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# WELL-POSEDNESS AND DECAY OF SOLUTIONS FOR A TRANSMISSION PROBLEM WITH HISTORY AND DELAY

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Abstract. In this article, we consider a transmission problem in the presence of history and delay terms. Under appropriate hypothesis on the relaxation function and the relationship between the weight of the damping and the weight of the delay, we prove well-posedness by using the semigroup theory. Also we establish a decay result by introducing a suitable Lyaponov functional.

## 1. Introduction

In this article, we study the following transmission system with a past history and a delay term

$$
\begin{gathered}
u_{tt}(x,t) - au_{xx}(x,t) + \int_0^\infty g(s)u_{xx}(x,t-s)\mathrm{d}s \\
+ \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\
v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty),
\end{gathered} \tag{1.1}
$$

under the boundary and transmission conditions

$$
\begin{gathered}
u(0,t) = u(L_3,t) = 0, \\
u(L_i,t) = v(L_i,t), \quad i = 1,2, \\
au_x(L_i,t) - \int_0^\infty g(s)u_x(L_i,t-s)\mathrm{d}s = bv_x(L_i,t), \quad i = 1,2,
\end{gathered} \tag{1.2}
$$

and the initial conditions

$$
\begin{gathered}
u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega,\ t \in (0,\tau), \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2),
\end{gathered} \tag{1.3}
$$

where $0 < L_1 < L_2 < L_3$, $\Omega = (0,L_1) \cup (L_2,L_3)$, $a$, $b$, $\mu_1$, $\mu_2$ are positive constants, $u_0$ is given history, and $\tau > 0$ represents the delay.

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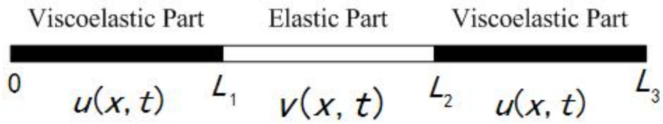


FIGURE 1. The configuration for problem (1.1)–(1.3).

Transmission problems arise in several applications of physics and biology. We note that problem (1.1)-(1.3) is related to the wave propagation over a body which consists of two different type of materials: the elastic part and the viscoelastic part that has the past history and time delay effect.

For wave equations with various dissipations, many results concerning stabilization of solutions have been proved. Recently, wave equations with viscoelastic damping have been investigated by many authors, see [6, 7, 8, 17, 18, 19, 26, 28, 29] and the references therein. It is showed that the dissipation produced by the viscoelastic part can produce the decay of the solution. For example, Cavalcanti et al [8] studied the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau + a(x)u_t + |u|^\gamma u = 0, \quad \text{in } \Omega \times (0, \infty),$$

where $a : \Omega \to \mathbb{R}_+$. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with $\omega$ satisfying some geometry restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

the authors showed the exponential decay. Berrimi and Messaoudi [6] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau = |u|^\gamma u, \quad \text{in } \Omega \times (0, \infty)$$

with only the viscoelastic dissipation and proved that the solution energy decays exponentially or polynomially depending on the rate of the decay of the relaxation function $g$. Guesmia [11] considered the asymptotic behavior of solutions to an abstract linear dissipative integrodifferential equation with infinite memory (past history) and introduced a new approach which allows a larger class of past-history kernels and consequently a more general decay result for a class of hyperbolic problems with past history is obtained. For other past (infinite) history problems, see [2, 12, 21, 24, 25] and the references therein.

In recent years, the control of PDEs with time delay effects has become an active area of research. The presence of delay may be a source of instability. For example, it was proved in [10, 16, 23] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms were used. Kirane and Said-Houari [14] considered the viscoelastic wave equation with a delay

$$\begin{aligned} &u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,t-s)\mathrm{d}s \\ &+ \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad \text{in } \Omega \times (0, \infty), \end{aligned}$$

where $\mu_1$ and $\mu_2$ are positive constants. They established an energy decay result under the condition that $0 \leq \mu_2 \leq \mu_1$. Later, Liu [15] improved this result by considering the equation with a time-varying delay term, with not necessarily positive coefficient $\mu_2$ of the delay term.

Transmission problems related to (1.1)-(1.3) have also been extensively studied. Bastos and Raposo [4] investigated the transmission problem with frictional damping and showed the well-posedness and exponential stability of the total energy. Muñoz Rivera and Portillo Oquendo [22] considered the transmission problem of viscoelastic waves and proved that the dissipation produced by the viscoelastic part can produce exponential decay of the solution, no matter how small its size is. Bae [3] studied the transmission problem, which one component is clamped and the other is in a viscoelastic fluid producing a dissipative mechanism on the boundary, and established a decay result which depends on the rate of the decay of the relaxation function. The effect of the delay in the stability of transmission system (1.1) in the absence of the past history term has been investigated by Benseghir in [5]. In [27], the present authors studied the well-posedness and decay of solution for a transmission problem in a bounded domain with a viscoelastic term $\int_0^t g(t-s)u_{xx}(x,s)\mathrm{d}s$ and a delay term.

Motivated by the above results, we intend to study in this paper the well-posedness and the decay result of problem (1.1)-(1.3), in which the infinite memory (past history) term $\int_0^\infty g(s)u_{xx}(x,t-s)\mathrm{d}s$ is involved. The main difficulty we encounter here arises from the simultaneous appearance of the past history and the delay term. We need also pay more attention to the influence of the transmission boundary. To attain our goal, we use the semigroup theory to prove the well-posedness, and introduce a suitable Lyaponov functional to establish the decay result.

