Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 243, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LYAPUNOV-TYPE INEQUALITIES FOR ODD ORDER LINEAR DIFFERENTIAL EQUATIONS

SOUGATA DHAR, QINGKAI KONG

Abstract. In this article, we obtain Lyapunov-type inequalities for certain odd order linear boundary-value problems. Our inequalities involve integrals of both $q_+(t)$ and $q_-(t)$ in addition to that of $|q(t)|$. The Green's function for even order boundary-value problems plays a key role in our proofs. Also, using the Fredholm alternative theorem, we obtain a criterion for the existence and uniqueness of solutions to the corresponding nonhomogeneous linear boundary-value problems.

1. Introduction

For the second-order linear differential equation

$$x'' + q(t)x = 0 \tag{1.1}$$

with $q \in C([a,b],\mathbb{R})$, the following result is known as the Lyapunov inequality, see [3, 13].

Theorem 1.1. *Assume* (1.1) *has a solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Then*

$$\int_a^b |q(t)|dt > \frac{4}{b-a}. \tag{1.2}$$

It was first noticed by Wintner [21] and later by several other authors that inequality (1.2) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{0, q(t)\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}. \tag{1.3}$$

An extension of (1.3), due to Hartman [11, Chapter XI], to the more general equation

$$(r(t)x')' + q(t)x = 0 \tag{1.4}$$

with $q, r \in C([a,b],\mathbb{R})$ and $r(t) > 0$ for $t \in [a,b]$, is as follows.

2010 *Mathematics Subject Classification.* 34A40, 34B05, 34C11, 26D20.
Key words and phrases. Lyapunov-type inequalities; odd order; linear differential equations; Green's function; boundary value problem.

Submitted July 7, 2015. Published September 7, 2016.

Theorem 1.2. *Assume* (1.4) *has a solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a, b)$*. Then*

$$\int_a^b q_+(t)dt > \frac{4}{\int_a^b r^{-1}(t)dt}. \tag{1.5}$$

The above Lyapunov inequalities have been improved by replacing $\int_a^b q_+(t)dt$ by some integrals of $q(t)$ on parts of or the whole interval $[a, b]$, see Harris and Kong [10], Brown and Hinton [2] for the details.

Lyapunov-type inequalities have been further developed for higher order linear and half-linear differential equations by many authors. The reader is referred to Cakmak [4, 5], He and Tang [12], Pachpatte [14, 15], Parhi and Panigrahi [17, 18], Panigrahi [16], Tiryaki, Unal and Cakmak [20], Yang [23], Yang and Lo [22], and Zhang and He [24] for the higher order linear case. Also, Pinasco [19] provided an excellent survey on various Lyapunov-type inequalities.

Among the above, Parhi and Panigrahi [17] established the Lyapunov-type inequalities for the third-order linear differential equation

$$x''' + q(t)x = 0 \tag{1.6}$$

with $-\infty < a < b < c < \infty$ and $q \in C([a, c], \mathbb{R})$.

Theorem 1.3. *Assume* (1.6) *has a solution* $x(t)$ *satisfying* $x(a) = x(b) = x(c) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a, b) \cup (b, c)$*. Then*

$$\int_a^c |q(t)|dt > \frac{4}{(c-a)^2}. \tag{1.7}$$

Recently, Dhar and Kong [7] obtained Lyapunov-type inequalities for third-order half-linear differential equations. Restricted to the linear equation (1.6), inequality (1.7) becomes

$$\max_{\xi \in [a,c]} \Big\{ \int_a^\xi q_-(s)ds + \int_\xi^c q_+(s)ds \Big\} > \frac{4}{(c-a)^2}. \tag{1.8}$$

Clearly, (1.8) improves (1.7) by replacing $|q(t)|$ in the integral of the left-hand side by $q_-(t)$ and $q_+(t)$, the negative and positive pars of q, respectively. In a different direction, the constant 4 on the right-hand side of (1.7) has been improved based on the Green's function for a corresponding second-order Dirichilet problem. In particular, motivated by the approach in Aktas, Cakmak, and Tiryaki [1], Dhar and Kong [8] obtained the following result.

