Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 231, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

POSITIVE SOLUTIONS OF MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. This article concerns the boundary value problem consisting of the nonlinear differential equation

$$u'' + g(t)f(t, u(t)) = 0, \quad t \in (0, 1)$$

and the multi-point boundary conditions

$$u(0) = \alpha u'(0),$$
$$u(1) = \sum_{i=1}^{m} \beta_i u(\eta_i) + \sum_{i=1}^{m} \gamma_i u'(\eta_i),$$

where $0 \le \alpha \le \infty$, $0 < \eta_1 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $\beta_i > 0$, $\gamma_i < 0$ $(i = 1, 2, \dots, m)$. By using the fixed point index theory, we establish the existences of at least one positive solution and at least two positive solutions.

1. Introduction

We consider the boundary value problem (BVP) consisting of the nonlinear differential equation

$$u'' + g(t)f(t, u(t)) = 0, \quad t \in (0, 1) \tag{1.1}$$

and the multi-point boundary conditions

$$u(0) = \alpha u'(0), \tag{1.2}$$

$$u(1) = \sum_{i=1}^{m} \beta_i u(\eta_i) + \sum_{i=1}^{m} \gamma_i u'(\eta_i), \tag{1.3}$$

where $0 \le \alpha \le \infty$, $0 < \eta_1 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $\beta_i > 0$, $\gamma_i < 0$ $(i = 1, 2, \dots, m)$.

Boundary value problems of ordinary differential equations arise in a variety of areas of applied mathematics and physics [9, 10]. Since Il'in and Moiseev [4] first studied the existence of solutions for a linear boundary value problem, more and more papers have been devoted to studying the existence of positive solutions of BVPs. Many phenomena can be modeled by (1.1) such as the Emden-Fowler equation, the Thomas-Fermi equation, etc, see [11].

2010 *Mathematics Subject Classification.* 34B10, 34B18.
Key words and phrases. Boundary value problems; multi-point boundary conditions; second-order nonlinear differential equations; positive solutions; fixed point index.

Submitted December 30, 2015. Published August 24, 2016.

In 1999, Ma [7] studied the three-point boundary value problem consisting of (1.1) and

$$u(0) = 0, \quad u(1) = \alpha u(\eta), \quad 0 < \eta < 1, \alpha\eta < 1. \tag{1.4}$$

In 2001, Ma [8] discussed (1.1) and Neumann boundary conditions at $t = 0$ as follows:

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m} \beta_i u(\eta_i). \tag{1.5}$$

He used the theorem on compression and expansion of a cone to discuss the existence of positive solutions when f is either sublinear or superlinear.

In 2001, Webb [11] studied the three-point boundary value problems consisting of (1.1) and either (1.4) or

$$u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad 0 < \eta < 1, \alpha < 1. \tag{1.6}$$

He applied the classical fixed point index theory to prove the existence of at least two positive solutions.

In 2004, Zhang and Sun [14] considered the m-point boundary value problem of (1.1) and

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m} \beta_i u(\eta_i), \tag{1.7}$$

where $\beta_i > 0$, $\sum_{i=1}^{m} \beta_i < 1$, $0 < \eta_i < 1$ $(i = 1, 2, \dots, m)$. The existences of one positive solution and multiple positive solutions were obtained by means of fixed point index theorem under some conditions concerning the first eigenvalue with respect to the linear operator.

