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FILTER REGULARIZATION FOR AN INVERSE PARABOLIC PROBLEM IN SEVERAL VARIABLES

TUAN NGUYEN HUY, MOKHTAR KIRANE, LONG DINH LE, THINH VAN NGUYEN

Abstract. The backward heat problem is known to be ill possed, which has lead to the design of several regularization methods. In this article we apply the method of filtering out the high frequencies from the data for a parabolic equation. First we identify two properties that if satisfied they imply the convergence of the approximate solution to the exact solution. Then we provide examples of filters that satisfy the two properties, and error estimates for their approximate solutions. We also provide numerical experiments to illustrate our results.

1. Introduction

The forward heat conduction problem consists of predicting the temperature of an object at a future time from the present temperature, boundary conditions, and heat source. On the other hand, the backward heat problem is an inverse problem that consists of recovering the temperature at a past time from the present temperature. Inverse problems are of great importance in engineering applications, and aim to detect a previous status from its present information. They can be applied to several areas such as image processing, mathematical finance, mechanics of continuous media, etc. The equation $u_t - b(t)\Delta u = f(x,t)$ is a simple form of the well-known advection-convection equation that appears in groundwater pollution problems and have been studied in [3].

In this article, we consider the problem of finding a function $u(x,t)$ from the given data $u(x,T) = g(x)$ in the parabolic problem

$$\begin{gathered}\frac{\partial u}{\partial t} - b(t)L[u] = f(x,t), \quad (x,t) \in \Omega \times (0,T) \\ u|_{\partial\Omega} = 0, \quad t \in (0,T) \\ u(x,T) = g(x), \quad x \in \Omega\,.\end{gathered} \tag{1.1}$$

Here Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; $b(t)$, $g(x)$, $f(x,t)$ are given functions; and L is a symmetric elliptic operator. As an example of operator L we have the negative Laplacian $-\Delta = -(u_{xx} + u_{yy} + \dots)$.

It is well-known that the backward problem is ill-posed; i.e., its solution may not exist, and if it exists, it does not depend continuously on the given data. In

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fact, small noise on the measured data may lead to solutions with large errors. This makes the numerical computation difficult, hence a regularization process is needed.

Many studies have been devoted to the regularization of (1.1). For one dimension with $b(t) = 1$ and $f(x,t) = 0$, we have the following: John [17] introduced a the idea of prescribing a bound on the solution at $t = T$ with relation on the final data g. Lattes and Lions [18], Showalter [29], and Ewing [11] used quasi-reversibility method. Ames and Epperson [2], and Miller [24] used the least squares methods with Tikhonov regularization. Lee and Sheen [20, 21] used a parallel method for backward parabolic problems. Among other researcher in this area, we have: Clark and Oppenheimer [7], Ames et al [2], Denche and Bessila [8], Tautenhahn et al [36], Melnikova et al [22, 23], Fu [14, 15], Yildiz et al [37, 38]. When $f(x,t)$ is not necessarily zero, (1.1) has been regularized by Trong et al [32, 33]. When $b(t)$ is not necessarily constant, (1.1) has been studied in [19, 34, 39].

All the above studies are for the one-dimensional problems. A filter regularization for a 3-dimensional Helmholtz equation was studied in [31]. Here apply a filter regularization to the backward problem of a multi-dimensional parabolic equation. This can be seen as an extension of the work in [25, 34].

The outline of the rest of this article is as follows. In the next section, we establish the existence and uniqueness of a solution to (1.1). In Section 3, we present the theoretical foundations of the filter regularization, and state two conditions (3.4) and (3.5) that if satisfied, approximate solutions converge to the exact solution. Also error estimates are presented there. In Section 4, we consider four regularizing filters, and present numerical experiments for two of those filters.

2. Inverse problem

We assume that $b : [0,T] \to \mathbb{R}$ is a differentiable function, and that there exist constants b_1, b_2, c_1 such that

$$0 < b_1 \le b(t) \le b_2, \quad 0 < b'(t) \le c_1 \quad \text{for all } t \in [0,T]. \tag{2.1}$$

Also we assume that $f \in L^2((0,T); L^2(\Omega))$ and $g \in L^2(\Omega)$. In the space $L^2(\Omega)$ we denote the norm by $\|\cdot\|$, and the inner product by $\langle \cdot, \cdot \rangle$.

First, we recall some properties of the elliptic operator L on a bounded open domain Ω with Dirichlet boundary conditions (see [10, Section 6.5]).

- Each eigenvalue of L is real, and the family of eigenvalues $\{\lambda_p\}_{p=1}^\infty$ satisfies $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \to \infty$ as $p \to \infty$.
- There exists an orthonormal basis $\{X_p\}_{p=1}^\infty$ for the space $L^2(\Omega)$, where $X_p \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_p; i.e., for $n \in \mathbb{N}$,

$$\begin{gathered} L[X_p](x) = \lambda_p X_p(x), \quad \text{for } x \in \Omega \\ X_p(x) = 0, \quad \text{for } x \in \partial\Omega\,. \end{gathered}$$

For $0 \le q < \infty$, let $g_p = \int_\Omega g(x) X_p(x) dx$. Then we denote by $S^q(\Omega)$ the space of functions $g \in L^2(\Omega)$ satisfying

$$\sum_{p=1}^\infty (1+\lambda)^{2q} |g_p|^2 < \infty, \tag{2.2}$$

with the norm $\|g\|^2_{S^q(\Omega)} = \sum_{p=1}^\infty (1+\lambda_p)^{2q} |g_p|^2$. When $q = 0$, $S^q(\Omega) = L^2(\Omega)$ (see [5, Chapter V], [14, page 179]).

