),$$

which implies $y(t) \geq \ell C(t)$ or

$$f(y(\sigma(t))) \geq f(\ell)f(C(\sigma(t))). \tag{2.3}$$

On the other hand, integrating (1.1) from t_* to ∞, one gets

$$(2\ell)^\gamma \geq \int_{t_*}^{\infty} p(s)f(y(\sigma(s)))\,\mathrm{d}s,$$

which in view of (2.3) yields

$$(2\ell)^\gamma \geq f(\ell)\int_{t_*}^{\infty} p(s)f(C(\sigma(s)))\,\mathrm{d}s.$$

This contradicts (2.1) and we conclude that (2.2) holds.

Now, using that $a^{1/\gamma}(t)(b(t)y'(t))'$ is increasing, we see that for all $t \geq t_1 > t_*$,

$$\begin{aligned} b(t)y'(t) &= b(t_1)y'(t_1) + \int_{t_1}^{t} a^{1/\gamma}(s)\frac{(b(s)y'(s))'}{a^{1/\gamma}(s)}\,\mathrm{d}s \\ &\leq b(t_1)y'(t_1) + a^{1/\gamma}(t)(b(t)y'(t))'\int_{t_1}^{t}\frac{1}{a^{1/\gamma}(s)}\,\mathrm{d}s \\ &= b(t_1)y'(t_1) - a^{1/\gamma}(t)(b(t)y'(t))'\int_{t_*}^{t_1}\frac{1}{a^{1/\gamma}(s)}\,\mathrm{d}s \\ &\quad + a^{1/\gamma}(t)(b(t)y'(t))'\int_{t_*}^{t}\frac{1}{a^{1/\gamma}(s)}\,\mathrm{d}s. \end{aligned}$$

By (2.2), this implies

$$b(t)y'(t) \leq a^{1/\gamma}(t)(b(t)y'(t))'\int_{t_*}^{t}\frac{1}{a^{1/\gamma}(s)}\,\mathrm{d}s$$

for all t large enough, let us say $t \geq t_2 > t_1$, and therefore

$$\Big(\frac{b(t)y'(t)}{A(t)}\Big)' = \frac{(b(t)y'(t))'A(t) - b(t)y'(t)\frac{1}{a^{1/\gamma}(t)}}{A^2(t)} \geq 0.$$

Thus, $\frac{b(t)y'(t)}{A(t)}$ is increasing for $t \ge t_2 > t_*$. Then this fact yields

$$\begin{aligned} y(t) &= y(t_2) + \int_{t_2}^{t} \frac{A(u)b(u)y'(u)}{b(u)A(u)} \mathrm{d}u \\ &\le y(t_2) + \frac{b(t)y'(t)}{A(t)} \int_{t_2}^{t} \frac{A(u)}{b(u)}\,\mathrm{d}u \\ &= y(t_2) - \frac{b(t)y'(t)}{A(t)} \int_{t_*}^{t_2} \frac{A(u)}{b(u)}\,\mathrm{d}u + \frac{b(t)y'(t)}{A(t)} \int_{t_*}^{t} \frac{A(u)}{b(u)}\,\mathrm{d}u. \end{aligned} \tag{2.4}$$

On the other hand, by L'Hospital rule

$$\lim_{t\to\infty} \frac{b(t)y'(t)}{A(t)} = \lim_{t\to\infty} a^{1/\gamma}(t)(b(t)y'(t))' = \infty$$

and so in view of (2.4), there exists $t_3 > t_2$ such that

$$y(t) \le \frac{b(t)y'(t)}{A(t)} \int_{t_*}^{t} \frac{A(u)}{b(u)}\,\mathrm{d}u, \quad t \ge t_3.$$

Consequently,

$$\Big(\frac{y(t)}{C(t)}\Big)' = \frac{y'(t)C(t) - y(t)A(t)\frac{1}{b(t)}}{C^2(t)} \ge 0,$$

which implies that $y(t)/C(t)$ is eventually increasing. The proof is complete. □

Lemma 2.3. *Let $y(t) \in \mathscr{N}_1$ be a positive solution of* (1.1)*. Then $y(t)/B(t)$ is eventually decreasing.*