Karakostas and Tsamatos [5] considered the second order ordinary differential equation

$$(p(t)x'(t))' + \mu(t)f(x(t)) = 0, \quad t \in [0, 1], \tag{1.8}$$

associated with the nonlocal boundary conditions

$$x'(0) = \int_0^1 x'(s)ds dg(s), \quad x(1) = -\int_0^1 x'(s)ds dh(s), \tag{1.9}$$

or

$$x'(1) = \int_0^1 x'(s)ds dg_1(s), \quad x(0) = -\int_0^1 x'(s)ds dh_1(s). \tag{1.10}$$

They used the Krasnoselskii's fixed point theorem on a suitable cone, several existence results for multiple positive solutions of a Fredholm integral equation are provided. In 2006, Webb and Lan [12] discussed the existence of multiple positive solutions of a second order differential of the form

$$x''(t) + g(t)f(t, x(t)) = 0, \quad t \in [0, 1], \tag{1.11}$$

under a variety of boundary conditions which include separated boundary conditions and non-local boundary conditions known as m-point boundary conditions boundary conditions.

In 2014, Wong and Kong [13] considered the differential equation $u'' + f(t, u(t)) = 0$ with the following multi-point boundary conditions

$$\cos\theta u(0) = \sin\theta u'(0) \tag{1.12}$$

and

$$u(1)=\sum_{i=1}^{m}\beta_i u(\eta_i)+\sum_{i=1}^{m}\gamma_i u'(\eta_i), \tag{1.13}$$

where $\beta_i,\gamma_i\in R$, $0<\eta_1<\eta_1<\eta_2<\cdots<\eta_m<1$. They used the Leray Schauder Nonlinear Alternative to obtain the existence of one non-trivial solution.

Motivated by these facts, we shall use the kernel function and the fixed point index theorem to obtain the existences of at least one positive solution and at least two positive solutions for boundary value problems (1.1)–(1.3). This paper is organized as follows: after the introduction, some preliminary results are stated in Section 2, and main results are shown in Section 3.

2. Preliminaries

We start by presenting our results via the Hammerstein integral equation

$$Tu(t):=u(t)=\int_0^1 k(t,s)g(s)f(s,u(s))ds \quad t\in[0,1]. \tag{2.1}$$

In the Banach space $C[0,1]$, with the norm $\|u\|=\max_{0\le t\le 1}|u(t)|$, we set $P=\{u\in C[0,1]: u(t)\ge 0, t\in[0,1]\}$. P is a positive cone in $C[0,1]$. Throughout this paper, the partial ordering is always given by P. We denote by $B_r=\{u\in C[0,1]\mid \|u\|<r, r>0\}$ the open ball of radius r. We make the following assumptions:

- (A1) Function $g:[0,1]\to[0,\infty)$ is continuous, $g(t)\not\equiv 0$, and $\int_0^1 g(t)dt<\infty$;
- (A2) Function $f:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$ satisfies Carathéodory conditions, that is, $f(\cdot,u)$is measurable for each fixed $u\in\mathbb{R}_+$, $f(t,\cdot)$ is continuous for almost every $t\in[0,1]$, and for each $r>0$ there exists $\phi_r\in L^\infty[0,1]$ such that $0\le f(t,u)\le\phi_r$ for all $u\in[0,r]$ and almost all $t\in[0,1]$;
- (A3) $0<\Delta<1+\alpha$, $\Delta=1+\alpha(1-\sum_{i=1}^m\beta_i)-\sum_{i=1}^m\gamma_i-\sum_{i=1}^m\beta_i\eta_i$.

A function u is said to be a positive solution of (1.1) if $u\in C[0,1]\cap C^2(0,1)$, $u(t)>0, t\in(0,1)$ and satisfies (1.8). We set

$$k_1(t,s)=\begin{cases}\frac{(1-t)(s+\alpha)}{1+\alpha}, & 0\le s\le t\le 1,\\ \frac{(1-s)(t+\alpha)}{1+\alpha}, & 0\le t\le s\le 1,\end{cases}$$

and for $i=1,2,\dots,m$,

$$\tilde{k}_i(t,s)=\begin{cases}-\frac{(\beta_i(\eta_i-s)+\gamma_i)(t+\alpha)}{\Delta}, & 0\le s\le\eta_i,\ t\in[0,1],\\ 0 & \eta_i\le s\le 1, t\in[0,1],\end{cases}$$

$$k(t,s)=k_1(t,s)+\sum_{i=1}^{m}\tilde{k}_i(t,s)+\frac{(1+\alpha-\Delta)(t+\alpha)(1-s)}{(1+\alpha)\Delta}.$$