This article is organized as follows. In Section 2, we give some materials needed for our work and state our main results. In Section 3,we prove the well-posedness of the problem. The decay result is proved in Section 4.

## 2. Preliminaries and statement of main results

In this section, we present some materials that shall be used for proving our main results. For the relaxation function $g$, we have the folloing assumptions:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^1$ function satisfying

$$g(0) > 0, \quad a - \int_0^\infty g(s)\mathrm{d}s = a - g_0 = l > 0.$$

(A2) There exists a non-increasing differentiable function $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0 \quad \text{and} \quad \int_0^\infty \xi(t)\mathrm{d}t = +\infty.$$

As in [23], we introduce the variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

Then

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

Following the ideal in [9], we set

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,t,s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{2.1}$$

Then

$$\eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), \quad (x,t,s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, system (1.1) becomes

$$\begin{gathered}
u_{tt}(x,t) - lu_{xx}(x,t) - \int_0^\infty g(s)\eta^t_{xx}(x,s)\mathrm{d}s \\
+ \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\
v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty), \\
\eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), \quad (x,s,t) \in \Omega \times (0,+\infty) \times (0,+\infty),
\end{gathered} \tag{2.2}$$

the boundary and transmission conditions (1.2) become

$$\begin{gathered}
u(0,t) = u(L_3,t) = 0, \\
u(L_i,t) = v(L_i,t), \quad i = 1,2,\ t \in (0,+\infty), \\
lu_x(L_i,t) + \int_0^\infty g(s)\eta^t_x(L_i,s)\mathrm{d}s = bv_x(L_i,t), \quad i = 1,2,\ t \in (0,+\infty),
\end{gathered} \tag{2.3}$$

and the initial conditions (1.3) become

$$\begin{gathered}
u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x),\ x \in \Omega, \\
z(x,0,t) = u_t(x,t), \quad z(x,1,t) = f_0(x,t-\tau), \quad (x,t) \in \Omega \times (0,+\infty), \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2),
\end{gathered} \tag{2.4}$$

It is clear that

$$\begin{gathered}
\eta^t(x,0) = 0, \quad \text{for all } x > 0, \\
\eta^t(0,s) = \eta^t(L_3,s) = 0, \quad \text{for all } s > 0, \\
\eta^0(x,s) = \eta_0(s), \quad \text{for all } s > 0.
\end{gathered} \tag{2.5}$$

Let $V := (u,v,\varphi,\psi,z,\eta^t)^T$, then $V$ satisfies the problem

$$\begin{gathered}
V_t = \mathscr{A}V(t), \quad t > 0, \\
V(0) = V_0,
\end{gathered} \tag{2.6}$$

where $V_0 := (u_0(\cdot,0), v_0, u_1, v_1, f_0(\cdot,-\tau), \eta_0)^T$ and the operator $\mathscr{A}$ is defined by

$$\mathscr{A} = \begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ lu_{xx} + \int_0^{+\infty} g(s)w_{xx}(s)\mathrm{d}s - \mu_1\varphi - \mu_2 z(.,1) \\ bv_{xx} \\ -\frac{1}{\tau}z_\rho \\ -w_s + \varphi \end{pmatrix}.$$

where

$$\begin{aligned}
X_* = \Big\{ &(u,v) \in H^1(\Omega) \cap H^1(L_1,L_2) : u(0,t) = u(L_3,t) = 0, u(L_i,t) = v(L_i,t), \\
&lu_x(L_i,t) + \int_0^\infty g(s)\eta^t_x(L_i,s)\mathrm{d}s = bv_x(L_i,t), i = 1,2 \Big\}
\end{aligned}$$

and $L^2_g(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of $H^1$-valued functions on $\mathbb{R}_+$, endowed with the inner product

$$(\phi, \vartheta)_{L^2_g(\mathbb{R}_+, H^1(\Omega))} = \int_\Omega \int_0^{+\infty} g(s)\phi_x(s)\vartheta_x(s)\mathrm{d}s\mathrm{d}x.$$

Set

$$V = (u, v, \varphi, \psi, z, w)^T, \quad \bar{V} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\psi}, \bar{z}, \bar{w})^T.$$

We define the inner product in $\mathscr{H}$,

$$\begin{aligned}\langle V, \bar{V}\rangle_{\mathscr{H}} = &\int_\Omega \varphi\bar{\varphi}\mathrm{d}x + \int_{L_1}^{L_2} \psi\bar{\psi}\mathrm{d}x + \int_\Omega l u_x \bar{u}_x \mathrm{d}x + \int_{L_1}^{L_2} b v_x \bar{v}_x \mathrm{d}x \\ &+ \int_\Omega \int_0^{+\infty} g(s) w_x(s) \bar{w}_x(s)\mathrm{d}s\mathrm{d}x + \zeta \int_\Omega \int_0^1 z\bar{z}\mathrm{d}\rho\mathrm{d}x.\end{aligned}$$

The domain of $\mathscr{A}$ is

$$\begin{aligned}D(\mathscr{A}) = \Big\{ &(u, v, \varphi, \psi, z, w)^T \in \mathscr{H} : u \in H^2(\Omega) \cap H^1(\Omega), \\ &v \in H^2(L_1, L_2) \cap H^1(L_1, L_2), \varphi \in H^1(\Omega), \psi \in H^1(L_1, L_2), \\ &w \in L^2_g\left(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)\right), w_s \in \left(\mathbb{R}_+, H^1(\Omega)\right), \\ &z_\rho \in L^2((0,1), L^2(\Omega)), w(x, 0) = 0, z(x, 0) = \varphi(x)\Big\}.\end{aligned}$$

The well-posedness of problem (2.2)-(2.3) is ensured by the following theorem.