Proof. Assume, that $y(t)$ is an eventually positive solution of (1.1) satisfying $y(t) \in \mathscr{N}_1$ for $t \ge t_*$. Then $b(t)y'(t)$ is decreasing and we see that

$$y(t) \ge \int_{t_*}^{t} b(s)y'(s)\frac{1}{b(s)}\,\mathrm{d}s \ge b(t)y'(t) \int_{t_*}^{t} \frac{1}{b(s)}\,\mathrm{d}s.$$

This implies

$$\Big(\frac{y(t)}{B(t)}\Big)' = \frac{y'(t)B(t) - y'(t)\frac{1}{b(t)}}{A^2(t)} \le 0, \quad t \ge t_*.$$

Thus, $y(t)/B(t)$ is eventually decreasing and the proof is complete. □

Remark 2.4. For $a(t) = b(t) \equiv 1$ and $\gamma = 1$ Lemmas 2.2 and 2.3 reduce to the results by Koplatadze et al. So we extended their result from linear differential equations to nonlinear equations with the extra factor b.

3. Criteria for property B

Now, we provide several criteria for the class $\mathscr{N}_1$ of (1.1) to be empty. In the literature such case is referred to as *property B* of (1.1).

Theorem 3.1. *Assume that*

$$\int_{t_*}^{\infty} \frac{1}{b(v)} \int_{v}^{\infty} \frac{1}{a^{1/\gamma}(u)} \Big[\int_{u}^{\infty} p(s)\,\mathrm{d}s\mathrm{d}u\Big]^{1/\gamma} \mathrm{d}v = \infty, \tag{3.1}$$

and

$$\lim_{u\to\pm\infty} \frac{u}{f^{1/\gamma}(u)} = K_1 < \infty. \tag{3.2}$$

If

$$\limsup_{t\to\infty}\Big\{f^{1/\gamma}(\frac{1}{B(\sigma(t))})\int_{t_*}^{t}f^{1/\gamma}(B(\sigma(s)))B(s)P(s)\,\mathrm{d}s + \int_t^{\sigma(t)}B(s)P(s)\,\mathrm{d}s+B(\sigma(t))\int_{\sigma(t)}^{\infty}P(s)\,\mathrm{d}s\Big\}>K_1,$$

then (1.1) *has property B.*

Proof. Assume on the contrary, that (1.1) possesses an eventually positive solution $y(t)\in\mathscr{N}_1$, $t\ge t_*$. Integration (1.1) twice from t to ∞ yields

$$\begin{aligned} b(t)y'(t) &\ge \int_{t_*}^{\infty}\frac{1}{a^{1/\gamma}(s)}\Big[\int_s^{\infty}p(x)f(y(\sigma(x)))\,\mathrm{d}x\Big]^{1/\gamma}\mathrm{d}s\\ &\ge \int_{t_*}^{\infty}f^{1/\gamma}(y(\sigma(s)))\frac{1}{a^{1/\gamma}(s)}\Big[\int_s^{\infty}p(x)\,\mathrm{d}x\Big]^{1/\gamma}\mathrm{d}s\\ &= \int_{t_*}^{\infty}f^{1/\gamma}(y(\sigma(s)))P(s)\,\mathrm{d}s, \end{aligned}$$

where we have used the monotonicity of $f(y(\sigma(t)))$. Integrating the last inequality from t_* to t and then changing the order of integration, one obtains

$$\begin{aligned} y(t) &\ge \int_{t_*}^{t}\frac{1}{b(u)}\int_u^{\infty}f^{1/\gamma}(y(\sigma(s)))P(s)\,\mathrm{d}s\mathrm{d}u\\ &= \int_{t_*}^{t}f^{1/\gamma}(y(\sigma(s)))P(u)B(u)\,\mathrm{d}u+B(t)\int_t^{\infty}f^{1/\gamma}(y(\sigma(s)))P(s)\,\mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} y(\sigma(t)) &\ge \int_{t_*}^{t}f^{1/\gamma}(y(\sigma(s)))P(u)B(u)\,\mathrm{d}u\\ &\quad+\int_t^{\sigma(t)}f^{1/\gamma}(y(\sigma(s)))P(u)B(u)\,\mathrm{d}u+B(\sigma(t))\int_{\sigma(t)}^{\infty}f^{1/\gamma}(y(\sigma(s)))P(s)\,\mathrm{d}s. \end{aligned}$$