Obviously, $k(t,s)$ is continuous on $[0,1]\times[0,1]$ and $k(t,s)\ge 0 (0\le t,s\le 1)$ by the assumption (A3). We define a linear operator

$$Lu(t):=\int_0^1 k(t,s)g(s)u(s)ds, \quad t\in[0,1]. \tag{2.2}$$

Lemma 2.1 ([2]). *Let E be a Banach space, and P be a cone in E, and $\Omega(P)$ be a bounded open set in P. Suppose that $T:\overline{\Omega(P)}\to P$ is a completely continuous operator.*

(1) *If there exists $u_0 \in P\backslash\{\theta\}$ such that*
$$u - Tu \neq \mu u_0, \quad \forall u \in \partial\Omega(P),\ \mu \geq 0,$$
then the fixed point index $i(T, \Omega(P), P) = 0$.
(2) *If $Tu \neq \mu u$ for all $u \in \partial\Omega(P)$, $\mu \geq 1$, then $i(T, \Omega(P), P) = 1$.*
(3) *Let P_1 be an open set in E such that $\bar{P_1} \subset P$. If $i(T, \Omega(P), P) = 1$ and $i(T, \Omega(P_1), P) = 0$, Then T has a fixed point in $\bar{P_1}\backslash P$. The same result holds if $i(T, \Omega(P), P) = 0$ and $i(T, \Omega(P_1), P) = 1$.*

Lemma 2.2 ([1])**.** *Let E be a Banach space, and P be a cone in E, and $\Omega(P)$ be a boundary open set in P. Suppose that $T : \overline{\Omega(P)} \to P$ is a completely continuous operator.*

(1) *If $\|Tu\| > \|u\|$ for all $u \in \partial\Omega(P)$, then the fixed point index $i(T, \Omega(P), P) = 0$;*
(2) *If $\theta \in \Omega(P)$ and $\|Tu\| \leq \|u\|$ for all $u \in \partial\Omega(P)$, then the fixed point index $i(T, \Omega(P), P) = 1$.*

Lemma 2.3 ([6])**.** *Under hypotheses* (A1)–(A3)*, the map T defined in* (2.1) *maps P to P and is completely continuous operator.*

It is easy to prove that the linear operator $L : C[0, 1] \to C[0, 1]$ is completely continuous and $L(P) \subset P$. We shall use the well known Krein-Rutman theorem [3].

Lemma 2.4. *Suppose that $L : [0, 1] \to C[0, 1]$ is a completely continuous linear operator and $L(P) \subset P$. If there exist $\Psi \in C[0, 1]\backslash(-P)$ and a constant $c > 0$ such that $cL\Psi \geq \Psi$, then the spectral radius $r(L) \neq 0$ and L has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$, that is $\varphi_1 = \lambda_1 L\varphi_1$.*

Lemma 2.5 ([14])**.** *Suppose that* (A1)–(A3) *are satisfied, then for the operator L defined by* (2.2)*, the spectral radius $r(L) \neq 0$ and L has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$.*

3. Main results

In this article, we use the following definitions:
$$\bar{f}(u) := \sup_{t\in[0,1]} f(t, u), \quad \underline{f}(u) := \inf_{t\in[0,1]} f(t, u),$$
$$f^0 = \limsup_{u\to 0^+} \bar{f}(u)/u, \quad f_0 = \liminf_{u\to 0^+} \underline{f}(u)/u,$$
$$f^\infty = \limsup_{u\to\infty} f(u)/u, \quad f_\infty = \liminf_{u\to\infty} \underline{f}(u)/u.$$