Theorem 1.4. *Assume* (1.6) *has a solution* $x(t)$ *satisfying* $x(a) = x(b) = x(c) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a, b) \cup (b, c)$*. Then one of the following holds:*

(a) $\int_a^c q_-(t)dt > \frac{8}{(c-a)^2}$,
(b) $\int_a^c q_+(t)dt > \frac{8}{(c-a)^2}$,
(c) $\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{8}{(c-a)^2}$.

As a result,

$$\int_a^c |q(t)|dt > \frac{8}{(c-a)^2}.$$

In this article, we use some ideas from [1] and [8] for third-order equations to derive Lyapunov-type inequalities for odd order equations. More specifically, we use the Green's function for even order linear boundary value problems (BVPs) to obtain Lyapunov-type inequalities for certain types of BVPs associated with odd order linear equations. Furthermore, by using the Fredholm alternative theorem, we obtain a criterion for the existence and uniqueness of solutions to nonhomogeneous linear boundary value problems of odd order.

2. Main results

We let $-\infty < a < b < \infty$ and consider the odd order linear differential equation

$$x^{(2n+1)} + (-1)^{n-1} q(t) x = 0 \tag{2.1}$$

with $n \in \mathbb{N}$ and $q \in C([a,b],\mathbb{R})$. To simplify the notation, in the following, we denote

$$S_n = \sum_{j=0}^{n-1} \sum_{k=0}^{j} 2^{2k-2j} \binom{n-1+j}{j} \binom{j}{k} B(n+1, n+k-j), \tag{2.2}$$

where $B(\alpha,\beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} dz$ is the Beta function for $\alpha, \beta > 0$.

Theorem 2.1. *Assume* (2.1) *has a nontrivial solution* $x(t)$ *satisfying*

$$x^{(i+1)}(a) = x^{(i+1)}(b) = 0, \quad i = 0,1,\dots,n-1 \tag{2.3}$$

and $x(c) = 0$ *for* $c \in [a,b]$. *Then*

$$\int_a^b |q(t)| dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}. \tag{2.4}$$

Proof. As shown in [6], the Green's function for the BVP

$$\begin{gathered} y^{(2n)} + (-1)^{n-1} h(t) = 0, \\ y^{(i)}(a) = y^{(i)}(b) = 0, \quad i = 0,1,\dots,n-1 \end{gathered} \tag{2.5}$$

is

$$G(t,s) = \begin{cases} \frac{1}{(2n-1)!}\Big(\frac{(t-a)(b-s)}{b-a}\Big)^n \sum_{j=0}^{n-1} \binom{n+j-1}{j} (s-t)^{n-j-1} \Big(\frac{(b-t)(s-a)}{b-a}\Big)^j, \\ \quad a \le t \le s \le b; \\ \frac{1}{(2n-1)!}\Big(\frac{(s-a)(b-t)}{b-a}\Big)^n \sum_{j=0}^{n-1} \binom{n+j-1}{j} (t-s)^{n-j-1} \Big(\frac{(t-a)(b-s)}{b-a}\Big)^j, \\ \quad a \le s \le t \le b. \end{cases} \tag{2.6}$$

Hence the solution $y(t)$ of BVP (2.5) satisfies

$$y(t) = \int_a^b G(t,s) h(s) ds. \tag{2.7}$$

We note that for the solution $x(t)$ of (2.1), $y(t) := x'(t)$ satisfies (2.5) with $h(t) = q(t)x(t)$. By (2.7)

$$x'(t) = \int_a^b G(t,s) q(s) x(s) ds. \tag{2.8}$$

Integrating (2.8) from c to t and noting that $x(c) = 0$, we have

$$x(t) = \int_c^t \int_a^b G(\tau,s) q(s) x(s) ds d\tau = \int_a^b \Big(\int_c^t G(\tau,s) d\tau \Big) q(s) x(s) ds. \tag{2.9}$$

It is easy to see that $G(t,s)\geq 0$ on $[a,b]\times[a,b]$. It follows that

$$|x(t)| = \Big|\int_a^b \Big(\int_c^t G(\tau,s)d\tau\Big)q(s)x(s)ds\Big| \leq \int_a^b \Big(\int_a^b G(\tau,s)d\tau\Big)|q(s)||x(s)|ds. \tag{2.10}$$