As is well known, the forward problem

$$\begin{gathered}\frac{\partial u}{\partial t} - b(t)L[u] = f(x,t), \quad (x,t) \in \Omega \times (0,T) \\ u|_{\partial\Omega} = 0, \quad t \in (0,T) \\ u(x,0) = g(x), \quad x \in \Omega\,,\end{gathered} \tag{2.3}$$

with $f \in L^2((0,T);L^2(\Omega))$ and $g \in L^2(\Omega)$, has a unique solution. However, for the backward problem (1.1), with $f \in L^2((0,T);L^2(\Omega))$ and $g \in L^2(\Omega)$, there is no guarantee that the solution exists.

Next we obtain a solution by the Fourier series method. For a fixed t, we use the eigenfunctions of the Laplacian to write Fourier series for f, g, and u. Then (1.1) determines an ordinary first-order differential equation for the Fourier coefficients. Then the coefficients of the solution to this equation at time T are equated to coefficients of g. This process yields the following result.

Theorem 2.1. *Problem* (1.1) *has a unique solution if and only if*

$$\sum_{p=1}^{\infty} \exp\Big(2\lambda_p \int_t^T b(\xi)d\xi\Big)\Big[g_p - \int_0^T \exp\Big(-\lambda_p \int_s^T b(\xi)d\xi\Big) f_p(s)ds\Big]^2 < \infty, \tag{2.4}$$

where

$$g_p = \int_\Omega g(x)X_p(x)dx, \quad f_p(s) = \int_\Omega f(x,s)X_p(x)dx. \tag{2.5}$$

In this case the exact solution is

$$u(x,t) = \sum_{p=1}^{\infty} \exp\Big(\lambda_p \int_t^T b(\xi)d\xi\Big)\Big[g_p - \int_0^T \exp\Big(-\lambda_p \int_s^T b(\xi)d\xi\Big) f_p(s)ds\Big]\,. \tag{2.6}$$

Remark 2.2. Examples of functions f and g satisfying (2.4) are given in the section for numerical experiments. When $b(t) = 1$ and $f(x,t) = 0$, problem (1.1) has a unique solution if and only if

$$\sum_{p=1}^{\infty} e^{2T\lambda_p} |\langle g(\cdot), X_p(\cdot)\rangle|^2 < \infty,$$

as stated in [7, Lemma 1].

The proof of uniqueness uses the bounds in (2.1) and is similar to the one in [12, Corollary 2.6] and [19, page 434]; so we omit it.

In spite of the solution to problem (1.1) begin unique, it is still ill-posed and some regularization methods are necessary. In the next section, we use a regularization method for solving the problem.

3. Filter regularization method

In this section, we assume that the measured data f^ϵ and g^ϵ belong to $L^2(\Omega)$ and satisfy

$$2\|g^\epsilon - g\|^2 + 2\Big\|\int_0^T |f^\epsilon(\cdot,s) - f(\cdot,s))|\,ds\Big\|^2 \le \epsilon^2. \tag{3.1}$$

The main idea of the filter method is to multiply f^ϵ and g^ϵ by functions $R_f(\alpha,p)$ and $R_g(\alpha,p)$, respectively. These two function are called regularizing filters, and α a regularization parameter. If these two functions approach zero as $p \to \infty$, the effect that "high frequency" data have on the solution will be diminished. For simplicity we set $R(\alpha,p) = R_f(\alpha,p) = R_g(\alpha,p)$. Therefore, the approximate solution is

$$\begin{aligned} U_\alpha^\epsilon(x,t) = \sum_{p=1}^\infty & \exp\Big(\lambda_p \int_t^T b(\xi)d\xi\Big) R(\alpha,p) \\ & \times \Big[g_p^\epsilon - \int_t^T \exp\Big(-\lambda_p \int_s^T b(\xi)d\xi\Big) f_p(s)ds\Big] X_p(x)\,, \end{aligned} \tag{3.2}$$

where f_p and g_p are defined by (2.5).

Theorem 3.1. *Assume that for the exact solution u of* (1.1) *there exist constants M_p and E such that*

$$\sum_{p=1}^\infty M_p^2 |\langle u(x,t), X_p(x)\rangle|^2 \le E^2 \quad \forall t \in [0,T]\,. \tag{3.3}$$

Also assume that there exist functions $K_1(\alpha)$ and $K_2(\alpha)$ such that

$$\exp\Big(\lambda_p \int_t^T b(\xi)d\xi\Big) |R(\alpha,p)| \le K_1(\alpha) \tag{3.4}$$

$$|R(\alpha,p) - 1| \le K_2(\alpha) M_p \tag{3.5}$$

for all $p \in \mathbb{N}$ and all $t \in [0,T]$. Then U_α^ϵ, defined by (3.2)*, satisfies*

$$\|U_\alpha^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} \le K_1(\alpha)\epsilon + K_2(\alpha)E. \tag{3.6}$$

A filter $R(\alpha,p)$ is admissible if $\alpha(\epsilon)$, $K_1(\alpha)$ and $K_2(\alpha)$ tend to zero as ϵ tends to zero. By Theorem 3.1 this implies the convergence of the approximate solution to the exact solution,

$$\lim_{\epsilon \to 0} \|U_\alpha^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} = 0\,,$$

for any $t \in [0,T]$.