Using that $y(t)$ is increasing and $y(t)/B(t)$ is decreasing, we have

$$\begin{aligned} y(\sigma(t)) &\ge f^{1/\gamma}\Big(\frac{y(\sigma(t))}{B(\sigma(t))}\Big)\int_{t_*}^{t}f^{1/\gamma}(B(\sigma(s)))P(u)B(u)\,\mathrm{d}u\\ &\quad+f^{1/\gamma}(y(\sigma(t)))\int_t^{\sigma(t)}P(u)B(u)\,\mathrm{d}u\\ &\quad+f^{1/\gamma}(y(\sigma(t)))B(\sigma(t))\int_{\sigma(t)}^{\infty}P(s)\,\mathrm{d}s. \end{aligned}\tag{3.3}$$

That is,

$$\begin{aligned} \frac{y(\sigma(t))}{f^{1/\gamma}(y(\sigma(t)))} &\ge f^{1/\gamma}(\frac{1}{B(\sigma(t))})\int_{t_*}^{t}f^{1/\gamma}(B(\sigma(s)))P(u)B(u)\,\mathrm{d}u\\ &\quad+\int_t^{\sigma(t)}P(u)B(u)\,\mathrm{d}u+B(\sigma(t))\int_{\sigma(t)}^{\infty}P(s)\,\mathrm{d}s. \end{aligned}$$

It follows from (3.1) that $y(t)\to\infty$ as $t\to\infty$. Taking $\limsup$ as $t\to\infty$ on both sides of the previous inequality, we are led to a contradiction with the assumptions of the theorem. The proof is complete. □

Theorem 3.2. *Assume that*

$$\int_{t_*}^{\infty} \frac{1}{a^{1/\gamma}(u)} \Big[\int_u^{\infty} p(s) f(B(\sigma(s)))\, ds\Big]^{1/\gamma} du = \infty, \tag{3.4}$$

$$\lim_{u\to 0} \frac{u}{f^{1/\gamma}(u)} = K_2 < \infty. \tag{3.5}$$

If

$$\begin{aligned} \limsup_{t\to\infty} \Big\{ &\frac{1}{B(\sigma(t))} \int_{t_*}^{t} f^{1/\gamma}(B(\sigma(s))) B(s) P(s)\, ds \\ &+ \frac{f^{1/\gamma}(B(\sigma(t)))}{B(\sigma(t))} \int_t^{\sigma(t)} B(s)P(s)\, ds + f^{1/\gamma}(B(\sigma(t))) \int_{\sigma(t)}^{\infty} P(s)\, ds \Big\} > K_2, \end{aligned}$$

then (1.1) *has property B.*

Proof. Assume that (1.1) possesses an eventually positive solution $y(t) \in \mathcal{N}_1$, $t \geq t_*$. By Lemma 2.3, function $y(t)/B(t)$ is decreasing and we shall prove that (3.4) implies

$$\lim_{t\to\infty} \frac{y(t)}{B(t)} = 0. \tag{3.6}$$

On the contrary assume that $\lim_{t\to\infty} y(t)/B(t) = \ell > 0$. Then $y(t)/B(t) \geq \ell$; therefore

$$f(y(\sigma(t))) = f\Big(\frac{y(\sigma(t))}{B(\sigma(t))} B(\sigma(t))\Big) \geq f(\ell) f(B(\sigma(t))).$$

Moreover, integrating (1.1) twice yields

$$\begin{aligned} b(t_*) y'(t_*) &\geq \int_{t_*}^{\infty} \frac{1}{a^{1/\gamma}(u)} \Big[\int_u^{\infty} p(s) f(y(\sigma(s)))\, ds\Big]^{1/\gamma} du \\ &\geq f(\ell) \int_{t_*}^{\infty} \frac{1}{a^{1/\gamma}(u)} \Big[\int_u^{\infty} p(s) f(B(\sigma(s)))\, ds\Big]^{1/\gamma} du. \end{aligned} \tag{3.7}$$

This contradict the assumptions of the theorem; we conclude that (3.6) holds.