We first show that for $s\in[a,b]$

$$\int_a^b G(\tau,s)d\tau \leq \frac{(b-a)^{2n}S_n}{2^{2n}(2n-1)!}\,, \tag{2.11}$$

where S_n is defined in (2.2). In fact,

$$\int_a^b G(\tau,s)d\tau = \int_a^s G(\tau,s)d\tau + \int_s^b G(\tau,s)d\tau. \tag{2.12}$$

We consider each integral separately. To ease the notation, we denote

$$A_j(s) = \frac{1}{(2n-1)!}\binom{n+j-1}{j}\frac{(b-s)^n(s-a)^j}{(b-a)^{n+j}}. \tag{2.13}$$

Then from (2.6),

$$\begin{aligned}\int_a^s G(\tau,s)d\tau &= \int_a^s (\tau-a)^n\sum_{j=0}^{n-1}A_j(s)(s-\tau)^{n-j-1}(b-\tau)^j d\tau\\ &= \sum_{j=0}^{n-1}A_j(s)\int_a^s(\tau-a)^n(s-\tau)^{n-j-1}(b-\tau)^j d\tau.\end{aligned} \tag{2.14}$$

We write

$$(b-\tau)^j = (b-s+s-\tau)^j = \sum_{k=0}^{j}\binom{j}{k}(b-s)^{j-k}(s-\tau)^k.$$

Substituting it into (2.14), we obtain

$$\begin{aligned}\int_a^s G(\tau,s)d\tau &= \sum_{j=0}^{n-1}A_j(s)\int_a^s(\tau-a)^n(s-\tau)^{n-j-1}\sum_{k=0}^{j}\binom{j}{k}(b-s)^{j-k}(s-\tau)^k d\tau\\ &= \sum_{j=0}^{n-1}A_j(s)\sum_{k=0}^{j}\binom{j}{k}(b-s)^{j-k}\int_a^s(\tau-a)^n(s-\tau)^{n-j+k-1}d\tau.\end{aligned} \tag{2.15}$$

To evaluate the integral in (2.15), we use the transformation $u=(\tau-a)/(s-a)$ which implies $1-u=(s-\tau)/(s-a)$. Hence

$$\begin{aligned}\int_a^s(\tau-a)^n(s-\tau)^{n-j+k-1}d\tau &= (s-a)^{2n-j+k}\int_0^1 u^n(1-u)^{n-j+k-1}du\\ &= (s-a)^{2n-j+k}B(n+1,n-j+k).\end{aligned}$$

Then by (2.15),

$$\int_a^s G(\tau,s)d\tau = \sum_{j=0}^{n-1}A_j(s)\sum_{k=0}^{j}\binom{j}{k}(b-s)^{j-k}(s-a)^{2n-j+k}B(n+1,n-j+k). \tag{2.16}$$

Using the expression for $A_j(s)$ in (2.16) and rearranging terms we obtain

$$\begin{aligned}\int_a^s G(\tau,s)d\tau = \frac{1}{(2n-1)!(b-a)^n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n+j-1}{j}\binom{j}{k}B(n+1,n-j+k)\\ \times\frac{(b-s)^{n+j-k}(s-a)^{2n+k}}{(b-a)^j}.\end{aligned} \tag{2.17}$$

Using the same technique, we also have

$$\begin{aligned}\int_s^b G(\tau,s)d\tau = \frac{1}{(2n-1)!(b-a)^n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n+j-1}{j}\binom{j}{k}B(n+1,n-j+k)\\ \times\frac{(b-s)^{2n+k}(s-a)^{n+j-k}}{(b-a)^j}.\end{aligned} \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.12) we obtain

$$\begin{aligned}\int_a^b G(\tau,s)d\tau = \frac{1}{(2n-1)!(b-a)^n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n+j-1}{j}\binom{j}{k}B(n+1,n-j+k)\\ \times\Big\{\frac{(b-s)^{n+j-k}(s-a)^{2n+k}}{(b-a)^j}+\frac{(b-s)^{2n+k}(s-a)^{n+j-k}}{(b-a)^j}\Big\}.\end{aligned} \tag{2.19}$$

