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STABILITY FOR NONCOERCIVE ELLIPTIC EQUATIONS

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Abstract. In this article, we consider the stability for elliptic problems that have degenerate coercivity in their principal part,

$$-\operatorname{div}\Big(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\Big)+|u|^{q-1}u=f,\quad x\in\Omega,$$
$$u(x)=0,\quad x\in\partial\Omega,$$

where $\theta>0$, $\Omega\subseteq\mathbb{R}^N$ is a bounded domain. Let K be a compact subset in Ω with zero r-capacity ($p<r\le N$). We prove that if f_n is a sequence of functions which converges strongly to f in $L^1_{\rm loc}(\Omega\backslash K)$ and $q>r(p-1)[1+\theta(p-1)]/(r-p)$, and u_n is the sequence of solutions of the corresponding problems with datum f_n. Then u_n converges to the solution u.

1. Introduction and statement of main results

Let $\Omega\subseteq\mathbb{R}^N$ be a bounded smooth domain. We are interested in the stability of quasilinear elliptic problems with principal part having degenerate coercivity,

$$\begin{gathered}-\operatorname{div}a(x,u,\nabla u)+|u|^{q-1}u=f,\quad x\in\Omega,\\ u(x)=0,\quad x\in\partial\Omega,\end{gathered}\tag{1.1}$$

where $\theta>0$, $1<p<N$ and $f\in L^1(\Omega)$. The function $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function (that is, $a(\cdot,s,\xi)$ measurable on Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and $a(x,\cdot,\cdot)$ continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) satisfying the following assumptions:

$$a(x,s,\xi)\xi\ge\alpha_1h^{p-1}(|s|)|\xi|^p,\quad\alpha_1>0,\tag{1.2}$$

$$|a(x,s,\xi)|\le\alpha_2|\xi|^{p-1},\quad\alpha_2>0,\tag{1.3}$$

$$\langle a(x,s,\xi)-a(x,s,\eta),\xi-\eta\rangle>0,\quad\xi\ne\eta,\tag{1.4}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R},\xi\in\mathbb{R}^N,\eta\in\mathbb{R}^N$, $h(t)$ is defined as

$$h(t)=\frac{1}{(1+|t|)^\theta}.\tag{1.5}$$

The interest in removable singularities for elliptic equations goes back to the pioneering work of Brezis[13]. Actually, Brezis shown that if $\{u_n\}$ are the sequence of solutions of the nonlinear elliptic problems

$$-\Delta u_n+|u_n|^{q-1}u_n=f_n,\quad x\in\Omega,$$

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$$u_n(x) = 0, \quad x \in \partial\Omega,$$

where $0 \in \Omega$, $q \geq \frac{N}{N-2}$ and $\{f_n\}$ be a sequence of $L^1(\Omega)$ functions satisfying

$$\lim_{n\to\infty} \int_{\Omega\setminus B_\rho(0)} |f_n - f| = 0.$$

Then u_n converges to the unique solution u of the equation

$$-\Delta u + |u|^{q-1}u = f.$$

In particular, surprisingly enough, let $\{f_n\}$ be a sequence in $L^1(\Omega)$ such that $f_n \subset B(0, \frac{1}{n})$ and $f_n \to \delta$, then $u_n \to 0$. While we would expect u_n converges to the solution u of

$$-\Delta u + |u|^{q-1}u = \delta.$$

but lt is well known that such a u does not exists if $q \geq \frac{N}{N-2}$, see [7].

The results in [13] were extended by Orsina and Prignet [21] for more general uniformly elliptic, coercive and pseudomonotone operator and where f is a measure which is concentrated on a set E of zero r-capacity. Continuing the studies in [21, 13], Orsina and Prignet [22] obtained stability results of elliptic equations

$$\begin{gathered} -\operatorname{div} a(x, u, \nabla u) + |u|^{q-1}u = f, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered}$$

where a is a Carathéodory function satisfying (1.3), (1.4) and

$$a(x, s, \xi)\xi \geq \alpha_1 |\xi|^p, \quad \alpha_1 > 0.$$

With motivation from the results of the above cited papers, the main purpose of this paper is to investigate the stability results of problem (1.1). The main results show that how the nonlinear term $|u|^{q-1}u$ and the singular term $h(u)^{p-1}$ affect the existence of solutions to (1.1).

The main results of this article is the following theorem.