**Theorem 2.1.** *Assume that $\mu_2 \le \mu_1$,* (A1) *and* (A2) *hold. Let $V_0 \in \mathscr{H}$, then there exists a unique weak solution $V \in C(\mathbb{R}_+, \mathscr{H})$ of problem* (2.6)*. Moreover, if $V_0 \in D(\mathscr{A})$, then*

$$V \in C(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

For a solution $u$ of (1.1)-(1.3), we define the energy

$$\begin{aligned}E(t) = &\frac{1}{2}\int_\Omega [u_t^2(x,t) + l u_x^2(x,t)]\mathrm{d}x + \frac{1}{2}\int_{L_1}^{L_2} [v_t^2(x,t) + b v_x^2(x,t)]\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x + \frac{\zeta}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x,\end{aligned} \tag{2.7}$$

where $\zeta$ is the positive constant satisfying

$$\begin{gathered}\zeta\mu_2 < \zeta < \tau(2\mu_1 - \mu_2), \quad \mu_2 < \mu_1, \\ \zeta = \tau\mu_2, \quad \mu_2 = \mu_1.\end{gathered} \tag{2.8}$$

Our decay result reads as follows.

**Theorem 2.2.** *Let $(u, v)$ be the solution of* (1.1)-(1.3)*. Assume that* (A1), (A2) *hold, that $\mu_2 \le \mu_1$, that for some $m_0 \ge 0$,*

$$\int_\Omega u_{0x}^2(x,s)\mathrm{d}x \le m_0, \quad \forall s > 0 \tag{2.9}$$

*and that*

$$a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2} l \tag{2.10}$$

*hold, then there exists constants $\gamma_0, \gamma_2 > 0$ such that, for all $t \in \mathbb{R}_+$ and for all $\gamma_1 \in (0, \gamma_0)$,*

$$E(t) \le \gamma_2\Big(1 + \int_0^t (g(s))^{1-\gamma_1}\mathrm{d}s\Big)e^{-\gamma_1 \int_0^t \xi(s)\mathrm{d}s} + \gamma_2 \int_t^{+\infty} g(s)\mathrm{d}s. \tag{2.11}$$

**Remark 2.3.** Here we consider some examples to illustrate our estimate (2.11).

(1) Let

$$g(t) = k_1 e^{-k_2(1+t)^q}, \quad 0 < q < 1,\ k_1 > 0,\ k_2 > 0,$$

then it is clear that (A2) holds for $\xi(t) = k_2 q(1+t)^{q-1}$. Consequently, applying (2.11), we obtain the decay result

$$E(t) \le \tilde{c_1} e^{-\tilde{c_2} k_2 (1+t)^q},$$

where $\tilde{c_1}$ and $\tilde{c_2}$ are positive constants.

(2) Let

$$g(t) = k_3 e^{-k_4[\ln(1+t)]^p}, \quad k_3 > 0,\ k_4 > 0,$$

then our assumption (A2) holds with $\xi(t) = \frac{k_4 p[\ln(1+t)]^{p-1}}{1+t}$. (2.11) gives us

$$E(t) \le \tilde{c_3} e^{-\tilde{c_4} k_4 [\ln(1+t)]^p},$$

where $\tilde{c_3}$ and $\tilde{c_4}$ are positive constants.

## 3. Well-posedness of the problem

In this section, by combining the frameworks of [5] with some necessary modifications due to the problem treated here, we use the semigroup approach and the Hille-Yosida theorem to prove the existence and uniqueness of problem (1.1)-(1.3).

*Proof of Theorem 2.1.* First, we prove that the operator $\mathscr{A}$ is dissipative. In fact, for $(u, v, \varphi, \psi, z, w)^T \in D(\mathscr{A})$, where $\varphi(L_i) = \psi(L_i), i = 1, 2$ and $\zeta$ is a positive constant satisfying (2.8), we have

$$\begin{aligned}
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= \int_\Omega l u_{xx}\varphi \mathrm{d}x + \int_\Omega \Big(\int_0^{+\infty} g(s) w_{xx}(s)\mathrm{d}s - \mu_1\varphi - \mu_2 z(.,1)\Big)\varphi \mathrm{d}x \\
&\quad + \int_\Omega l u_x \varphi_x \mathrm{d}x + \int_{L_1}^{L_2} b v_x \psi_x \mathrm{d}x + \int_{L_1}^{L_2} b v_{xx}\psi \mathrm{d}x \\
&\quad + \int_\Omega \int_0^{+\infty} g(s) w_x(s)(-w_s + \varphi)_x \mathrm{d}s\mathrm{d}x \\
&\quad - \frac{\zeta}{\tau}\int_\Omega \int_0^1 z z_\rho(x,\rho)\mathrm{d}\rho\mathrm{d}x.
\end{aligned} \tag{3.1}$$