Theorem 3.1. *If $\lambda_1 < f_0 < \infty$, then there exists $R_1 > 0$ such that $i(T, B_R \cap P, P) = 0$ for each $R \in (0, R_1]$.*

Proof. If $\lambda_1 < f_0 < \infty$, let $\epsilon > 0$ satisfy $f_0 > (\lambda_1 + \epsilon)u$ and then there exists $R_1 > 0$ such that
$$f(t, u) \geq (\lambda_1 + \epsilon)u, \quad \forall u \in [0, R_1], \text{ and almost all } t \in [0, 1]. \tag{3.1}$$
Let $R \in (0, R_1]$. We show that $u \neq Tu + \beta\varphi_1$ for all $\beta \geq 0, u \in \partial B_R \cap P$, where $\varphi_1 \in P$ is the positive eigenfunction of L with $\|\varphi_1\| = 1$ corresponding to the eigenvalue $1/r(L)$, which implies that $i(T, B_{R_1} \cap P, P) = 0$. In fact, if not, then

there exist u with $\|u\| = R$ and $\beta \geq 0$ such that $u = Tu + \beta\varphi_1$, it implies that $u \geq \beta\varphi_1$ and $Lu \geq \beta L\varphi_1 \geq \beta\frac{1}{\lambda_1}\varphi_1$. Together with (3.1), we have

$$u \geq (\lambda_1 + \epsilon)Lu + \beta\varphi_1 \geq (\lambda_1 + \epsilon)\frac{\beta}{\lambda_1}\varphi_1 + \beta\varphi_1 > 2\beta\varphi_1.$$

Repeating the process leads to $u \geq n\beta\varphi_1$ for $n \in \mathbb{N}$, a contradiction with $\|u\| = R$. Hence, we have $i(T, B_R \cap P, P) = 0$. □

Theorem 3.2. *If $0 < f^0 < \lambda_1$, then there exists $r_1 > 0$ such that $i(T, B_r \cap P, P) = 1$ for each $r \in (0, r_1]$.*

Proof. If $0 < f^0 < \lambda_1$, let $\epsilon > 0$ satisfy $f^0 < (\lambda_1 - \epsilon)u$ and then there exists $r_1 > 0$ such that

$$f(t, u) \leq (\lambda_1 - \epsilon)u, \quad \forall u \in [0, r_1], \text{ and almost all } t \in [0, 1]. \tag{3.2}$$

Let $r \in (0, r_1]$. We show that $Tu \neq \beta u$ for $u \in \partial B_r \cap P$ and $\beta \geq 1$, which implies the result. In fact, if it does not hold, then there exist $u \in \partial B_r \cap P, \beta \geq 1$ such that $Tu = \beta u$, and

$$\beta u(t) = \int_0^1 k(t, s)g(s)f(s, u(s))ds \leq (\lambda_1 - \epsilon)\int_0^1 k(t, s)g(s)u(s)ds \leq (\lambda_1 - \epsilon)Lu(t).$$

Thus, we have $u(t) \leq (\lambda_1 - \epsilon)Lu(t)$ which indicates

$$u(t) \leq (\lambda_1 - \epsilon)L[(\lambda_1 - \epsilon)Lu(t)] = (\lambda_1 - \epsilon)^2 L^2 u(t),$$

and iterating gives $u(t) \leq (\lambda_1 - \epsilon)^n L^n u(t)$ for $n \in \mathbb{N}$. It follows that

$$1 \leq (\lambda_1 - \epsilon)^n \|L^n\|.$$

We can see $1 \leq (\lambda_1 - \epsilon)\lim_{n\to\infty} \|L^n\|^{1/n} = (\lambda_1 - \epsilon)\frac{1}{\lambda_1} < 1$. This is obviously a contradiction. Hence, we have $i(T, B_r \cap P, P) = 1$. □

Theorem 3.3. *If $0 < f^\infty < \lambda_1$, then there exists $r_2 > r_1$ such that $i(T, B_r \cap P, P) = 1$ for each $r > r_2$, where r_1 is the same as in Theorem 3.2.*