Note that $\alpha\beta \le (\alpha+\beta)^2/4$ and $\alpha^l+\beta^l \le (\alpha+\beta)^l$ for $\alpha,\beta>0$ and $l\in\mathbb{N}$. Letting $\alpha=b-s$, $\beta=s-a$ and $l=n-j+2k$ we have

$$\begin{aligned}&\frac{(b-s)^{n+j-k}(s-a)^{2n+k}}{(b-a)^j}+\frac{(b-s)^{2n+k}(s-a)^{n+j-k}}{(b-a)^j}\\ &=\frac{(b-s)^{n+j-k}(s-a)^{n+j-k}}{(b-a)^j}\Big[(s-a)^{n-j+2k}+(b-s)^{n-j+2k}\Big]\\ &\le\frac{(b-a)^{2n+2j-2k}}{2^{2n+2j-2k}(b-a)^j}(b-a)^{n-j+2k}\\ &=\frac{(b-a)^{3n}}{2^{2n+2j-2k}}.\end{aligned}$$

Then (2.11) follows from (2.19).

We then show that (2.4) holds. Define $m:=\max\{|x(t)|: t\in[a,b]\}$. Then taking maximum of $|x(t)|$ in (2.10) and using the fact that $x(t)\not\equiv m$ on $[a,b]$, we have

$$m < m\int_a^b\Big(\int_a^b G(\tau,s)d\tau\Big)|q(s)|ds.$$

Canceling m from both sides and using (2.11), we obtain (2.4). □

If, in addition to the assumptions of Theorem 2.1, we assume $x(t)\neq 0$ for $t\in(a,c)\cup(c,b)$, then stronger Lyapunov-type inequalities can be derived. We present the results in the next Theorem.

Theorem 2.2. *Assume* (2.1) *has a solution* $x(t)$ *satisfying*

$$x^{(i+1)}(a)=x^{(i+1)}(b)=0,\quad i=0,1,\dots,n-1.$$

(a) Suppose $x(c) = 0$ for $c \in (a, b)$ and $x(t) \neq 0$ for $t \in [a, c) \cup (c, b]$. Then one of the following holds:

(i) $\int_a^b q_-(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}$,

(ii) $\int_a^b q_+(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}$,

(iii) $\int_a^c q_-(t)dt + \int_c^b q_+(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}$.

(b) Suppose $x(a) = 0$ and $x(t) \neq 0$ for $t \in (a, b]$. Then

$$\int_a^b q_+(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}.$$

(c) Suppose $x(b) = 0$ and $x(t) \neq 0$ for $t \in [a, b)$. Then

$$\int_a^b q_-(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}.$$

Proof. As in the proof of Theorem 2.1, we see that (2.9) and (2.11) hold.

(a) Since $x(t)$ is continuous and $x(c) = 0$ for $c \in (a, b)$, there exist $t_1 \in (a, c)$ and $t_2 \in (c, b)$ such that $|x(t_1)| = \max\{|x(t)| : t \in [a, c]\}$ and $|x(t_2)| = \max\{|x(t)| : t \in [c, b]\}$. Without loss of generality, we may assume $x(t)$ satisfies one of the following cases:

- (I) $x(t) > 0$ on $(a, c) \cup (c, b)$ and $x(t_1) \geq x(t_2)$;
- (II) $x(t) > 0$ on $(a, c) \cup (c, b)$ and $x(t_1) < x(t_2)$;
- (III) $x(t) > 0$ on (a, c) and $x(t) < 0$ on (c, b), and $x(t_1) \geq -x(t_2)$;
- (IV) $x(t) > 0$ on (a, c) and $x(t) < 0$ on (c, b), and $x(t_1) < -x(t_2)$.

In the sequel, we denote $m = \max\{|x(t_1)|, |x(t_2)|\}$.

Case I: $m = x(t_1)$**.** Then (2.9) with $t = t_1$ shows that

$$m = \int_a^b \Big(\int_{t_1}^c G(\tau, s)d\tau\Big)(-q(s))x(s)ds\,.$$

Using that $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$, and $-q(t) \leq q_-(t)$, we have

$$m < m\int_a^b \Big(\int_a^b G(\tau, s)d\tau\Big)q_-(s)ds.$$

Canceling m from both sides and using (2.11) we obtain

$$\int_a^b q_-(s)ds > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}\,;$$

i.e., conclusion (i) in Part (a) holds.