For the last term of the right side of (3.1), we obtain

$$\zeta \int_\Omega \int_0^1 z z_\rho(x,\rho)\mathrm{d}\rho\mathrm{d}x = \zeta \int_\Omega \int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho} z^2(x,\rho)\mathrm{d}\rho\mathrm{d}x = \frac{\zeta}{2}\int_\Omega (z^2(x,1) - z^2(x,0))\mathrm{d}x.$$

Noticing that $z(x,0,t) = \varphi(x,t)$, $w(x,0) = 0$ and $\varphi(L_i) = \psi(L_i), i = 1, 2$, we obtain

$$\begin{aligned}
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= \Big[l u_x \varphi + \int_0^{+\infty} g(s) w_x(s)\mathrm{d}s\varphi\Big]_{\partial\Omega} + [b v_x \psi]_{L_1}^{L2} \\
&\quad + \int_\Omega (-\mu_1\varphi - \mu_2 z(.,1))\varphi \mathrm{d}x - \Big[\frac{1}{2}\int_0^{+\infty} g(s)|w_x(x,s)|^2 \mathrm{d}s\Big]_{\partial\Omega}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|w_x(x,s)|^2\mathrm{d}s\mathrm{d}x-\frac{\zeta}{\tau}\int_\Omega\int_0^1 zz_\rho\mathrm{d}\rho\mathrm{d}x\\
&\le -\big(\mu_1-\frac{\zeta}{2\tau}\big)\int_\Omega\varphi^2(x)\mathrm{d}x-\frac{\zeta}{2\tau}\int_0^1 z^2(x,1)\mathrm{d}x\\
&-\mu_2\int_\Omega z(x,1)\varphi(x)\mathrm{d}x+\frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|w_x(x,s)|^2\mathrm{d}s\mathrm{d}x,
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
&[lu_x\varphi+\int_0^{+\infty} g(s)w_x(s)\mathrm{d}s\varphi]_{\partial\Omega}\\
&=\Big(lu_x(L_1,t)+\int_0^{+\infty} g(s)w_x(L_1,s)\mathrm{d}s\Big)\varphi(L_1,t)\\
&\quad-\Big(lu_x(L_2,t)+\int_0^{+\infty} g(s)w_x(L_2,s)\mathrm{d}s\Big)\varphi(L_2,t)\\
&=-[bv_x\psi]_{L_1}^{L2}
\end{aligned}
$$

Using Young's inequality, we have

$$
\begin{aligned}
\langle\mathscr{A}V,V\rangle_{\mathscr{H}}&=-\Big(\mu_1-\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\Big)\int_\Omega\varphi^2(x)\mathrm{d}x-\Big(\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\Big)\int_0^1 z^2(x,1)\mathrm{d}x\\
&\quad+\frac{1}{2}\int_\Omega\int_0^{+\infty} g'(s)|w_x(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}
$$

Consequently, taking (2.8) and (A2) into account, we conclude that

$$
\langle\mathscr{A}V,V\rangle_{\mathscr{H}}\le 0;
$$

that is, $\mathscr{A}$ is dissipative.

Next, we prove that $-\mathscr{A}$ is maximal. Actually, let $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in\mathscr{H}$, we prove that there exists $V=(u,v,\varphi,\psi,z,w)^T\in D(\mathscr{A})$ satisfying

$$
(\lambda I-\mathscr{A})V=F, \tag{3.2}
$$

which is equivalent to

$$
\begin{gathered}
\lambda u-\varphi=f_1,\\
\lambda v-\psi=f_2,\\
\lambda\varphi-lu_{xx}-\int_0^\infty g(s)w_{xx}(s)\mathrm{d}s+\mu_1\varphi+\mu_2 z(.,t)=f_3,\\
\lambda\psi-bv_{xx}=f_4,\\
\lambda z+\frac{1}{\tau}z_\rho=f_5,\\
\lambda w+w_s-\varphi=f_6.
\end{gathered}\tag{3.3}
$$

Assume that with the suitable regularity we have found $u$ and $v$, then

$$
\begin{gathered}
\varphi=\lambda u-f_1,\\
\psi=\lambda v-f_2.
\end{gathered}\tag{3.4}
$$

So we have $\varphi\in H^1(\Omega)$ and $\psi\in H^1(L_1,L_2)$. Moreover, we can find $z$ with

$$
z(x,0)=\varphi(x),\quad\text{for } x\in\Omega.
$$

Following the same approach in [23] and using the equation in (3.3), we obtain

$$z(x,\rho)=\varphi(x)e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma.$$

From (3.4), we obtain

$$z(x,\rho)=\lambda u e^{-\lambda\rho\tau}-f_1 e^{-\lambda\rho\tau}\tau+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma. \tag{3.5}$$

It is easy to see that the last equation in (3.3) with $w(x,0)=0$ has a unique solution

$$\begin{aligned} w(x,s)&=\Big(\int_0^s e^{\lambda y}(f_6(x,y)+\varphi(x))\mathrm{d}y\Big)e^{-\lambda s} &(3.6)\\ &=\Big(\int_0^s e^{\lambda y}(f_6(x,y)+\lambda u(x)-f_1(x))\mathrm{d}y\Big)e^{-\lambda s}. &(3.7)\end{aligned}$$