Proof. If $0 < f^\infty < \lambda_1$, let $\epsilon > 0$ satisfy $f^\infty < \lambda_1 - \epsilon$ and then there exists r_0 such that

$$f(t, u) < (\lambda_1 - \epsilon)u, \quad \forall u \geq r_0.$$

By assumption (A2) , there exists $\phi_{r_0} \in L^\infty[0, 1]$ such that

$$f(t, u) < \phi_{r_0}, \quad \forall u \in [0, r_0].$$

Hence, we have

$$f(t, u) < (\lambda_1 - \epsilon)u + \phi_{r_0}, \quad \forall u \in \mathbb{R}_+.$$

Since λ_1 is the first eigenvalue of L, the first eigenvalue of $L_1, (I/(\lambda_1 - \epsilon) - L)^{-1}$ exists. Let

$$M = \sup_{u \in \bar{B}_{r_2} \cap P} \int_0^1 \big(\max_{0 \leq t, s \leq 1} k(t, s)\big)g(s)\phi_{r_0} ds$$

and

$$r_2 = (I/(\lambda_1 - \epsilon) - L)^{-1}(M/(\lambda_1 - \epsilon)).$$

We show that for each $r > r_2$, $Tu \neq \beta u$, $u \in \partial B_r \cap P$ and $\beta \geq 1$, which implies the result. In fact, if it does not hold, then there exist $u \in \partial B_r \cap P$ and $\beta \geq 1$ such that $Tu = \beta u$, and then

$$u(t) = (\lambda_1 - \epsilon)Lu(t) + M,$$

$$(I/(\lambda_1-\epsilon)-L)u(t) \le M/(\lambda_1-\epsilon),$$
$$u(t) \le (I/(\lambda_1-\epsilon)-L)^{-1}(M/(\lambda_1-\epsilon)) = r_2.$$

Thus, we have $\|u\| \le r_2 < r$, a contradiction. Further, we have $i(T, B_R \cap P, P) = 0$. □

We define the operator

$$\tilde{L}u(t) := \int_a^b k(t,s)g(s)u(s)ds, \quad t \in [0,1], \tag{3.3}$$

where $[a,b] \subset [0,1]$. Then $\tilde{L}$ is a completely continuous linear operator and $\tilde{L}(P) \subseteq P$. So $r(\tilde{L})$ is an eigenvalue of $\tilde{L}$ with corresponding eigenfunction $\tilde{\varphi_1}$ in P. Let $\tilde{\lambda_1} := 1/r(\tilde{L})$. Note that $\tilde{\lambda_1} \ge \lambda_1$, so the condition in the following theorem is more stringent than if we could use $r(L)$.

Theorem 3.4. *If $\tilde{\lambda_1} < f_\infty < \infty$, then there exists $R_2 > R_0$ such that $i(T, B_R \cap P, P) = 0$ for each $R > R_2$.*

Proof. If $\tilde{\lambda_1} < f_\infty < \infty$, we let $R > 0$ satisfy $f^\infty > \tilde{\lambda_1} + \epsilon$, and then there exists $u > R_2$ such that

$$f(t,u) > (\tilde{\lambda_1}+\epsilon)u, \quad \forall u > R_2 \text{ and almost all } t \in [0,1].$$

Let $R > R_2$, we show that $u \ne Tu + \beta\tilde{\varphi_1}$ for all $\beta \ge 0$, which indicates $u \in \partial B_R \cap P$. If it does not hold, we have $u(t) = Tu(t) + \beta\tilde{\varphi_1}$. Then

$$\begin{aligned} u(t) = Tu(t) + \beta\tilde{\varphi_1} &\ge \int_a^b k(t,s)g(s)f(s,u(s))ds + \beta\tilde{\varphi_1} \\ &> (\tilde{\lambda_1}+\epsilon)\tilde{L}u(t) + \beta\tilde{\varphi_1} \\ &> (\tilde{\lambda_1}+\epsilon)\frac{1}{\tilde{\lambda_1}}\beta\tilde{\varphi_1} + \beta\tilde{\varphi_1} \\ &> 2\beta\tilde{\varphi_1} \end{aligned}$$

and iterating gives

$$u(t) \ge n\beta\tilde{\varphi_1}(t) \quad \text{for } t \in [a,b], n \in \mathbb{N}.$$