On the other hand, setting

$$z(t) = \frac{y(\sigma(t))}{B(\sigma(t))},$$

condition (3.3) and (H2) imply

$$\begin{aligned} \frac{z(t)}{f^{1/\gamma}(z(t))} \geq &\frac{1}{B(\sigma(t))} \int_{t_*}^{t} f^{1/\gamma}(B(\sigma(s))) P(u) B(u)\, du \\ &+ \frac{f^{1/\gamma}(B(\sigma(t)))}{B(\sigma(t))} \int_t^{\sigma(t)} P(u)B(u)\, du + f^{1/\gamma}(B(\sigma(t))) \int_{\sigma(t)}^{\infty} P(s)\, ds. \end{aligned}$$

Taking the lim sup as $t \to \infty$ on both sides of the previous inequality, we have a contradiction with the assumptions of our theorem. The proof is complete. □

Now we apply the criteria obtained to superlinear, sublinear and half-linear cases of (1.1), where δ is quotient of odd positive integers.

Corollary 3.3. *Let* (3.1) *hold and*

$$\limsup_{t\to\infty} \Big\{ B^{-\delta/\gamma}(\sigma(t)) \int_{t_*}^{t} B^{\delta/\gamma}(\sigma(s)) B(s) P(s)\, ds$$

$$+\int_t^{\sigma(t)} B(s)P(s)\,\mathrm{d}s + B(\sigma(t))\int_{\sigma(t)}^{\infty} P(s)\,\mathrm{d}s\Big\} > 0,$$

then the superlinear differential equation

$$\big[a(t)\big(b(t)(y'(t))^{\gamma}\big)'\big]' - p(t)y^{\delta}(\sigma(t)) = 0, \quad \delta > \gamma.$$

has property B.

Corollary 3.4. *Let* (3.1) *hold and*

$$\begin{aligned}\limsup_{t\to\infty}\Big\{B^{-1}(\sigma(t))\int_{t_*}^{t} B(\sigma(s))B(s)P(s)\,\mathrm{d}s\\ +\int_t^{\sigma(t)} B(s)P(s)\,\mathrm{d}s + B(\sigma(t))\int_{\sigma(t)}^{\infty} P(s)\,\mathrm{d}s\Big\} > 1,\end{aligned}$$

then the halflinear differential equation

$$[a(t)(b(t)(y'(t))^{\gamma})']' - p(t)y^{\gamma}(\sigma(t)) = 0. \tag{3.8}$$

has property B.

Corollary 3.5. *Let* (3.4) *hold. If*

$$\begin{aligned}\limsup_{t\to\infty}\Big\{\frac{1}{B(\sigma(t))}\int_{t_*}^{t} B^{\delta/\gamma}(\sigma(s))B(s)P(s)\,\mathrm{d}s\\ +\frac{B^{\delta/\gamma}(\sigma(t))}{B(\sigma(t))}\int_t^{\sigma(t)} B(s)P(s)\,\mathrm{d}s + B^{\delta/\gamma}(\sigma(t))\int_{\sigma(t)}^{\infty} P(s)\,\mathrm{d}s\Big\} > 0,\end{aligned}$$

then the sublinear differential equation

$$\big[a(t)\big(b(t)(y'(t))^{\gamma}\big)'\big]' - p(t)y^{\delta}(\sigma(t)) = 0, \quad \gamma > \delta. \tag{3.9}$$

has property B.

Note that corollaries 3.3–3.5 essentially improve the results known for (1.2).

4. Oscillation

Our previous results concern property B of (1.1). To achieve oscillation, we need to eliminate also the class $\mathcal{N}_3$.