Proof of Theorem 3.1. The strategy is to define a function U_α and use the triangle inequality. Let

$$\begin{aligned} U_\alpha(x,t) = \sum_{p=1}^\infty & R(\alpha,p) \exp\Big(\lambda_p \int_t^T b(\xi)d\xi\Big) \\ & \times \Big[g_p - \int_t^T \exp\Big(-\lambda_p \int_s^T b(\xi)d\xi\Big) f_p(s)ds\Big] X_p(x). \end{aligned}$$

Then we have

$$
\begin{aligned}
&\|U_\alpha^\epsilon(\cdot,t)-U_\alpha(\cdot,t)\|^2\\
&=\sum_{p=1}^\infty |R(\alpha,p)|^2\exp\Big(2\lambda_p\int_t^T b(\xi)d\xi\Big)\\
&\quad\times\Big(g_p^\epsilon-g_p-\int_t^T e^{-\lambda_p\int_s^T b}\big(f_p^\epsilon(s)-f_p(s)\big)ds\Big)^2\\
&\le\sum_{p=1}^\infty |R(\alpha,p)|^2\exp\Big(2\lambda_p\int_t^T b(\xi)d\xi\Big)\\
&\quad\times 2\Big((g_p^\epsilon-g_p)^2+\Big(\int_t^T|f_p^\epsilon(s)-f_p(s)|ds\Big)^2\Big)\\
&\le|K_1(\alpha)|^2 2\Big(\|g^\epsilon-g\|^2+\Big\|\int_t^T|f^\epsilon(\cdot,s)-f(\cdot,s)|\,ds\Big\|^2\Big)
\end{aligned}
\tag{3.7}
$$

Here we used that $(\alpha+\beta)^2\le\frac12(\alpha^2+\beta^2)$ and that $e^{-\lambda_p\int_s^T b}\le 1$ because λ_p and b are non-negative. The squared integral is estimated using Fubini's theorem as follows. Let $\phi(x)=\int_0^T h(x,s)\,ds$ then the Fourier coefficients satisfy

$$
\phi_p=\int_\Omega\int_t^T h(x,s)\,dsX_p(x)\,dx=\int_t^T\int_\Omega h(x,s)X_p(x)\,dx\,ds=\int_t^T h_p(s)\,ds
$$

By Parseval's equality,

$$
\|\phi\|^2=\sum_{p=1}^\infty\phi_p^2=\sum_{p=1}^\infty\Big(\int_0^T h_p(s)\,ds\Big)^2=\|\int_0^T h(\cdot,s)\,ds\|^2\,.
$$

From (3.1) and (3.7), we have

$$
\|U_\alpha^\epsilon(\cdot,t)-U_\alpha(\cdot,t)\|\le K_1(\alpha)\epsilon \tag{3.8}
$$

From the definition of U_α, we have

$$
\begin{aligned}
\|U_\alpha(\cdot,t)-u(\cdot,t)\|^2&=\sum_{p=1}^\infty[R(\alpha,p)-1]^2\exp\Big(2\lambda_p\int_t^T b(\xi)d\xi\Big)\\
&\quad\times\Big[g_p-\int_t^T\exp\Big(-\lambda_p\int_s^T b(\xi)d\xi\Big)f_p(s)ds\Big]^2\\
&=\sum_{p=1}^\infty[R(\alpha,p)-1]^2|\langle u(x,t),X_p(x)\rangle|^2\\
&\le|K_2(\alpha)|^2\sum_{p=1}^\infty M_p^2|\langle u(x,t),X_p(x)\rangle|^2\\
&\le|K_2(\alpha)|^2E^2.
\end{aligned}
$$

This inequality, (3.8), and the triangle inequality complete the proof. □

Remark 3.2. Assumption (3.3) holds naturally when $M_p=\lambda_p^k$ for any $k>0$. In this case

$$
\sum_{p=1}^\infty M_p^2|\langle u(x,t),X_p(x)\rangle|^2=\sum_{p=1}^\infty\lambda_p^{2k}|\langle u(x,t),X_p(x)\rangle|^2=E^2=\|u\|^2_{S^k(\Omega)}\,,
$$

where $S^k(\Omega)$ is defined in Section 2.

Next we present specific filters and their regularized solutions.