Hence, $\tilde{\varphi_1}(t)$ is strictly positive on $[a,b]$. This is a contradiction. So we have $i(T, B_R \cap P, P) = 0$ for each $R > R_2$. □

Theorem 3.5. *Suppose that* (A1)–(A3) *hold together with one of the following two conditions:*

(1) $0 \le f^0 < \lambda_1(L)$ *and* $\tilde{\lambda}_1(\tilde{L}) < f_\infty \le \infty$*;*
(2) $0 \le f^\infty < \lambda_1(L)$ *and* $\lambda_1(L) < f_0 \le \infty$*.*

Then the multi-point boundary value problem (1.1)-(1.3) *has at least a positive solution.*

Proof. When (1) holds, by Theorems 3.2 and 3.4, there exist r_1 and $r_2 > r_1$ such that $i(T, B_{r_1}, P) = 1$ and $i(T, B_{r_2}, P) = 0$. Since $r_2 > r_1$, we have $(B_{r_1} \cap P) \subset (B_{r_2} \cap P)$. By applying the additivity of Lemma 2.1, we have

$$i(T, (B_{r_2} \cap P)\backslash(B_{r_1} \cap P), P) = i(T, B_{r_2} \cap P, P) - i(T, B_{r_1} \cap P, P) = -1 \ne 0.$$

Then T has at least one fixed point on $(B_{r_2} \cap P)\backslash(B_{r_1} \cap P)$. The proof for (2) is similar, we omit it. □

Theorem 3.6. *Suppose that* (A1)–(A3) *hold together with one of the following two conditions:*

(1) $0 \le f^0 < \lambda_1(L)$, $f(t,u) > \eta_1 r_0$, *where* $\eta_1 = \int_a^b g(s)ds$ *for some* $r_0 > 0$, $0 \le f^\infty < \lambda_1(L)$;

(2) $\lambda_1(L) < f_0 \le \infty$, $f(t,u) < \eta_2 R_0$, *where* $\eta_2 = \frac{\Delta}{(1+\alpha)\int_0^1 g(s)ds}$ *for some* $R_0 > 0$, $\tilde{\lambda}_1(\tilde{L}) < f_\infty \le \infty$.

Then the multi-point boundary value problem (1.1)-(1.3) *has at least two positive solutions.*

Proof. Under assumption (1), there exist $0 < r_1 < r_0$ and $r_2 > r_0$ such that $i(T, B_{r_1} \cap P, P) = 1$, $(0 \le u \le r_1)$ and $i(T, B_{r_2} \cap P, P) = 1$, $(0 \le u \le r_2)$. Next, we suppose that T has no fixed point on $\partial B_{r_1} \cap P$ and $\partial B_{r_2} \cap P$. If not, the proof is completed.

For $f(t,u) > \eta_1 r_0, u \in \partial B_{r_0} \cap P$, we have

$$Tu(t) \ge \int_a^b k_1(t,s)g(s)f(s,u(s))ds > \int_a^b k_1(s,s)g(s)\eta r_0 ds = r_0, \quad t \in [0,1].$$