Theorem 1.1. *Let $p < r \leq N$, $f = f^+ - f^-$ be a function in $L^1(\Omega)$, u_n be a solution to problems*

$$\begin{gathered} -\operatorname{div} a(x, u_n, \nabla u_n) + |u_n|^{q-1}u_n = f_n, \quad x \in \Omega, \\ u_n(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.6}$$

where $f_n = f_n^{\oplus} - f_n^{\ominus}$, $f_n^{\oplus}$ and $f_n^{\ominus}$ be two sequences of nonnegative $L^\infty(\Omega)$ functions such that

$$\lim_{n\to\infty} \int_{\Omega\setminus I(K^+)} |f_n^{\oplus} - f^+| = 0, \quad \lim_{n\to\infty} \int_{\Omega\setminus I(K^-)} |f_n^{\ominus} - f^-| = 0, \tag{1.7}$$

for every neighbourhood $I(K^+)$ of K^+ and $I(K^-)$ of K^-, where K^+ and K^- be two disjoint compact subsets of Ω of zero r-capacity. Then, up to subsequences still denoted by u_n, u_n converges to a solution in the sense of distributions of the problems (1.1) *with datum f provided*

$$q > \frac{r(p-1)[1 + \theta(p-1)]}{r - p}. \tag{1.8}$$

Remark 1.2. We emphasize that we do not assume that $f_n^{\oplus}$ and $f_n^{\ominus}$ are the positive and negative part of f_n, but only that they are nonnegative. This is the reason why we use the unconventional notation $f_n^{\oplus}$ and $f_n^{\ominus}$.

Remark 1.3. The preceding theorem can be seen as a non-existence result for problem (1.1): A particular case of Theorem 1.1 is when the sequence $f_n^{\oplus}$ is convergent to f in the tight topology of measures f, where f is a bounded Radon measure concentrated on a set K of zero harmonic capacity and $f_n^{\ominus} = 0$, In this case, Theorem 1.1 states that the sequence u_n tends to zero almost everywhere in Ω. This is exactly the result [11, Theorem 4.1].

Remark 1.4. The result of preceding theorem can also be seen as a result of removable singularities for problem (1.1). Indeed it states that sets of zero r-capacity are not seen by the equation if q satisfies (1.8). Some other results about removable singularities of elliptic equations, see [1, 2, 9, 14, 20, 17, 24, 25].

Remark 1.5. With minor technical modifications in the proof of [15, Theorem 1.6], we can obtain the existences of distributional solutions $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to problem (1.6). Indeed, lower order term $|u|^{q-1}u$ has a regularizing effect. Roughly speaking, large values of q can compensate the "bad coercivity" of the principal part and the poor summability of the right hand side.

Remark 1.6. The principal part left-hand of (1.1) is defined on $W_0^{1,p}(\Omega)$, but it may not be coercive on the same space as u becomes large, due to this lack of coercivity, standard existence theorems for solutions of nonlinear elliptic equations cannot be applied. Furthermore, $\frac{\nabla u}{(1+|u|)^{\theta(p-1)}}$ tends to zero as u tends to infinity, which produces a saturation effect. Some other results of elliptic equations with principal part having degenerate coerciveness, see [5, 6, 10, 12, 19].

Remark 1.7. In this article, we only consider $\theta > 0$. The case $\theta \equiv 0$ has been considered by Orsina and Prignet [22],

The plan of this article is as follows. In Section 2, we briefly recall some notations and known results about measures. Section 3 contains the proof of Theorem 1.1.

2. Preliminaries

In this section, we first recall some notation and definitions. In the following, C will be a constant that may change from an inequality to another, to indicate a dependence of C on the real parameters δ, we shall write $C = C(\delta)$.

For each real number s, we define $s^+ = \max(s, 0)$ and $s^- = -\max(-s, 0)$. Obviously, $s = s^+ - s^-$ and $|s| = s^+ + s^-$.

For $k > 0$, denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k; that is,

$$T_k(s) = \max\{-k, \min\{k, s\}\}.$$

The "remainder" of the truncation $T_k(s)$ is defined as $G_k(s) = s - T_k(s)$.