Proposition 3.3. *As in* [19], *let*

$$R_1(\alpha, p) = \frac{1}{1 + \alpha \exp\big(\lambda_p \int_0^T b(\xi)d\xi\big)}\,,$$

and $\alpha(\epsilon) = \epsilon^{(1-m)b_1/b_2}$ *with* $m \in (0,1)$ *and* b_1 *and* b_2 *as in* (2.1). *Then* R_1 *satisfies* (3.4) *and* (3.5) *with*

$$K_1(\alpha) = \alpha^{-b_2/b_1}, \quad K_2(\alpha) = \frac{\int_0^T b(\xi)d\xi}{\ln\big(\lambda_1 \int_0^T b(\xi)d\xi/\alpha\big)}, \quad M_p = \lambda_p\,.$$

From [19, Lemma 2], we have

$$\begin{aligned}
R_1(\alpha, p)\exp\Big(\lambda_p \int_t^T b(\xi)d\xi\Big) &= \frac{\exp\big(-\lambda_p \int_0^t b(\xi)d\xi\big)}{\alpha + \exp\big(-\lambda_p \int_0^T b(\xi)d\xi\big)}\\
&\le \alpha^{\frac{b_2 t}{b_1} - \frac{b_2}{b_1}}\\
&\le \alpha^{-b_2/b_1} = K_1(\alpha)\,.
\end{aligned}$$

To verify condition (3.5), we apply the elementary estimate

$$\frac{1}{\alpha z + e^{-Mz}} \le \frac{M}{\alpha \ln(M/\alpha)}$$

for $M > 0$ and α small enough. Therefore,

$$\begin{aligned}
|R_1(\alpha, p) - 1| &= \frac{\alpha}{\alpha + \exp\big(-\lambda_p \int_0^T b(\xi)d\xi\big)}\\
&= \frac{\alpha\lambda_p}{\alpha\lambda_p + \lambda_p \exp\big(-\lambda_p \int_0^T b(\xi)d\xi\big)}\\
&\le \frac{\alpha\lambda_p}{\alpha\lambda_p + \lambda_1 \exp\big(-\lambda_p \int_0^T b(\xi)d\xi\big)}\\
&\le \frac{\int_0^T b(\xi)d\xi}{\ln\big(\lambda_1 \int_0^T b(\xi)d\xi/\alpha\big)}\lambda_p = K_2(\alpha)M_p.
\end{aligned}$$

Proposition 3.4. *For* $k \ge 1$, *Let*

$$R_2(\alpha, p) = \frac{1}{1 + \epsilon\lambda_p^k \exp\big(\lambda_p \int_0^T b(\xi)d\xi\big)}\,.$$

Then R_2 *satisfies* (3.4) *and* (3.5) *with* $\alpha(\epsilon) = \epsilon$,

$$K_1(\alpha) = b_4\epsilon^{-1}\Big(\ln\big(\frac{b_3}{\epsilon}\big)\Big)^{-k}, \quad K_2(\alpha) = b_4\Big(\ln\big(\frac{b_3}{\epsilon}\big)\Big)^{-k}, \quad M_p = \lambda_p^k\,,$$

where $b_3 = (b_2 T)^k/k$ *and* $b_4 = (k b_2 T)^k$.

To prove the above proposition, we need the following Lemma.

Lemma 3.5. *For* $M, \epsilon, x > 0$, $k \ge 1$, *we have the inequality*

$$\frac{1}{\epsilon x^k + e^{-Mx}} \le \frac{(kM)^k}{\epsilon \ln^k\big(\frac{M^k}{k\epsilon}\big)}\,.$$

Proof. Let $f(x) = \frac{1}{\epsilon x^k + e^{-Mx}}$. Then

$$f'(x) = \frac{\epsilon k x^{k-1} - Me^{-Mx}}{-(\epsilon x^k + e^{-Mx})^2}\,.$$

The only critical point x_0 satisfies $x_0^{k-1}e^{Mx_0} = \frac{M}{k\epsilon}$ and yields a maximum. Hence

$$f(x) \le \frac{1}{\epsilon x_0^k + e^{-Mx_0}} = \frac{1}{\epsilon x_0^k + \frac{k\epsilon}{M}x_0^{k-1}}\,.$$

By using the inequality $e^{Mx_0} \ge Mx_0$, we obtain

$$\frac{M}{k\epsilon} = x_0^{k-1}e^{Mx_0} \le \Big(\frac{e^{Mx_0}}{M}\Big)^{(k-1}e^{Mx_0} = \frac{1}{M^{k-1}}e^{kMx_0}.$$

This gives $e^{kMx_0} \ge \frac{M^k}{k\epsilon}$ and $kMx_0 \ge \ln(\frac{M^k}{k\epsilon})$. Therefore $x_0 \ge \frac{1}{kM}\ln(\frac{M^k}{k\epsilon})$. Hence, we obtain

$$f(x) \le \frac{1}{\epsilon x_0^k} \le \frac{(kM)^k}{\epsilon \ln^k(\frac{M^k}{k\epsilon})}\,.$$

□

Proof of Proposition 3.4. Condition (3.4) is obtained as follows

$$\begin{aligned} R_2(\epsilon,p)\exp\Big(\lambda_p\int_t^T b(\xi)d\xi\Big) &= \frac{\exp\big(-\lambda_p\int_0^t b(\xi)d\xi\big)}{\epsilon\lambda_p^k + \exp\big(-\lambda_p\int_0^T b(\xi)d\xi\big)} \\ &\le \frac{1}{\epsilon\lambda_p^k + \exp\big(-b_2T\lambda_p\big)}\,. \end{aligned}$$

Using the inequality

$$\frac{1}{\epsilon x^k + e^{-b_2Tx}} \le (kTb_2)^k\epsilon^{-1}\Big(\ln(\frac{(b_2T)^k}{k\epsilon})\Big)^{-k} = b_4\epsilon^{-1}\Big(\ln(\frac{b_3}{\epsilon})\Big)^{-k},$$

we conclude that

$$R_2(\epsilon,p)\exp\Big(\lambda_p\int_t^T b(\xi)d\xi\Big) \le b_4\epsilon^{-1}\Big(\ln(\frac{b_3}{\epsilon})\Big)^{-k} = K_1(\epsilon)\,.$$