Then $\|Tu\| > \|u\|$, so $i(T, B_{r_0} \cap P, P) = 0$. By Lemma 2.1, we have

$$\begin{aligned} i(T, (B_{r_0} \cap P)\backslash(B_{r_1} \cap P), P) &= i(T, B_{r_0} \cap P, P) - i(T, B_{r_1} \cap P, P) \\ &= -1 \ne 0, \end{aligned}$$

and

$$\begin{aligned} i(T, (B_{r_2} \cap P)\backslash(B_{r_0} \cap P), P) &= i(T, B_{r_2} \cap P, P) - i(T, B_{r_0} \cap P, P) \\ &= 1 \ne 0. \end{aligned}$$

Then T has at least two fixed points on $(B_{r_0} \cap P)\backslash(B_{r_1} \cap P)$ and $(B_{r_2} \cap P)\backslash(B_{r_0} \cap P)$. This means that the multi-point boundary value problem (1.1)-(1.3) has at least two positive solutions.

Under assumption (2), there exist $0 < R_1 < R_0$ and $R_2 > R_0$ such that $i(T, B_{R_1} \cap P, P) = 1$, $(0 \le u \le R_1)$ and $i(T, B_{R_2} \cap P, P) = 1$, $(0 \le u \le R_2)$. Next, we suppose that T has no fixed point on $\partial B_{R_1} \cap P$ and $\partial B_{R_2} \cap P$. If not, the proof is complete.

For $f(t,u) < \eta_2 R_0, u \in \partial B_{R_0} \cap P$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 k_1(t,s)g(s)f(s,u(s))ds + \int_0^1 \sum_{i=1}^m \tilde{k}_i(t,s)g(s)f(s,u(s))ds \\ &\quad + \int_0^1 \frac{(1+\alpha-\Delta)(t+\alpha)(1-s)}{(1+\alpha)\Delta} g(s)f(s,u(s))ds \\ &\le \int_0^1 k_1(t,s)g(s)f(s,u(s))ds \\ &\quad + \int_0^1 \frac{(1+\alpha-\Delta)(t+\alpha)(1-s)}{(1+\alpha)\Delta} g(s)f(s,u(s))ds \\ &< \int_0^1 g(s)f(s,u(s))ds + \frac{1+\alpha-\Delta}{\Delta}\int_0^1 g(s)f(s,u(s))ds \\ &= \frac{1+\alpha}{\Delta}\int_0^1 g(s)f(s,u(s))ds \end{aligned}$$

$$
\begin{aligned}
&< \frac{1+\alpha}{\Delta}\int_0^1 g(s)\eta_2 R_0 ds \\
&= R_0, \quad t\in[0,1].
\end{aligned}
$$

Then $\|Tu\| \le \|u\|$, so $i(T, B_{R_0}\cap P, P) = 0$. By applying Lemma 2.1, we have

$$
\begin{aligned}
i(T,(B_{R_0}\cap P)\backslash(B_{R_1}\cap P),P) &= i(T,B_{R_0}\cap P,P) - i(T,B_{R_1}\cap P,P) \\
&= -1 \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
i(T,(B_{R_2}\cap P)\backslash(B_{R_0}\cap P),P) &= i(T,B_{R_2}\cap P,P) - i(T,B_{R_0}\cap P,P) \\
&= 1 \neq 0.
\end{aligned}
$$

Then T has at least two fixed points on $(B_{R_0}\cap P)\backslash(B_{R_1}\cap P)$ and $(B_{R_2}\cap P)\backslash(B_{R_0}\cap P)$. This means that the multi-point boundary value problem (1.1)-(1.3) has at least two positive solutions. □

Remark 3.7. If we use the change of variable $t \to 1-t$ in (1.2) and (1.3), we have

$$u(1) = \alpha u'(1), \tag{3.4}$$

$$u(0) = \sum_{i=1}^m \beta_i u(\eta_i) + \sum_{i=1}^m \gamma_i u'(\eta_i), \tag{3.5}$$

where $0 \le \alpha \le \infty$, $0 < \eta_1 < \eta_1 < \eta_2 < \dots < \eta_m < 1$, $\beta_i > 0$ and $\gamma_i < 0$ $(i = 1, 2, \dots, m)$. We can use the similar method to obtain an analogous result.