Theorem 4.1. *Let the assumptions of* (2.1) *hold. Assume that*

$$\lim_{u\to\pm\infty}\frac{u}{f^{1/\gamma}(u)} = K_3 < \infty. \tag{4.1}$$

If

$$\limsup_{t\to\infty}\frac{1}{C(\sigma(t))}\int_t^{\sigma(t)}\frac{1}{b(v)}\int_v^t\frac{1}{a^{1/\gamma}(u)}\Big[\int_u^t p(s)f(C(\sigma(s)))\,\mathrm{d}s\mathrm{d}u\Big]^{1/\gamma}\mathrm{d}v > K_3,$$

then the class $\mathcal{N}_3 = \emptyset$ *for* (1.1).

Proof. Assume that (1.1) possesses an eventually positive solution $y(t) \in \mathcal{N}_3$, $t \ge t_*$. An integration of (1.1) from s to $t < s$ yields

$$\begin{aligned}\big[(b(s)y'(s))'\big]^{\gamma} &\ge \frac{1}{a(s)}\int_t^s p(x)f\Big(\frac{y(\sigma(x))}{C(\sigma(x))}\,C(\sigma(x))\Big)\,\mathrm{d}x\\ &\ge f\Big(\frac{y(\sigma(t))}{C(\sigma(t))}\Big)\frac{1}{a(s)}\int_t^s p(x)\,\mathrm{d}x.\end{aligned}$$

Integrating in s, we have

$$y'(s) \geq f^{1/\gamma}\Big(\frac{y(\sigma(t))}{C(\sigma(t))}\Big)\frac{1}{b(s)}\int_t^s \frac{1}{a^{1/\gamma}(u)}\Big[\int_t^u p(x)\,\mathrm{d}x\Big]^{1/\gamma}\mathrm{d}u.$$

Integrating once more, we obtain

$$y(s) \geq f^{1/\gamma}\Big(\frac{y(\sigma(t))}{C(\sigma(t))}\Big)\int_t^s \frac{1}{b(v)}\int_v^t \frac{1}{a^{1/\gamma}(u)}\Big[\int_t^u p(s)\,\mathrm{d}s\Big]^{1/\gamma}\mathrm{d}u\mathrm{d}v.$$

Setting $s = \sigma(t)$ and $z(t) = y(\sigma(t))/C(\sigma(t))$, we obtain

$$\frac{z(t)}{f^{1/\gamma}(z(t))} \geq \frac{1}{C(\sigma(t))}\int_{\sigma(t)}^t \frac{1}{b(v)}\int_v^t \frac{1}{a^{1/\gamma}(u)}\Big[\int_u^t p(s)\,\mathrm{d}s\mathrm{d}u\Big]^{1/\gamma}\mathrm{d}v.$$

Taking $\limsup$ as $t \to \infty$ on both sides of the previous inequality, we are led to a contradiction with the assumption of the theorem. The proof is complete. $\square$

Combining the criteria obtained for both classes $\mathscr{N}_1$ and $\mathscr{N}_3$ to be empty, we obtain results for oscillation of (1.1).

Theorem 4.2. *Let all conditions of Theorem 3.1 (Theorem 3.2) and Theorem 4.1 hold. Then* (1.1) *is oscillatory.*

5. Examples

We support the results obtained above with the following illustrative example.

Example 5.1. We consider the third-order advanced differential equation

$$\Big(t^{1/4}\big[\big(t^{1/3}y'(t)\big)'\big]^{1/3}\Big)' - \frac{a}{t^{47/36}}y^{1/3}(\lambda t) = 0,$$

where $a > 0$ and $\lambda > 1$. Simple computation shows that

$$A(t) \sim 4t^{1/4}, \quad B(t) \sim \frac{3t^{2/3}}{2}, \quad C(t) \sim \frac{48t^{11/12}}{11}.$$

By Corollary 3.4, condition

$$\frac{3}{2}(\frac{36}{11})^3 a^3(3 + \ln\lambda) > 1, \tag{5.1}$$

guarantees property B of (5.1).