Note that we will deal with functions u that may not belong to Sobolev spaces, we need to give a suitable definition of gradient. Consider a measurable function $u : \Omega \to \mathbb{R}$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. According to [8, Lemma 2.1], there exists an unique measurable function $v : \Omega \to \overline{\mathbb{R}}^N$ such that, for each $k > 0$,

$$\nabla T_k(u) = v\chi_{|u|\le k} \quad \text{almost everywhere in } \Omega,$$

where $\chi_{|u|\le k}$ is the characteristic function of $\{|u| \le k\}$. We define the gradient ∇u of u as this function v, and denote $\nabla u = v$.

Remark 2.1. It is worth pointing out that the gradient defined in this way is not, in general, the gradient used in the definition of Sobolev spaces, However, v is the distributional gradient of u provided v belongs to $(L^1_{\rm loc}(\Omega))^N$, which also implies that u belongs to $W^{1,1}_{\rm loc}(\Omega)$.

Remark 2.2. As point out in [8], the set of functions u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k>0$ is not a linear space. That is, if u and v are such that both $T_k(u)$ and $T_k(v)$ belong to $W_0^{1,p}(\Omega)$ for every $k>0$, while $\nabla(u+v)$ may not be defined.

Denote by $|\Omega|$ the N-dimensional Lebesgue measure of a measurable set Ω. Let $f(x), g(x)$ are functions defined in $\mathbb{R}^N$ and a, b are constants, we set

$$\{f(x)>a\}:=\{x\in\mathbb{R}^N: f(x)>a\},\quad \{g(x)\le b\}:=\{x\in\mathbb{R}^N: g(x)\le b\}.$$

The r-capacity $\operatorname{cap}_{1,p}(K,\Omega)$ of a compact set $K\subset\Omega$ with respect to Ω is defined by

$$\operatorname{cap}_{1,p}(K,\Omega)=\inf\Big\{\int_\Omega |\nabla\phi|^p dx : \phi\in C_0^\infty(\Omega), \phi\ge\chi_E\Big\}.$$

The following technical propositions will be be useful throughout the paper[18].

Proposition 2.3. *Let K^+ and K^- be two disjoint compact subsets of Ω of zero r-capacity and $p<r\le N$. Then, for every $\delta>0$ there exist A_δ^+ and A_δ^-, two disjoint open subsets of Ω, and ψ_δ^+ and ψ_δ^- in $C_c^\infty(\Omega)$ such that*

$$0\le\psi_\delta^+(x)\le 1,\quad 0\le\psi_\delta^-(x)\le 1,\quad x\in\Omega, \tag{2.1}$$
$$\psi_\delta^+(x)\equiv 1,\quad x\in K^+,\quad \psi_\delta^-(x)\equiv 1,\quad x\in K^-, \tag{2.2}$$
$$\operatorname{supp}(\psi_\delta^+(x))=A_\delta^+,\quad \operatorname{supp}(\psi_\delta^-(x))=A_\delta^-, \tag{2.3}$$
$$\int_\Omega|\nabla\psi_\delta^+(x)|^r dx\le\delta,\quad \int_\Omega|\nabla\psi_\delta^-(x)|^r dx\le\delta, \tag{2.4}$$
$$\operatorname{meas}(A_\delta^+)\le\delta,\quad \operatorname{meas}(A_\delta^-)\le\delta. \tag{2.5}$$

3. Proof of Theorem 1.1

The following arguments are similar to these in [22], and the proof will be done with the aid of the following two lemmas.

Lemma 3.1. *There exists $0<C<\infty$ such that for any $k>0$,*

$$\int_\Omega|\nabla T_k(u_n)|^p dx < Ck^{q+1+\theta(p-1)}. \tag{3.1}$$

Proof. Choose $T_k(u_n)(1-\psi_\delta)^s$ as a test function in (1.6), here and elsewhere in the paper

$$\psi_\delta=\psi_\delta^+ + \psi_\delta^-,\quad s=\frac{\beta}{\beta-p+1}.$$

where β appears in (3.8). Thus

$$\begin{aligned}
&\int_\Omega a(x,u_n\nabla u_n)\cdot\nabla T_k(u_n)(1-\psi_\delta)^s dx+\int_\Omega |u_n|^{q-1}u_nT_k(u_n)(1-\psi_\delta)^s dx\\
&= s\int_\Omega a(x,u_n\nabla u_n)\nabla\psi_\delta T_k(u_n)(1-\psi_\delta)^{s-1}dx \qquad (3.2)\\
&\quad+\int_\Omega f_n^{\oplus}T_k(u_n)(1-\psi_\delta)^s dx-\int_\Omega f_n^{\ominus}H(T_k(u_n)(1-\psi_\delta)^s dx.
\end{aligned}$$

By (1.2), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)d\mu_\delta \ge \alpha_1\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}}d\mu_\delta, \qquad (3.3)$$

here and the rest of this paper we use the note $d\mu_\delta=(1-\psi_\delta)^s dx$.