By using (3.3), (3.4) and (3.6), the functions $u$ and $v$ satisfy

$$\begin{gathered}\left(\lambda^2+\mu_1\lambda+\mu_2\lambda e^{-\lambda\tau}\right)u-\tilde{l}u_{xx}=\tilde{f},\\ \lambda^2 v-bv_{xx}=f_4+\lambda f_2,\end{gathered} \tag{3.8}$$

where

$$\tilde{l}=l+\lambda\int_0^\infty g(s)e^{-\lambda s}\Big(\int_0^s e^{\lambda y}\mathrm{d}y\Big)\mathrm{d}s$$

and

$$\begin{aligned}\tilde{f}=&\int_0^\infty g(s)e^{-\lambda s}\Big(\int_0^s e^{\lambda y}(f_6(x,y)-f_1(x,y))_{xx}\mathrm{d}y\Big)\mathrm{d}s\\ &-\mu_2\tau e^{-\lambda\tau}\int_0^1 f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma+\left(\lambda+\mu_1+\mu_2 e^{-\lambda\tau}\right)f_1+f_3.\end{aligned}$$

We just need to prove that (3.8) has a solution $(u,v)\in X_*$ and replace in (3.4), (3.5) and (3.6) to get $V=(u,v,\varphi,\psi,z,w)^T\in D(\mathscr{A})$ satisfying (3.2). Consequently, problem (3.8) is equivalent to the problem

$$\Phi((u,v),(\omega_1,\omega_2))=l(\omega_1,\omega_2), \tag{3.9}$$

where the bilinear form $\Phi:(X_*,X_*)\to\mathbb{R}$ and the linear form $l:X_*\to\mathbb{R}$ are defined by

$$\begin{aligned}\Phi((u,v),(\omega_1,\omega_2))=&\int_\Omega\Big[\left(\lambda^2+\mu_1\lambda+\mu_2\lambda e^{-\lambda\tau}\right)u\omega_1+\tilde{l}u_x(\omega)_x\Big]\mathrm{d}x-[\tilde{l}u_x\omega_1]_{\partial\Omega}\\ &+\int_{L_1}^{L_2}\left(\lambda^2 v\omega_2+bv_x(\omega_2)_x\right)\mathrm{d}x-[bv_x\omega_2]_{L_1}^{L_2}\end{aligned}$$

and

$$l(\omega_1,\omega_2)=\int_\Omega\tilde{f}\omega_1\mathrm{d}x+\int_{L_1}^{L_2}(f_4+\lambda f_2)\omega_2\mathrm{d}x.$$

Using the properties of the space $X_*$, it is easy to see that $\Phi$ is continuous and coercive, and $l$ is continuous. Applying the Lax-Milgram theorem, we infer that for all $(\omega_1,\omega_2)\in X_*$, problem (3.9) has a unique solution $(u,v)\in X_*$. It follows from (3.8) that $(u,v)\in\left\{\left(H^2(\Omega)\times H^2(L_1,L_2)\right)\cap X_*\right\}$. Thence, the operator $\lambda I-\mathscr{A}$ is surjective for any $\lambda>0$. At last, the result of Theorem 2.1 follows from the Hille-Yosida theorem. $\square$

## 4. Decay of the solution

In this section, we consider a decay result of problem (1.1)-(1.3). We will discuss two cases, the case $\mu_2 < \mu_1$ and $\mu_2 = \mu_1$. We will separate two cases since the proof are slightly different.

### 4.1. The case $\mu_2 < \mu_1$.

For the proof of Theorem 2.2, we need some lemmas.

**Lemma 4.1.** *Let $(u, v, z)$ be the solution of (2.2)-(2.3). Assume that $\mu_2 < \mu_1$ and $E(t)$ satisfies (2.7). Then we have the inequality*

$$\begin{aligned} \frac{d}{dt}E(t) \le -c_1\Big[\int_\Omega u_t^2(x,t)\mathrm{d}x + \int_\Omega z^2(x,1,t)\mathrm{d}x\Big] \\ + \frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{4.1}$$

*Proof.* Multiplying the first equation in (2.2) by $u_t$, the second equation by $v_t$, using the fourth equation in (2.2), integrating by parts and using (2.3), we obtain

$$\begin{aligned} &\int_\Omega \Big(u_t u_{tt} + l u_x u_{xt} + \int_0^\infty g(s)\eta_x^t(x,s)u_{xt}\mathrm{d}s + \mu_1 u_t^2 + \mu_2 z(x,1,t)u_t\Big)\mathrm{d}x \\ &- \Big[\Big(l u_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u_t\Big]_{\partial\Omega} - [b v_x v_t]_{L_1}^{L_2} + \int_{L_1}^{L_2}(v_t v_{tt} + b v_x v_{xt})\mathrm{d}x \\ &= \frac{1}{2}\frac{d}{dt}\Big[\int_\Omega [u_t^2(x,t) + l u_x^2(x,t)]\mathrm{d}x\Big] + \frac{1}{2}\frac{d}{dt}\Big[\int_{L_1}^{L_2}[v_t^2(x,t) + b v_x^2(x,t)]\mathrm{d}x\Big] \\ &\quad + \frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x + \Big[\frac{1}{2}\int_\Omega g(s)|\eta_x^t(x,s)|^2\mathrm{d}x\Big]_0^\infty \\ &\quad + \mu_1\int_\Omega u_t^2(x,t)\mathrm{d}x + \mu_2\int_\Omega u_t(x,t)z(x,1,t)\mathrm{d}x \\ &\quad - \frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x = 0, \end{aligned} \tag{4.2}$$

where we have used that

$$\begin{aligned} &- \Big[\Big(l u_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u_t\Big]_{\partial\Omega} \\ &= \Big(l u_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)\mathrm{d}s\Big)u_t(L_2,t) \\ &\quad - \Big(l u_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)\mathrm{d}s\Big)u_t(L_1,t) \\ &= [b v_x v_t]_{L_1}^{L_2}, \end{aligned}$$

and

$$\Big[\frac{1}{2}\int_\Omega g(s)|\eta_x^t(x,s)|^2\mathrm{d}x\Big]_0^\infty = 0.$$