We derive Condition (3.5) as follows

$$\begin{aligned} |R_2(\epsilon,p)-1| &= \frac{\epsilon\lambda_p^k}{\epsilon\lambda_p^k + \exp\big(-\lambda_p\int_0^T b(\xi)d\xi\big)} \\ &\le \epsilon\lambda_p^k b_4\epsilon^{-1}\Big(\ln(\frac{b_3}{\epsilon})\Big)^{-k} \\ &= b_4\Big(\ln(\frac{b_3}{\epsilon})\Big)^{-k} = K_2(\epsilon)M_p\,. \end{aligned}$$

□

Proposition 3.6. *Let*

$$R_3(\alpha,p) = \begin{cases} 1, & \text{if } \lambda_p \le 1/\alpha, \\ 0, & \text{if } \lambda_p > 1/\alpha\,. \end{cases}$$

where $\alpha = b_2T/\ln(1/\epsilon)$. *Then* R_3 *satisfies* (3.4) *and* (3.5) *with* $\alpha(\epsilon) = \epsilon$,

$$K_1(\alpha) = \epsilon^{-1}, \quad K_2(\alpha) = \alpha, \quad M_p = \lambda_p.$$

Proof. Condition (3.4) is obtained as

$$
\begin{aligned}
R_3(\alpha,p)\exp\Big(\lambda_p\int_t^T b(\xi)d\xi\Big) &= \begin{cases}\exp\Big(\lambda_p\int_t^T b(\xi)d\xi\Big), & \text{if } \lambda_p\le 1/\alpha\\ 0, & \text{if } \lambda_p>1/\alpha\end{cases}\\
&\le \exp\Big(\frac{1}{\alpha}\int_t^T b(\xi)d\xi\Big)\le \epsilon^{-1}\,.
\end{aligned}
$$

Condition (3.5) follows from

$$
\begin{aligned}
|R_3(\alpha,p)-1| &= \begin{cases}0, & \text{if } \lambda_p\le 1/\alpha\\ 1, & \text{if } \lambda_p>1/\alpha\end{cases}\\
&\le \alpha\lambda_p = K_2(\alpha)M_p\,.
\end{aligned}
$$

□

Proposition 3.7. *Let*

$$
R_4(\alpha,p)=\exp\Big(-\alpha\lambda_p^2\int_0^T b(\xi)d\xi\Big).
$$

Then R_4 satisfies (3.4) *and* (3.5) *with* $\alpha(\epsilon)=\epsilon$,

$$
K_1(\alpha)=\exp\Big(\frac{\int_0^T b(\xi)d\xi}{4\alpha}\Big),\quad K_2(\alpha)=\alpha,\quad M_p=\lambda_p^2\,.
$$

Proof. Conditions (3.4) and (3.5) follow from

$$
\begin{aligned}
R_4(\alpha,p)\exp\Big(\lambda_p\int_t^T b(\xi)d\xi\Big) &= \exp\Big((\lambda_p-\alpha\lambda_p^2)\int_t^T b(\xi)d\xi\Big)\\
&\le \exp\Big(\frac{\int_0^T b(\xi)d\xi}{4\alpha}\Big)=K_1(\alpha),
\end{aligned}
$$

where we used that $\lambda_p-\alpha\lambda_p^2\le\frac{1}{4\alpha}$. Using the inequality $1-e^{-z}\le z$ for $z>0$, we obtain

$$
|R_4(\alpha,p)-1| = 1-\exp\Big(-\alpha\lambda_p^2\int_0^T b(\xi)d\xi\Big)\le \alpha\lambda_p^2\int_0^T b(\xi)d\xi\le K_2(\alpha)M_p\,,
$$

where $M_p=\lambda_p^2$. □

4. Numerical experiments

Since numerical experiments were implemented for filter R_1 in [19], we implement experiments only for R_2 and R_3. The efficiency of the methods is observed by comparing the errors between numerical and exact solutions. In both examples, we choose the exponent $k=1$, and consider (1.1) in a two-dimensional region. Let $\Omega=(0,a)\times(0,b)$ be an open rectangle in $\mathbb{R}^2$, and $T>0$. Let us consider

$$
\begin{gathered}
u_t-b(t)\big(u_{xx}+u_{yy}\big)=f(x,y,t),\quad (x,y)\in\Omega,\ t\in[0,T]\\
u(x,y,t)=0,\quad (x,y)\in\partial\Omega,\quad t\in[0,T]\\
u(x,y,T)=g(x,y),\quad (x,y)\in\Omega.
\end{gathered}
$$

The eigenfunctions and eigenvalues of the Laplacian are

$$
\psi_{mn}(x,y)=\frac{2}{\sqrt{ab}}\sin\big(\frac{m\pi x}{a}\big)\sin\big(\frac{n\pi y}{b}\big),
$$