Remark 3.8. Let $\alpha = 0, \gamma_i = 0$ $(i = 1, 2, \dots, m)$, then the boundary conditions (1.2) and (1.3) reduce to (1.7). Under the same conditions, if $\alpha = \infty$ and $\gamma_i = 0$ $(i = 1, 2 \dots m)$, we can also derive the same results as presented in [12].

Acknowledgments. This work is supported by the NNSF of China, under grants 11271379 and 11671406.

References

[1] K. Deimling; *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[2] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[3] D. Guo, J. Sun; *Nonlinear Integral Equations*, Shandong Science and Technology Press, Jinan, 1987 (in Chinese).

[4] V. Il'in, E. Moiseev; *Nonlocal boundary value problem of the first kind for Sturm-Liouville operator in its differential and finite difference aspects*, J. Differential Equations 23 (1987), 803-810.

[5] G. Karakostas, P. Tsamatos; *Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems*, Electron. J. Differential Equations 30 (2002), 1-17 .

[6] K. Lan, J. R. L. Webb; *Positive solutions of semilinear differential equations with singularities*, J. Differential Equations 148 (1998), 407-421.

[7] R. Ma; *Positive solutions of a nonlinear three-point boundary value problem*, Electronic J. Differentail Equation 34 (1999), 1-8.

[8] R. Ma; *Existence of solutions of nonlinear m-point boundary value problems*, J. Math. Anal. Appl. 256 (2001), 556-567.

[9] M. Moshinsky; *Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas*, Bol. Soc. Mat. Mexicana 7 (1950), 1-25.

[10] T. Timoshenko; *Theory of Elastic Theory*, McGraw-Hill, New York, 1971.

[11] J. R. L. Webb; *Positive solutions of some three point boundary value problems via fixed point index theory*, Nonlinear Anal. 47 (2001), 4319-4332.
[12] J. R. L. Webb, K. Q. Lan; *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type*, Topol. Methods Nonlinear Anal. 27 (2006), 91-115.
[13] J. S. W. Wong, L. Kong; *Solvability of second order nonlinlinear multi-point boundary value problems*, Dynam. Syst. Appl. 23 (2014), 145-154.
[14] G. Zhang, J. Sun; *Positive solutions of m-point boundary value problems*, J. Math. Anal. Appl. 291 (2004), 406-418.

4. Addendum posted by the editor on February 7, 2017

Professors J. Webb and K. Lan, the authors in reference [2], sent to the editors the following statements on August 25, 2016.

(1) There is lack of ethics by the authors by repeating, almost verbatim, large parts from reference [12], without given the proper credit:

Theorem 3.1 is Theorem 3.4 of [12], Theorem 3.2 is Theorem 3.2 of [12], Theorem 3.3 is Theorem 3.3 of [12] (with an added typo at end of proof), Theorem 3.5 is Theorem 4.1 of [12].

(2) Theorem 3.4 is attempting to prove Theorem 3.5 of [12] but for that proof one has to work in a smaller cone K (exactly as in [12]) which requires extra knowledge of (a, b) and $k(t, s)$ which are not stated in this paper.

(3) Theorem 3.6 is attempting to prove Theorem 4.4 of [12] but has several mistakes. One is that the conditions (1) are not written properly, they must be pointwise conditions, the range of u must be specified, but when written correctly they are impossible to satisfy, which is why one has to work in the smaller cone K. A second is that an upper bound on k_1 is used as if it was a lower bound. A third is confusing index =1 and index =0, which leads to other index mis-statements.

(4) Remark 3.7 has a sign error on derivative terms.

The editor contacted the authors who did not accept item (1), and did not want to send a suggested apology. The authors sent corrections for items (2)-(4), but these corrections were deemed insufficient. Two more rounds of corrections were also insufficient; so the editor decided to post this note.

End of addendum.

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