Case II: $m = x(t_2)$**.** Then (2.9) with $t = t_2$ shows that

$$m = \int_a^b \Big(\int_c^{t_2} G(\tau, s)d\tau\Big)q(s)x(s)ds.$$

Again using the facts that $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$, and $q(t) \leq q_+(t)$, we have

$$m < m\int_a^b \Big(\int_a^b G(\tau, s)d\tau\Big)q_+(s)ds.$$

Canceling m from both sides and using (2.11) we obtain

$$\int_a^b q_+(s)ds > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}.$$

i.e., conclusion (ii) in Part (a) holds.

Case III: $m = x(t_1)$**.** Then (2.9) with $t = t_1$ shows that

$$\begin{aligned} m &= \int_a^b \Big(\int_{t_1}^c G(\tau,s)d\tau\Big)(-q(s))x(s)ds \\ &= \int_a^c \Big(\int_{t_1}^c G(\tau,s)d\tau\Big)(-q(s))x(s)ds + \int_c^b \Big(\int_{t_1}^c G(\tau,s)d\tau\Big)q(s)(-x(s))ds. \end{aligned}$$

Note that $x(t) > 0$ on $[a,c)$ and $x(t) < 0$ on $(c,b]$. Then by a similar argument to Cases I and II, we see that

$$\int_a^c q_-(s)ds + \int_c^b q_+(s)ds > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}\,;$$

i.e., conclusion (iii) in Part (a) holds.

Case IV. The same argument as in Case III shows that conclusion (iii) in Part (a) holds. We omit the details.

(b) Note that $x(a) = 0$. Then it follows from (2.9) that

$$x(t) = \int_a^b \Big(\int_a^t G(\tau,s)d\tau\Big)q(s)x(s)ds. \tag{2.20}$$

Without loss of generality, we may assume $x(t) > 0$ in $(a,b]$. Then there exists $t_2 \in (a,b]$ such that $m = x(t_2) = \max\{x(t) : t \in [a,b]\}$. Using $t = t_2$ in (2.20) we obtain

$$m = \int_a^b \Big(\int_a^{t_2} G(\tau,s)d\tau\Big)q(s)x(s)ds \le \int_a^b \Big(\int_a^b G(\tau,s)d\tau\Big)q_+(s)x(s)ds.$$

Using (2.11) and a similar technique as before, we see that conclusion in Part (b) holds.

(c) In this case, a similar argument as Part (b) holds with $m = x(t_1) = \max\{|x(t)| : t \in [a,b]\}|$. We omit the details. □

Now we interpret the results in Theorems 2.1 and 2.2 to the special case with $n = 1$, i.e., the third-order linear differential equation

$$x''' + q(t)x = 0. \tag{2.21}$$

From (2.2),

$$S_1 = \sum_{j=0}^{0}\sum_{k=0}^{j} 2^{2k-2j}\binom{j}{j}\binom{j}{k}B(2,1+k-j) = B(2,1) = \frac{1}{2}.$$

Corollary 2.3. *Assume* (2.21) *has a nontrivial solution* $x(t)$ *satisfying*

$$x'(a) = x'(b) = 0$$

and $x(c) = 0$ *for* $c \in [a,b]$*. Then*

$$\int_a^b |q(t)|dt > \frac{8}{(b-a)^2}.$$

Corollary 2.4. *Assume* (2.21) *has a solution* $x(t)$ *satisfying*

$$x'(a) = x'(b) = 0.$$

(a) Suppose $x(c) = 0$ *for* $c \in (a,b)$ *and* $x(t) \ne 0$ *for* $t \in [a,c) \cup (c,b]$*. Then one of the following holds:*

(i) $\int_a^b q_-(t)dt > \frac{8}{(b-a)^2}$,

(ii) $\int_a^b q_+(t)dt > \frac{8}{(b-a)^2}$,

(iii) $\int_a^c q_-(t)dt + \int_c^b q_+(t)dt > \frac{8}{(b-a)^2}$.