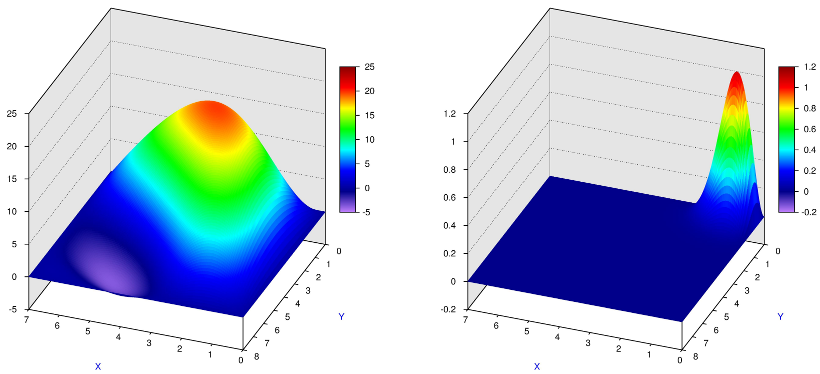

FIGURE 1. Exact solution at $t = 0$, and $t = 1$

TABLE 1. Absolute error estimate for mesh resolution $M = N = 127$, $\Delta x = 2.7559E - 02$, $\Delta y = 3.1496E - 02$.

	$\epsilon = 10^{-1}$		$\epsilon = 10^{-2}$		$\epsilon = 10^{-3}$		$\epsilon = 10^{-4}$	
t	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{1,2}$	$\delta_{1,3}$
0.00	5.616E-01	7.045E-01	7.263E-02	1.338E-01	2.878E-02	7.796E-02	3.954E-02	6.457E-02
0.11	2.034E-01	3.487E-01	2.678E-02	3.911E-02	9.921E-03	2.116E-02	2.078E-02	1.995E-02
0.22	1.432E-01	4.184E-01	1.976E-02	1.085E-01	6.500E-03	2.284E-02	1.299E-02	1.356E-02
0.33	1.120E-01	3.563E-01	1.635E-02	1.512E-01	4.972E-03	4.830E-02	8.450E-03	1.552E-02
0.44	9.195E-02	2.867E-01	1.418E-02	1.584E-01	4.096E-03	6.950E-02	5.649E-03	2.551E-02
0.55	7.775E-02	2.307E-01	1.263E-02	1.494E-01	3.540E-03	8.006E-02	3.905E-03	3.573E-02
0.66	6.709E-02	1.882E-01	1.144E-02	1.352E-01	3.167E-03	8.281E-02	2.855E-03	4.296E-02
0.77	5.875E-02	1.559E-01	1.047E-02	1.203E-01	2.907E-03	8.104E-02	2.279E-03	4.704E-02
0.88	5.205E-02	1.311E-01	9.669E-03	1.066E-01	2.718E-03	7.696E-02	2.011E-03	4.866E-02
0.99	4.654E-02	1.118E-01	8.978E-03	9.441E-02	2.575E-03	7.189E-02	1.917E-03	4.860E-02

$$\lambda_{mn} = \big(\frac{m\pi}{a}\big)^2 + \big(\frac{n\pi}{b}\big)^2,$$

for $(m, n) \in \mathbb{N}^2$. When

$$b(t) = \frac{1}{100 + \exp(t^2)}$$

this problem has exact solution

$$u(x, y, t) = e^{-t(x^2+y^2)} \sin\big(\frac{xy}{a + t}\big)(a - x)(b - y)\,.$$

For the numerical computations we use $a = 7$, $b = 8$, and $T = 1$. The source function f and the final datum $g(x, y) = u(x, y, T)$ are such that u is the exact solution of the problem.