Recall that $u_nT_k(u_n)\ge 0$, which leads to

$$\begin{aligned}
\int_\Omega |u_n|^{q-1}u_nT_k(u_n)(1-\psi_\delta)^s dx &\ge \int_{\{|u_n|\ge k\}}|u_n|^{q-1}u_nT_k(u_n)d\mu_\delta\\
&\ge k^{q+1}\mu_\delta(\{|u_n|\ge k\}).
\end{aligned} \qquad (3.4)$$

Using (1.3) and Young's inequality, we find

$$\begin{aligned}
&\int_\Omega |a(x,u_n,\nabla u_n)\nabla\psi_\delta T_k(u_n)(1-\psi_\delta)^{s-1}|dx\\
&\le \alpha_2 k\int_\Omega |\nabla u_n|^{p-1}(|\nabla\psi_\delta^+|+|\nabla\psi_\delta^-|)(1-\psi_\delta)^{s-1}dx \qquad (3.5)\\
&\le Ck\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\psi_\delta)^{(s-1)r'}dx+Ck\int_\Omega(|\nabla\psi_\delta^+|^r+|\nabla\psi_\delta^-|^r)dx.
\end{aligned}$$

Combining (2.4) and (3.2)-(3.5), we obtain

$$\begin{aligned}
&\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}}d\mu_\delta+k^{q+1}\mu_\delta(\{|u_n|\ge k\})\\
&\le Ck(\delta+I_1(n,\delta)+I_2(n,\delta)),
\end{aligned} \qquad (3.6)$$

where

$$I_1(n,\delta)=\int_\Omega (f_n^{\oplus}+f_n^{\ominus})d\mu_\delta,\quad I_2(n,\delta)=\int_\Omega |\nabla u_n|^{(p-1)r'}(1-\psi_\delta)^{(s-1)r'}dx.$$

For a fixed $\rho\ge 0$, thanks to (3.6), we have

$$\begin{aligned}
&\mu_\delta(\{|\nabla u_n|\ge\rho\})\\
&=\mu_\delta\big(\{|\nabla u_n|\ge\rho\}\cup\{|u_n|<k\}\big)+\mu_\delta\big(\{|\nabla u_n|\ge\rho\}\cup\{|u_n|\ge k\}\big)\\
&\le \frac{1}{\rho^p}\int_\Omega |\nabla T_k(u_n)|^p d\mu_\delta+\mu_\delta(\{|u_n|\ge k\})\\
&\le \frac{(1+k)^{\theta(p-1)}}{\rho^p}\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^{\theta(p-1)}}d\mu_\delta+\mu_\delta(\{|u_n|\ge k\})\\
&\le C(\delta+I_1(n,\delta)+I_2(n,\delta))\Big(\frac{k^{1+\theta(p-1)}}{\rho^p}+\frac{1}{k^q}\Big),
\end{aligned}$$

which implies

$$\mu_\delta(\{|\nabla u_n|\ge\rho\})\le C\rho^{-\frac{pq}{q+1+\theta(p-1)}}(\delta+I_1(n,\delta)+I_2(n,\delta)). \qquad (3.7)$$

Let β be such that

$$(p-1)r' < \beta < \frac{pq}{q+1+\theta(p-1)}. \tag{3.8}$$

It can be easily seen that such a β exists by (1.8). In view of (3.7), we have

$$\int_\Omega |\nabla u_n|^\beta d\mu_\delta \le C(\delta + I_1(n,\delta) + I_2(n,\delta)). \tag{3.9}$$

This fact and Hölder's inequality imply

$$I_2(n,\delta) \le C\Big(\int_\Omega |\nabla u_n|^\beta d\mu_\delta\Big)^{\frac{(p-1)r'}{\beta}} \le C(\delta + I_1(n,\delta) + I_2(n,\delta))^{\frac{(p-1)r'}{\beta}},$$

which, combined with the fact that $X^\gamma \le C + X$ imply that X is bounded provided $\gamma > 1$; this yields

$$I_2(n,\delta) \le C(\delta + I_1(n,\delta)) \le C(\delta), \tag{3.10}$$

since $1-\psi_\delta$ is zero both on a neighbourhood of K^+ and of K^-, this fact and (1.7) show that $I_1(n,\delta)$ is bounded with respect to δ.