Multiplying the third equation in (2.2) by $\zeta z/\tau$, integrating the result over $\Omega \times (0,1)$ with respect to $x$ and $\rho$ respectively, we have

$$\frac{\zeta}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x = -\frac{\zeta}{2\tau}\int_\Omega (z^2(x,1) - z^2(x,0))\mathrm{d}x. \tag{4.3}$$

From (2.7), (4.2) and (4.3), we arrive at

$$\begin{aligned} \frac{d}{dt}E(t) &= -\Big(\mu_1 - \frac{\zeta}{2\tau}\Big)\int_\Omega u_t^2(x,t)\mathrm{d}x - \frac{\zeta}{2\tau}\int_\Omega z^2(x,1,t)\mathrm{d}x \\ &\quad - \mu_2\int_\Omega u_t(x,t)z(x,1,t)\mathrm{d}x + \frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{4.4}$$

Young's inequality gives us

$$\begin{aligned} \frac{d}{dt}E(t) &\le -\Big(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\Big)\int_\Omega u_t^2(x,t)\mathrm{d}x - \Big(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\Big)\int_\Omega z^2(x,1,t)\mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \end{aligned}$$

Thanks to (2.8) and (2.7), our conclusion holds. The proof is complete. □

Now we define the functional

$$\mathscr{D}(t) = \int_\Omega uu_t\mathrm{d}x + \frac{\mu_1}{2}\int_\Omega u^2\mathrm{d}x + \int_{L_1}^{L_2} vv_t\mathrm{d}x.$$

Then we have the following lemma.

**Lemma 4.2.** *The functional $\mathscr{D}(t)$ satisfies*

$$\begin{aligned} \frac{d}{dt}\mathscr{D}(t) &\le \int_\Omega u_t^2\mathrm{d}x + \int_{L_1}^{L_2} v_t^2\mathrm{d}x + (L^2\varepsilon + \varepsilon - l)\int_\Omega u_x^2\mathrm{d}x - \int_{L_1}^{L_2} bv_x^2\mathrm{d}x \\ &\quad + \frac{g_0}{4\varepsilon}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x + \frac{\mu_2^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x. \end{aligned} \tag{4.5}$$

*Proof.* Taking the derivative of $\mathscr{D}(t)$ with respect to $t$ and using (2.2), we have

$$\begin{aligned} \frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2\mathrm{d}x - l\int_\Omega u_x^2\mathrm{d}x - \mu_2\int_\Omega z(x,1,t)u\mathrm{d}x + [bv_xv]_{L_1}^{L_2} + \int_{L_1}^{L_2} v_t^2\mathrm{d}x \\ &\quad + \Big[\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u\Big]_{\partial\Omega} \\ &\quad - \int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\mathrm{d}x - \int_{L_1}^{L_2} bv_x^2\mathrm{d}x \\ &= \int_\Omega u_t^2\mathrm{d}x - l\int_\Omega u_x^2\mathrm{d}x - \mu_2\int_\Omega z(x,1,t)u\mathrm{d}x + \int_{L_1}^{L_2} v_t^2\mathrm{d}x \\ &\quad - \int_{L_1}^{L_2} bv_x^2\mathrm{d}x - \int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{4.6}$$

where we used that

$$\begin{aligned} \Big[\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u\Big]_{\partial\Omega} &= \Big(lu_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)\mathrm{d}s\Big)u(L_1,t) \\ &\quad - \Big(lu_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)\mathrm{d}s\Big)u(L_2,t) \\ &= -[bv_xv_t]_{L_1}^{L_2}. \end{aligned}$$

By the boundary condition (1.2), we have

$$u^2(x,t) = \Big(\int_0^x u_x(x,t)\mathrm{d}x\Big)^2 \le L_1\int_0^{L_1} u_x^2(x,t)\mathrm{d}x, \quad x \in [0, L_1],$$

$$u^2(x,t) \le (L_3 - L_2) \int_{L_2}^{L_3} u_x^2(x,t)\mathrm{d}x, \quad x \in [L_2, L_3],$$

which implies

$$\int_\Omega u^2(x,t)\mathrm{d}x \le L^2 \int_\Omega u_x^2 \mathrm{d}x, \quad x \in \Omega, \tag{4.7}$$

where $L = \max\{L_1, L_3 - L_2\}$. By making use of Young's inequality and (4.7), for any $\varepsilon > 0$, we obtain

$$\mu_2 \int_\Omega z(x,1,t) u \mathrm{d}x \le \frac{\mu_2^2}{4\varepsilon} \int_\Omega z^2(x,1,t)\mathrm{d}x + L^2 \varepsilon \int_\Omega u_x^2 \mathrm{d}x. \tag{4.8}$$

Young's inequality, Hölder's inequality and (A2) imply that

$$\int_\Omega u_x(x,t) \int_0^\infty g(s) \eta_x^t(x,s)\mathrm{d}s\mathrm{d}x \le \varepsilon \int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{g_0}{4\varepsilon} \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x. \tag{4.9}$$

Inserting the estimates (4.8) and (4.9) into (4.6), then (4.5) is fulfilled. □

Next, enlightened by [20], we introduce the functional

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x \in [0, L_1], \\ \frac{L_1}{2} - \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}(x - L_1), & x \in (L_1, L_2), \\ x - \frac{L_2 + L_3}{2}, & x \in [L_2, L_3]. \end{cases}$$

It is easy to see that $q(x)$ is bounded: $|q(x)| \le M$, where $M = \max\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\}$.