On the other hand, by Theorem 4.2, condition

$$\begin{aligned}&\frac{3}{2}a^3\lambda^{11/12}[(4 - \frac{12}{11})\ln^3\lambda - (3.4^2 - \frac{3.12^2}{11^2})\ln^2\lambda \\ &+ (3.4^3 - \frac{6.12^3}{11^3})\ln\lambda - 6.4^4(1 - \frac{1}{\lambda^{1/4}}) + \frac{6.12^4}{11^4}(1 - \frac{1}{\lambda^{11/12}})] > 1,\end{aligned} \tag{5.2}$$

guarantees that $\mathscr{N}_3 = \emptyset$ for (5.1), By Theorem 4.2, Equation (5.1) is oscillatory if both conditions (5.1) and (5.2) are satisfied.

Thus, in particular when $\lambda = 2$,

$$\begin{aligned}a > 0.17269 &\Rightarrow \text{ property B of (5.1)},\\ a > 4.2262 &\Rightarrow \text{ oscillation of (5.1)}.\end{aligned}$$

Note that Koplatadze' criteria cannot be used nor for examination of property B nor for oscillation of (5.1).

Although our results are oriented for advanced differential equations, Corollaries 1–3 improve Chanturia's tests [11] for property B of ordinary differential equation without deviating argument which read as follows: If

$$\limsup_{t\to\infty} t \int_t^\infty sp(s)\,\mathrm{d}s > 2,$$

then (1.2) has property B. Note that for the Euler equation

$$y'''(t) - \frac{p}{t^3} y(t) = 0$$

Chanturia's criterion for property B requires $p > 2$, while Corollary 3.3 requires only $p > 1$. On the other hand, our results are applicable also for advanced differential equations and for property B of

$$y'''(t) - \frac{p}{t^3} y(\lambda t) = 0, \quad \lambda > 1,$$

Corollary 3.3 requires $2p + p \ln \lambda > 2$.

Summary. The results obtained are of high generality and improve earlier results known for special cases of (1.1). Moreover the monotonic properties of solutions presented in Lemmas 2.2 and 2.3 can be applied in various techniques (comparison principles, Riccati transformation, integral averaging technique, etc.) used in the theory of oscillation.

References

[1] R. Agarwal, M. F. Aktaş, A. Tiryaki; *On oscillation criteria for third order nonlinear advanced differential equations*, Arch. Math. 45 (2009), No. 4, 1–18.

[2] M. F. Aktaş, A. Tiryaki, A. Zafer; *Oscillation criteria for third-order nonlinear functional differential equations*, Applied Mathematics Letters 7 (2010), 756–762.

[3] B. Baculíková, J. Džurina; *Comparison theorems for the third-order advanced trinomial differential equations*, Advances in Difference Equations, 2010 (2010), 1–9.

[4] Dzurina J. Kotorova R.; *Zero points of the solutions of a differential equation*, Acta Electrotechnica et Informatica. 7 (2007), 26-29.

[5] M. Cecchi, Z. Došlá, M. Marini; *On third order differential equations with property A and B*, Journal of Mathematical Analysis and Applications 231 (1999), 509–525.

[6] J. Graef, S. H. Saker; *Oscillation theory of third-order nonlinear functional differential equations*, Hiroshima Mathematical Journal, 43 (2013), 49–72.

[7] I. T. Kiguradze, T. A. Chaturia; *Asymptotic Properties of Solutions of Nonatunomous Ordinary Differential Equations*, Kluwer Acad. Publ., Dordrecht 1993.

[8] A. Kneser; *Untersuchen uber die reelen Nullstellen der Integrale lineare Differentialgleichungen*, Math. Ann. 42(1893), 409435.

[9] R. Koplatadze, G. Kvinkadze, I. P. Stavroulakis; *Properties A and B of n-th order linear differential equations with deviating argument* Gorgian Math. J. 6(1999), 553-566

[10] G. S. Ladde, V. Lakshmikantham, B. G. Zhang; *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987. Zbl 0832.34071

[11] H. Liang; *Asymptotic behavior of solutions to higher order nonlinear advanced differential equations,* Electron J. differential Equ., 2014 (2014) no. 186, 1–12.

[12] A. Tiryaki, M. F. Aktaş; *Oscillation criteria of a certain class ot third order nonlinear advanced differential equations with damping*, J. Math. Anal. Appl. 325(2007), 54–68. Zbl 1110.34048

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