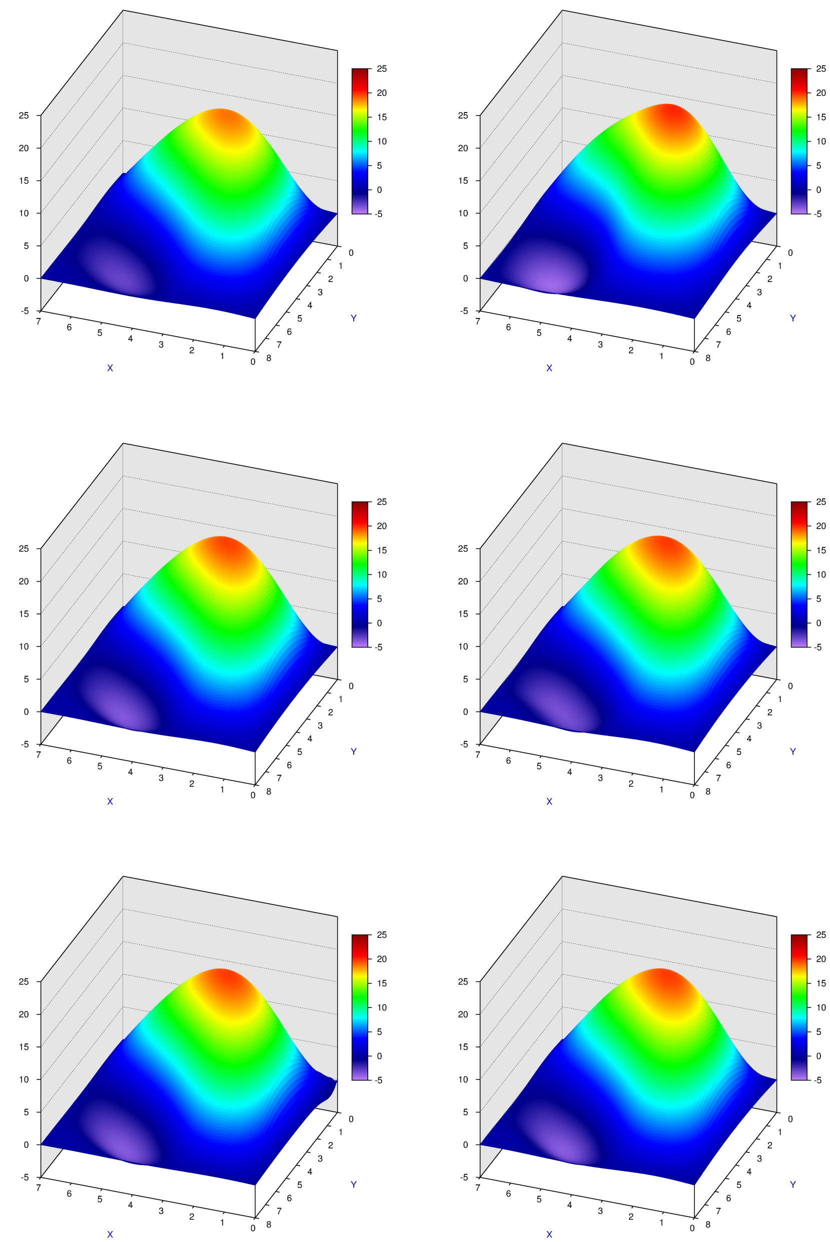

FIGURE 2. Numerical solutions at $t = 0$ for filters R_2 (left) and R_3 (right) with $\epsilon = 10^{-1}$, $\epsilon = 10^{-2}$, $\epsilon = 10^{-4}$ (from top to bottom)

For the measured data f^ϵ and g^ϵ, we use a random number generator rand() in $(-1, 1)$,

$$g^\epsilon(x, y) = g(x, y) + \frac{\epsilon}{\pi}\,\mathrm{rand}(), \quad f^\epsilon = f\,.$$

At a given ϵ and t, the absolute error between the exact solution and the regularized solutions is estimated by

$$\delta_{1,l} = \Big(\frac{\sum_{i=1}^{M}\sum_{j=1}^{N}|u^{l,\epsilon}(x_i,y_j,t)-u(x_i,y_j,t)|^2}{(M)(N)}\Big)^{1/2}. \tag{4.1}$$

Regularized solutions by filter R_2 correspond to $l=2$, and by filter R_3 to $l=3$. We choose a calculation grid of 127×127 interior points, with $x_i = i\pi/I$, $y_j = j\pi/J$, and $u^{l,\epsilon}(x,y,t)$. See Table 1.

Figure 1 shows the exact solution while Figure 2 shows the regularized solutions at $t=0$. From Table 1 we see that overall filter R_2 gives a better approximation than filter R_3. Both regularized solutions converge to the exact solution at $t=0$. However, when t close to 1 ($t=0.99$) the solution from filter R_3 is strongly oscillating and slowly converges to the exact solution. In comparison the convergence rate of filter R_2 is significant better than the convergence rate of the filter R_3.

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References

[1] K. Ames, G. Clark, F. Epperson; *A comparison of regularizations for an ill-posed problem*, Math. Comp., 67 (1998), No. 224, 1451-1471.

[2] K. A. Ames, J.F. Epperson; *A kernel-based method for the approximate solution of backward parabolic problems*, SIAM J. Numer. Anal., Vol. 34 (1997), No. 4, 127-145.

[3] J. Atmadja, A.C. Bagtzoglou; *Marching-jury backward beam equation and quasi-reversibility methods for hydrologic inversion: Application to contaminant plume spatial distribution recovery.* WRR 39, 1038C1047 (2003).

[4] F. Berntsson; *A spectral method for solving the sideways heat equation*, Inverse Problem 15, (1999) No. 4, 891-906.

[5] H. Brezis; *Analyse Fonctionelle,* Masson, Paris, 1983.

[6] V. Burmistrova; *Regularization method for parabolic equation with variable operator*, J. Appl. Math., 2005, No. 4, 382-392.

[7] G. W. Clark, S. F. Oppenheimer; *Quasireversibility methods for non-well posed problems*, Elect. J. Diff. Eqns., Vol. 301, 1994, No. 8 pp. 1-9.

[8] M. Denche, K. Bessila; *A modified quasi-boundary value method for ill-posed problems*, J. Math. Anal. Appl, Vol. 301 (2005), pp. 419-426.

[9] L. Elden, F. Berntsson, T. Reginska; *Wavelet and Fourier methods for solving the sideways heat equation*, SIAM J. Sci. Comput., 21 (2000), No. 6, 2187-2205.

[10] L. C. Evans; *Partial Differential Equation,* American Mathematical Society, Providence, Rhode Island, Volume 19 (1997).

[11] R. E. Ewing; *The approximation of certain parabolic equations backward in time by Sobolev equation*, SIAM J. Math. Anal., Vol. 6 (1975), No. 2, 283-294.

[12] X. L. Feng, Lars Elden, C. L. Fu; *Stability and regularization of a backward parabolic PDE with variable coefficient.* J. Inverse and Ill-posed Problems, Vol. 18 (2010), 217-243.