Using estimates (3.5), (3.6) and (3.10), we conclude that

$$\int_\Omega |\nabla T_k(u_n)|^p d\mu_\delta \le C(\delta) k^{1+\theta(p-1)}, \tag{3.11}$$

$$\int_\Omega |u_n|^{q-1} u_n T_k(u_n) d\mu_\delta \le C(\delta) k, \tag{3.12}$$

$$\int_\Omega |\nabla u_n|^{p-1} (|\nabla \psi_\delta^+| + |\nabla \psi_\delta^-|)(1-\psi_\delta)^{s-1} dx \le C(\delta). \tag{3.13}$$

Choose $T_k(u_n^+)(1-\psi_\delta^+)^s$ and $-T_k(u_n^-)(1-\psi_\delta^-)^s$ as a test function in (1.6) respectively. Similar arguments show that

$$\begin{gathered} \int_\Omega |\nabla T_k(u_n^+)|^p d\mu_\delta^+ \le C(\delta) k^{1+\theta(p-1)}, \\ \int_\Omega |\nabla T_k(u_n^-)|^p d\mu_\delta^- \le C(\delta) k^{1+\theta(p-1)}, \end{gathered} \tag{3.14}$$

and

$$\begin{gathered} \int_\Omega |u_n^+|^{q-1} u_n^+ T_k(u_n^+) d\mu_\delta^+ \le C(\delta) k, \\ \int_\Omega |u_n^-|^{q-1} u_n^- T_k(u_n^-) d\mu_\delta^- \le C(\delta) k, \end{gathered} \tag{3.15}$$

where $d\mu_\delta^+ = (1-\psi_\delta^+)^s dx$ and $d\mu_\delta^- = (1-\psi_\delta^-)^s dx$.

Now we choose $(k - T_k(u_n^+))(1-(1-\psi_\delta^+)^s)$ as a test function in (1.6). We must emphasize that

$$\begin{gathered} (k - T_k(u_n^+))(1-(1-\psi_\delta^+)^s) = k - T_k(u_n^+), \quad x \in K^+, \\ (k - T_k(u_n^+))(1-(1-\psi_\delta^+)^s) = 0. \end{gathered}$$

Apart from the support of ψ_δ^+, a simple calculation yields

$$\begin{aligned}
&-\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n^+)(1-(1-\psi_\delta^+)^s)dx\\
&+s\int_\Omega a(x,u_n,\nabla u_n)\nabla\psi_\delta^+(k-T_k(u_n^+))(1-\psi_\delta^+)^{s-1}dx\\
&+\int_\Omega |u_n|^{q-1}u_n(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx\\
=&\int_\Omega f_n^\oplus(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx\\
&-\int_\Omega f_n^\ominus(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx.
\end{aligned} \tag{3.16}$$

Obviously

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n^+)(1-(1-\psi_\delta^+)^s)dx\\
&\geq \frac{\alpha_1}{(1+k)^{\theta(p-1)}}\int_\Omega|\nabla T_k(u_n^+)|^p(1-(1-\psi_\delta^+)^s)dx,
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla\psi_\delta^+(k-T_k(u_n^+))(1-\psi_\delta^+)^{s-1}dx\\
&\leq k\int_\Omega |T_k(u_n^+)|^{p-1}|\nabla\psi_\delta^+|(1-\psi_\delta^+)^{s-1}dx\leq C(\delta)k,
\end{aligned} \tag{3.18}$$

here we have used (3.13) and the fact that $k-T_k(u_n^+)\leq k$.