We define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x) u_t \Big( l u_x + \int_0^\infty g(s) \eta_x^t(x,s)\mathrm{d}s \Big)\mathrm{d}x, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x) v_x v_t \mathrm{d}x.$$

Then we have the following results.

**Lemma 4.3.** *The functionals $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ satisfy*

$$\begin{aligned} &\frac{d}{dt}\mathscr{F}_1(t) \\ &\le \Big( \frac{l + g_0}{2} + \frac{M^2 \mu_1^2}{4\varepsilon_1} + \varepsilon_1 M^2 \Big) \int_\Omega u_t^2 \mathrm{d}x + \big( l^2 + 2 l^2 \varepsilon_1 \big) \int_\Omega u_x^2 \mathrm{d}x \\ &\quad + \frac{M^2 \mu_2^2}{4\varepsilon_1} \int_\Omega z^2(x,1,t)\mathrm{d}x + (g_0 + 2 g_0 \varepsilon_1) \int_\Omega \int_0^\infty g(s) |\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\ &- \frac{g(0)}{4\varepsilon_1} \int_\Omega \int_0^\infty g'(s) |\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x - \Big[ \frac{l + g_0}{2} q(x) u_t^2 \Big]_{\partial\Omega} \\ &- \Big[ \frac{q(x)}{2} \Big( l u_x(x,t) + \int_0^\infty g(s) \eta_x^t(x,s)\mathrm{d}s \Big)^2 \Big]_{\partial\Omega} \end{aligned} \tag{4.10}$$

*and*

$$\begin{aligned} \frac{d}{dt}\mathscr{F}_2(t) \le &-\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} \Big( \int_{L_1}^{L_2} v_t^2 \mathrm{d}x + \int_{L_1}^{L_2} b v_x^2 \mathrm{d}x \Big) + \frac{L_1}{4} v_t^2(L_1) \\ &+ \frac{L_3 - L_2}{4} v_t^2(L_2) + \frac{b}{4} \big( (L_3 - L_2) v_x^2(L_2, t) + L_1 v_x^2(L_1, t) \big). \end{aligned} \tag{4.11}$$

*Proof.* Taking the derivative of $\mathscr{F}_1(t)$ with respect to $t$ and using (2.2), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathscr{F}_1(t)\\
&= -\int_\Omega q(x)u_{tt}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&\quad - \int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&= -\int_\Omega q(x)\Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s\Big)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&\quad + \int_\Omega q(x)(\mu_1 u_t(x,t) + \mu_2 z(x,1,t))\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&\quad - \int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\Big)\mathrm{d}x.
\end{aligned}
\tag{4.12}
$$

We pay attention to

$$
\begin{aligned}
&-\int_\Omega q(x)\Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s\Big)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&= \frac{1}{2}\int_\Omega q'(x)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\mathrm{d}x\\
&\quad - \Big[\frac{q(x)}{2}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega}.
\end{aligned}
\tag{4.13}
$$

The last term in (4.12) can be treated as follows

$$
\begin{aligned}
&-\int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&= -l\int_\Omega q(x)u_tu_{xt}\mathrm{d}x - \int_\Omega q(x)u_t\int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\mathrm{d}x\\
&= \Big[-\frac{l}{2}q(x)u_t^2\Big]_{\partial\Omega} + \frac{l}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x\\
&\quad - \int_\Omega q(x)u_t\int_0^\infty g(s)\left(u_t - \eta_s^t\right)_x\mathrm{d}s\mathrm{d}x\\
&= \Big[-\frac{l}{2}q(x)u_t^2\Big]_{\partial\Omega} + \frac{l}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x - g_0\int_\Omega q(x)u_tu_{tx}\mathrm{d}x\\
&\quad + \int_\Omega q(x)u_t\int_0^\infty g(s)\eta_{sx}^t(x,s)\mathrm{d}s\mathrm{d}x\\
&= \Big[-\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega} + \frac{l+g_0}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x\\
&\quad - \int_\Omega q(x)u_t\int_0^\infty g'(s)\eta_x^t\mathrm{d}s\mathrm{d}x,
\end{aligned}
\tag{4.14}
$$

where we used that

$$
-\Big[\int_\Omega q(x)u_tg(s)\eta_x^t(x,s)\mathrm{d}x\Big]_0^\infty = 0.
$$

Inserting (4.13) and (4.14) in (4.12), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\mathscr{F}_1(t)\\
&= -\Big[\frac{q(x)}{2}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega} - \Big[\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega}\\
&\quad + \frac{1}{2}\int_\Omega q'(x)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\mathrm{d}x + \int_\Omega q(x)(\mu_1 u_t(x,t)\\
&\quad + \mu_2 z(x,1,t))\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&\quad + \frac{l+g_0}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x - \int_\Omega q(x)u_t\int_0^\infty g'(s)\eta_x^t\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{4.15}$$