[13] X. L. Feng, Lars Elden, Chu-Li Fu; *A quasi-boundary-value method for the Cauchy problem for elliptic equations with nonhomogeneous Neumann data* J. Inverse Ill-Posed Probl., 18 (2010), No. 6, 617-645.

[14] X. L. Feng, Z. Qian, C. L. Fu; *Numerical approximation of solution of nonhomogeneous backward heat conduction problem in bounded region.* J. Math. Comp. Simulation 79 (2008), No. 2, 177-188.

[15] C. L. Fu, X. T. Xiong, Z. Qian; *Fourier regularization for a backward heat equation* J. Math. Anal. Apl., 331 (2007), No. 1, 472-480.

[16] V. Isakov; *Inverse Problems for Partial Differential Equation*, Springer-Verlag, New York, 1998.

[17] F. John; *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math, 13 (1960), 551-585.

[18] R. Lattès, J.L. Lions; *Methode de Quasi-Reversibilité et Application*, Dunod, Paris, 1967.

[19] Triet M. Le, Q. H. Pahm, T. D. Dang, H. T. Nguyen; *A backward parabolic equation with a time-dependent coefficient: regularization and error estimates.* J. Comput. Appl. Math., 237 (2013), No. 1, 432-441.

[20] J. Lee, D. Sheen; *A parallel method for backward parabolic problem based on the Laplace transformation*, SIAM J.Nummer. Anal., 44 (2006), No. 4, pp 1466-1486.

[21] J. Lee, D. Sheen; *F. John's stability conditions versus A. Carasso's SECB constraint for backward parabolic problems*, Inverse Problem 25 (2009), No. 5, 055001.

[22] I. V. Melnikova, S. V. Bochkareva; *C-semigroups and regularization of an ill-posed Cauchy problem*, Dok. Akad. Nauk., 329 (1993), 270-273.

[23] I. V. Melnikova, A. I. Filinkov; *The Cauchy problem. Three approaches, Monograph and Surveys in Pure and Applied Mathematics* 120, London - New York: Chapman and Hall, 2001.

[24] K. Miller; *Least Squares Methods for Ill-Posed Problems with a prescribed bound*, SIAM J. Math. Anall., (1970), 52-74.

[25] Pham Hoang Quan, Dang Duc Trong, Le Minh Triet; Nguyen Huy Tuan; *A modified quasi-boundary value method for regularizing of a backward problem with time-dependent coefficient.* Inverse Probl. Sci. Eng., 19 (2011), No. 3, 409-423.

[26] P. T. Nam, D. D. Trong, N. H. Tuan; *The truncation method for a two-dimensional nonhomogeneous backward heat problem* Appl. Math. Comput., 216 (2010), No. 12, 3423-3432.

[27] L. E. Payne; *Some general remarks on improperly posed problems for partial differential equations* Symposium on Non-well Posed Problems and Logarithmic Convexity, Lecture Notes in Mathematics, 316 (1973), Springer-Verlarge, Berlin, 1-30

[28] A. Shidfar, A. Zakeri; *A numerical technique for backward inverse heat conduction problems in one - dimensional space* Appl. Math. Comput., 171 (2005), No. 2, 1016-1024.

[29] R. E. Showalter; *The final value problem for evolution equations*, J. Math. Anal. Appl, **47** (1974), 563-572.

[30] R. E. Showalter; *Cauchy problem for hyper - parabolic partial differential equations*, in Trends in the Theory and Practice of Non-Linear Analysis, Elsevier 1983.

[31] Q. V. Tran, H. T. Nguyen, V. T. Nguyen, D. T. Dang; *A general filter regularization method to solve the three dimensional Cauchy problem in inhomogeneous Helmholtz-type equatons: Theory and numerical simulation*, Apied Math. Modelling, 38 (2014), 4460-4479.

[32] D. D. Trong, N. H. Tuan; *Regularization and error estimates for nonhomogeneous backward heat problem*, Electron. J. Diff. Eqns., Vol. 2006 , No. 04, pp. 1-10.

[33] D. D. Trong, N. H. Tuan; *A nonhomogeneous backward heat problem: Regularization and error estimates*, Electron. J. Diff. Eqns., Vol. 2008 , No. 33, pp. 1-14.

[34] N. H. Tuan, P. H. Quan, D. D. Trong, T. Le Minh; *On a backward heat problem with time-dependent coefficient: Regularization and error estimates*, Appl. Math. Comp., 219 (2013) 6066–6073.

[35] N. H. Tuan, D. D. Trong; *A nonlinear parabolic equation backward in time: regularization with new error estimates*, Nonlinear Anal., 73 (2010), No. 6, 1842-1852.

[36] T. Schroter, U. Tautenhahn; *On optimal regularization methods for the backward heat equation*, Z. Anal. Anw., 15 (1996), 475-493

[37] B. Yildiz, M. Ozdemir; *Stability of the solution of backward heat equation on a weak conpactum*, Appl. Math. Comput., 111 (2000), 1-6.

[38] B. Yildiz, H. Yetis, A. Sever; *A stability estimate on the regularized solution of the backward heat problem*, Appl. Math. Comp., 135 (2003), 561-567.

[39] Y. X. Zhang, C.-L. Fu, Chu-Li; Y-J. Ma; *An a posteriori parameter choice rule for the truncation regularization method for solving backward parabolic problems*, J. Comput. Appl. Math., 255 (2014), 150-160.

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