It can be easily seen that

$$\begin{aligned}
&\int_\Omega |u_n|^{q-1}u_n(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx\\
&\leq \int_{\{0\leq u_n\leq k\}}|u_n|^{q-1}u_n(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx\\
&\leq C(\delta)k^{q+1},
\end{aligned} \tag{3.19}$$

$$0\leq\int_\Omega f_n^\oplus(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx\leq C(\delta)k, \tag{3.20}$$

$$0\leq\int_\Omega f_n^\ominus(k-T_k(u_n^+))(1-(1-\psi_\delta^+)^s)dx\leq C(\delta)k. \tag{3.21}$$

From (3.16)-(3.21), we obtain

$$\int_\Omega |\nabla T_k(u_n^+)|^p(1-(1-\psi_\delta^+)^s)dx\leq C(\delta)k^{q+1+\theta(p-1)}. \tag{3.22}$$

Similarly, choosing $(k+T_k(u_n^-))(1-(1-\psi_\delta^-)^s)$ as a test function in (1.6), we find

$$\int_\Omega |\nabla T_k(u_n^-)|^p(1-(1-\psi_\delta^-)^s)dx\leq C(\delta)k^{q+1+\theta(p-1)}. \tag{3.23}$$

Combining (3.14) with (3.22) and (3.23), and then choosing $\delta=1$ (for example), we have

$$\int_\Omega |\nabla T_k(u_n)|^pdx\leq Ck^{q+1+\theta(p-1)}, \tag{3.24}$$

which shows that (3.1) holds. Consequently, $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ independently of n. This implies that there exists a subsequence of u_n (still denoted by u_n) which is almost everywhere convergent in Ω to a measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ [8]. □

The next step of the proof is to state some propositions of limit function u.

Lemma 3.2. *There exists a constant C such that*

$$\int_\Omega |\nabla u|^{(p-1)r'} dx \le C, \tag{3.25}$$

$$\int_\Omega |u|^q dx \le C. \tag{3.26}$$

Proof. Firstly, we show that u_n is a Cauchy sequence in measure. To do this, we define

$$\Phi(t) = \int_0^t \frac{1}{(1+|s|)^\gamma} ds,$$

where $\gamma = 1 + (p-1)(1-\theta)$. It can be easily seen that

$$|\Phi(t)| \le \frac{1}{(p-1)|1-\theta|}.$$

Choose $\Phi(u_n)(1-\psi_\delta)^s$ as a test function in (1.6), we obtain

$$\begin{aligned} &\int_\Omega \frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^\gamma} \cdot \nabla u_n d\mu_\delta + \int_\Omega |u_n|^{q-1} u_n \Phi(u_n) d\mu_\delta \\ &= s\int_\Omega a(x,u_n,\nabla u_n)\nabla\psi_\delta \Phi(u_n)(1-\psi_\delta)^{s-1} dx + \int_\Omega f_n^\oplus \Phi(u_n) d\mu_\delta \\ &\quad - \int_\Omega f_n^\ominus \Phi(u_n) d\mu_\delta. \end{aligned} \tag{3.27}$$

Obviously, by (1.2),

$$\int_\Omega \frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^\gamma} \cdot \nabla u_n d\mu_\delta \ge \alpha_1 \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} d\mu_\delta, \tag{3.28}$$

$$\int_\Omega |u_n|^{q-1} u_n \Phi(u_n) d\mu_\delta \ge 0, \tag{3.29}$$

Consider the first terms of the right-hand side of (3.27), using (1.3), we have

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla\psi_\delta \Phi(u_n)(1-\psi_\delta)^{s-1} dx \\ &\le C\int_\Omega |\nabla u_n|^{p-1}(|\nabla\psi_\delta^+| + |\nabla\psi_\delta^-|)(1-\psi_\delta)^{s-1} dx \\ &\le C(\delta + I_2(n,\delta)). \end{aligned} \tag{3.30}$$

Therefore, using (3.27)–(3.30) and (3.10), we have

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p} d\mu_\delta \le C(\delta). \tag{3.31}$$

Similar arguments as the proof of Lemma 3.1, choose $\Phi(k - T_k(u_n^+))(1-(1-\psi_\delta)^s)$ as a test function, show that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p}(1-(1-\psi_\delta)^s) dx \le C(\delta). \tag{3.32}$$

Inequalities (3.31) and (3.32) yield

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^p}dx \le C(\delta). \tag{3.33}$$

Split $\{|u_n|\ge k\}$ as $\{|u_n|\ge k\}\cap A_\delta$ and $\{|u_n|\ge k\}\cap A_\delta^c$, where $A_\delta = A_\delta^+ + A_\delta^-$ and A_δ^+, A_δ^- appear in Proposition 2.3. In view of (2.5), we have

$$\operatorname{meas}(\{|u_n|\ge k\}\cap A_\delta) \le \operatorname{meas}(A_\delta) \le 2\delta. \tag{3.34}$$