(b) Suppose $x(a) = 0$ and $x(t) \neq 0$ for $t \in (a, b]$. Then

$$\int_a^b q_+(t)dt > \frac{8}{(b-a)^2}.$$

(c) Suppose $x(b) = 0$ and $x(t) \neq 0$ for $t \in [a, b)$. Then

$$\int_a^b q_-(t)dt > \frac{8}{(b-a)^2}.$$

We observe that the inequalities in Corollaries 2.3 and 2.4 supplement those in [8, Corollary 2.1] for different boundary conditions.

3. Applications to boundary-value problems

In the final section, we apply the results on the Lyapunov-type Inequalities obtained in Section 2 to study the nonexistence, uniqueness, and existence-uniqueness for solutions of certain BVPs. Consider the BVP consisting of (2.1) and the boundary conditions

$$\begin{gathered} x^{(i+1)}(a) = x^{(i+1)}(b) = 0, \quad i = 0, 1, \dots, n-1; \\ x(c) = 0, \quad c \in [a, b]. \end{gathered} \tag{3.1}$$

In the following, we let S_n be defined by (2.2). The first result is on the nonexistence of solutions of the boundary-value problem (2.1) (3.1).

Theorem 3.1. *Assume*

$$\int_a^b |q(t)|dt \leq \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}. \tag{3.2}$$

Then BVP (2.1) (3.1) has no nontrivial solution for any $c \in [a, b]$.

Proof. Assume the contrary, i.e., (2.1) (3.1) has a nontrivial solution $x(t)$. Then by Theorem 2.1, inequality (2.4) holds. This contradicts assumption (3.2). □

As a direct application of Theorem 2.2, we present the following result.

Theorem 3.2. *Assume*

$$\max_{\xi \in [a,b]} \Big\{ \int_a^\xi q_-(t)dt + \int_\xi^b q_+(t)dt \Big\} \leq \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}. \tag{3.3}$$

Then every nontrivial solution of BVP (2.1), (3.1) has at least two zeros in $[a, b]$.

Proof. Assume the contrary, i.e., (2.1), (3.1) has a nontrivial solution $x(t)$ with only one zero c_1 in $[a, b]$. Then $c_1 = c$ and $x(t) \neq 0$ for $t \in [a, c) \cup (c, b]$. It follows that one of the conclusions in Part (a) of Theorem 2.2 holds. This contradicts (3.3). □

Next we consider the odd order nonhomogeneous linear BVPs consisting of the equation

$$x^{(2n+1)} + (-1)^{n-1} q(t)x = f(t) \quad \text{on } (A, B) \tag{3.4}$$

with $-\infty < A < B < \infty$ and $q, f \in C((A,B),\mathbb{R})$; and boundary condition

$$\begin{gathered} x^{(i+1)}(a) = k_{i1}, \quad x^{(i+1)}(b) = k_{i2}, \quad i = 0,1,\dots,n-1 \\ x(c) = k_{i3}, \quad c \in [a,b] \end{gathered} \tag{3.5}$$

with

$$A < a < b < B \quad \text{and} \quad k_{i1}, k_{i2}, k_{i3} \in \mathbb{R}. \tag{3.6}$$

Based on Theorem 2.1, we obtain a criterion for BVP (3.4), (3.5) to have a unique solution.

Theorem 3.3. *Assume*

$$\int_a^b |q(t)|dt \le \frac{2^{2n}(2n-1)!}{(B-A)^{2n} S_n}.$$

Then BVP (3.4)*,* (3.5) *has a unique solution on* (A,B) *for any* $a, b \in (A,B)$*, and* $c \in [a,b]$*, and* k_{i1}, k_{i2}, k_{i3} *satisfying* (3.6)*.*

Proof. We first show that BVP (3.4), (3.5) has at most one solution for any a, b and k_1, k_2, k_3 satisfying (3.6). Assuming the contrary, it has two solutions $x_1(t)$ and $x_2(t)$ in (A,B). Define $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of BVP (2.1), (3.1). Then by Theorem 3.1, $x(t) \equiv 0$, i.e., $x_1(t) \equiv x_2(t)$. This shows the uniqueness of solution to BVP (3.4), (3.5).