As for $\{|u_n|\ge k\}\cap A_\delta^c$, using (3.11), (3.33) and Poincaré inequality, we have

$$\begin{aligned}
&\operatorname{meas}(\{|u_n|\ge k\}\cap A_\delta^c)\\
&\le \frac{1}{(\ln(1+k))^p}\int_{\{|u_n|\ge k\}\cap A_\delta^c}\big(\ln(1+|T_k(u_n)|)\big)^p dx\\
&= \frac{1}{(\ln(1+k))^p}\int_{\{|u_n|\ge k\}\cap A_\delta^c}\big(\ln(1+|T_k(u_n)|)\big)^p(1-\psi_\delta)^s dx\\
&= \frac{C}{(\ln(1+k))^p}\int_{\{|u_n|\ge k\}\cap A_\delta^c}\big(\ln(1+|T_k(u_n)|)(1-\psi_\delta)^{\frac{s}{p}}\big)^p dx\\
&\le \frac{C}{(\ln(1+k))^p}\int_\Omega \frac{\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^p}d\mu_\delta\\
&\quad + \frac{C}{(\ln(1+k))^p}\int_{\{|u_n|\ge k\}}|\nabla\psi_\delta|^p(1-\psi_\delta)^{s-p}(\ln(1+|T_k(u_n)|))^p dx\\
&\le \frac{C}{(\ln(1+k))^p}\int_\Omega \frac{\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^p}d\mu_\delta + C\int_\Omega|\nabla\psi_\delta|^p(1-\psi_\delta)^{s-p}dx\\
&\le \frac{C(\delta)}{(\ln(1+k))^p} + C\Big(\int_\Omega|\nabla\psi_\delta|^r dx\Big)^{p/r}\\
&\le \frac{C}{(\ln(1+k))^p} + C\delta^{p/r},
\end{aligned} \tag{3.35}$$

here we have used that $1-\psi_\delta\equiv 1$ on A_δ^c by Proposition 2.3.

Combining (3.34) and (3.35), we arrive at

$$\begin{aligned}
\operatorname{meas}(\{|u_n|\ge k\}) &= \operatorname{meas}(\{|u_n|\ge k\}\cap A_\delta) + \operatorname{meas}(\{|u_n|\ge k\}\cap A_\delta^c)\\
&\le 2\delta + \frac{C}{(\ln(1+k))^p} + C\delta^{p/r},
\end{aligned}$$

which implies that u_n is a Cauchy sequence in measure.

We thus have that (up to subsequences, still denoted by u_n) u_n converges almost everywhere in Ω to some function u and

$$\alpha_1\int_\Omega \frac{|\nabla T_k(u)|^p}{(1+|T_k(u)|)^{\theta(p-1)}}d\mu_\delta + k^{q+1}\mu_\delta(\{|u|\ge k\}) \le C(\delta)k.$$

Furthermore,

$$\int_\Omega |\nabla u|^{(p-1)r'}(1-\psi_\delta)^{(s-1)r'}dx \le C(\delta).$$

Letting δ tend to zero, we find

$$\int_\Omega |\nabla u|^{(p-1)r'}dx \le C,$$

which shows that (3.25) holds. In a similar way we can prove that

$$\int_\Omega |u|^q dx \le C,$$

which is (3.26). □

Proof of Theorem 1.1. By Lemmas 3.1 and 3.2, with similar arguments as the proof of [22], we choose $T_k(u_n - T_h(u))(1-\psi_\delta)^s$ as a test function in (1.6), and show that

$$\nabla u_n(1-\psi_\delta)^s \to \nabla u(1-\psi_\delta)^s, \quad \text{almost everywhere in } \Omega.$$

We choose

$$\frac{1}{\varepsilon}T_k(G_{k-\varepsilon}(u_n))(1-\psi_\delta)^s$$

as a test function in (1.6), and arrive at

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\int_\Omega |\nabla|u_n|^p(1-\psi_\delta)^s dx = 0.$$

Then choosing $v(1-\psi_\delta)^s$ as a test function in (1.6), where $v \in C_0^\infty(\Omega)$, we can pass to the limit. More details can be found in [22, steps 4, 5, 6], so we omit them here. □

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