Since the homogeneous linear BVP (2.1), (3.1) only has the zero solution, then by the Fredholm alternative theorem [9], we conclude that the nonhomogeneous linear BVP (3.4), (3.5) has a unique solution. □

References

[1] M. F. Aktas, D. Cakmak, A. Tiryaki; On the Lyapunov-type inequalities of a three point boundary value problem for third order linear differential equations, *Appl. Math. Letters.* **45**(2015), 1-6.

[2] R. C. Brown, D. B. Hilton; Opial's inequality and oscillation of second-order equations, *Proc. Amer. Math. Soc.* **125** (1997), 1123-1129.

[3] G. Borg; On a Liapunoff criterion of stability, *Amer. J. Math.* **71** (1949), 67-70.

[4] D. Cakmak; On Lyapunov-type inequality for a class of nonlinear systems, *Math. Inq. Appl.* **16** (2013), 101-108.

[5] D. Cakmak; Liapunov-type integral inequalities for certain higher order differential equations, *Appl. Math. Comput.* **216** (2010), 368-373.

[6] K. M. Das, A. S. Vatsala; Green's function for *n*-*n* boundary value problem and an analogue of Hartman's result, *J. Math. Anal. Appl.* **51** (1975), 670-777.

[7] S. Dhar, Q. Kong; Liapunov-type inequalities for third-order half-linear equations and applications to boundary value problems, *Nonlin. Anal.* **110** (2014), 170-181.

[8] S. Dhar, Q. Kong; Lyapunov-type inequalities for third-order linear differential equations, *Math. Inequal. Appl.* **19** (2016), 297-312.

[9] E. I. Fredholm; Sur une classe d'quations fonctionnelles, *Acta Math.* **27** (1903), 365-390.

[10] B. J. Harris, Q. Kong; On the oscillation of differential equations with an oscillatory coefficient, *Trans. Amer. Math. Soc.* **347** (1995), 1831-1839.

[11] P. Hartman; *Ordinary Differential Equations*, Wiley, New York, 1964, and Birkhauser, Boston 1982.

[12] X. He, X. H. Tang; Liapunov-type inequalities for even-order differential equations, *Comm. Pure Appl. Math.*, **11** (2012), 465-473.

[13] A. M. Liapunov; Probleme general de la stabilite du mouvement, *Ann. Math Stud.*, **17** (1947), 203-474.

[14] B. G. Pachpatte; Lyapunov-type integral inequalities for certain differential equations, *Georgian Math. J.* **4** (1997), 139-148.

[15] B. G. Pachpatte; On Lyapunov-type inequalities for certain higher order differential equations, *J. Math. Anal. Appl.* **195** (1995), 527-536.
[16] S. Panigrahi; Lyapunov-type integral inequalities for certain higher order differential equations, *Electron. J. Differential. Eq.* **2009** (2009), 1-14.
[17] N. Parhi, S. Panigrahi; On Liapunov-type inequality for third-order differential equations, *J. Math. Anal. Appl.* **233** (1999), 445-460.
[18] N. Parhi, S. Panigrahi; Liapunov-type inequality for higher order differential equations, *Math. Slovaca* **52** (2002), 31-46.
[19] J. P. Pinasco; *Lyapunov-type inequalities with Applications to Eigenvalue Problems*, Springer, 2010.
[20] A. Tiryaki, M. Unal, D. Cakmak; Liapunov-type inequalities for nonlinear systems, *J. Math. Anal. Appl.* **332** (2007), 497-511.
[21] A. Wintner; On the non-existence of conjugate points, *Amer. J. Math.* **73** (1951), 368-380.
[22] X. Yang, K. Lo; Liapunov-type inequality for a class of even-order differential equations, *Appl. Math. Comput.* **215** (2010), 3884-3890.
[23] X. Yang; On Lyapunov inequality for certain higher-order differential equations, *Appl. Math. Comput.* **134** (2003), 307-317.
[24] Q. Zhang, X. He; Liapunov-type inequalities for a class of even-order differential equations, *J. Ineq. Appl.* 2012, **2012**:5.

Sougata Dhar
Department of Mathematics, Northern Illinois University, DeKalb, IL 60115, USA
E-mail address: sdhar@niu.edu

Qingkai Kong
Department of Mathematics, Northern Illinois University, DeKalb, IL 60115, USA
E-mail address: qkong@niu.edu