Using Minkowski and Young's inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\mathrm{d}x\\
&\le l^2\int_\Omega u_x^2\mathrm{d}x + g_0\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{4.16}$$

Young's inequality gives us that for any $\varepsilon_1 > 0$,

$$\begin{aligned}
&\Big|\mu_1\int_\Omega q(x)u_t(x,t)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\Big|\\
&\le \frac{M^2\mu_1^2}{4\varepsilon_1}\int_\Omega u_t^2(x,t)\mathrm{d}x + l^2\varepsilon_1\int_\Omega u_x^2(x,t)\mathrm{d}x\\
&\quad + g_0\varepsilon_1\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\Big|\mu_2\int_\Omega q(x)z(x,1,t)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x\Big|\\
&\le \frac{M^2\mu_2^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x + l^2\varepsilon_1\int_\Omega u_x^2(x,t)\mathrm{d}x\\
&\quad + g_0\varepsilon_1\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{4.18}$$

It is clear that

$$\begin{aligned}
&\Big|\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta_x^t\mathrm{d}s\mathrm{d}x\Big|\\
&\le \varepsilon_1 M^2\int_\Omega u_t^2\mathrm{d}x - \frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{4.19}$$

Inserting (4.16)-(4.19) into (4.15), we obtain (4.10).

By the same method, taking the derivative of $\mathscr{F}_1(t)$ with respect to $t$, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t) &= -\int_{L_1}^{L_2} q(x)v_{xt}v_t\mathrm{d}x - \int_{L_1}^{L_2} q(x)v_x v_{tt}\mathrm{d}x\\
&= \Big[-\frac{1}{2}q(x)v_t^2\Big]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\mathrm{d}x + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2\mathrm{d}x\\
&\quad + \Big[-\frac{b}{2}q(x)v_x^2\Big]_{L_1}^{L_2}
\end{aligned}$$

$$
\begin{aligned}
&\le -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\mathrm{d}x+\int_{L_1}^{L_2} bv_x^2\mathrm{d}x\Big)+\frac{L_1}{4}v_t^2(L_1)\\
&\quad+\frac{L_3-L_2}{4}v_t^2(L_2)+\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right).
\end{aligned}
$$

Thus, the proof of Lemma 4.3 is complete.

As in [1], we define the functional

$$
\mathscr{F}_3(t)=\tau\int_\Omega\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x,
$$

then we have the following estimate.

**Lemma 4.4** ([1])**.** *The functionals $\mathscr{F}_3(t)$ satisfies*

$$
\frac{d}{dt}\mathscr{F}_3(t)\le -c_2\Big(\int_\Omega z^2(x,1,t)\mathrm{d}x+\tau\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\Big)+\int_\Omega u_t^2(x,t)\mathrm{d}x.
$$

*Proof of Theorem 2.2.* We define the Lyapunov functional

$$
L(t)=N_1E(t)+N_2\mathscr{D}(t)+\mathscr{F}_1(t)+N_4\mathscr{F}_2(t)+\mathscr{F}_3(t), \tag{4.20}
$$

where $N_1,N_2,N_4$ are positive constants that will be fixed later.

Taking the derivative of (4.20) with respect to $t$ and taking advantage of the above lemmas, we have

$$
\begin{aligned}
\frac{d}{dt}L(t)&\le -\Big\{N_1c_1-1-N_2-\Big(\frac{l+g_0}{2}+\frac{M^2\mu_1^2}{4\varepsilon_1}+\varepsilon_1M^2\Big)\Big\}\int_\Omega u_t^2\mathrm{d}x\\
&\quad-\Big\{N_1c_1+c_2-\frac{\mu_2^2N_2}{4\varepsilon}-\frac{M^2\mu_2^2}{4\varepsilon_1}\Big\}\int_\Omega z^2(x,1,t)\mathrm{d}x\\
&\quad-\big\{N_2(l-L^2\varepsilon-\varepsilon)-(l^2+2l^2\varepsilon_1)\big\}\int_\Omega u_x^2\mathrm{d}x\\
&\quad-\Big\{\frac{b(L_1+L_3-L_2)}{4(L_2-L_1)}N_4+N_2b\Big\}\int_{L_1}^{L_2} v_x^2\mathrm{d}x\\
&\quad-\Big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4-N_2\Big\}\int_{L_1}^{L_2} v_t^2\mathrm{d}x\\
&\quad-(b-N_4)\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right)\\
&\quad-(a-N_4)\Big[\frac{L_1}{4}v_t^2(L_1,t)+\frac{L_3-L_2}{4}v_t^2(L_2,t)\Big]\\
&\quad+c(N_2)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x\\
&\quad+\big(\frac{N_1}{2}-\frac{g(0)}{4\varepsilon_1}\big)\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}
\tag{4.21}
$$

At this moment, we wish all coefficients except the last two in (4.21) will be negative. We want to choose $N_2$ and $N_4$ to ensure that

$$
\begin{gathered}
a-N_4\ge 0,\quad b-N_4\ge 0,\\
\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4-N_2>0.
\end{gathered}
$$

For this purpose, since $\frac{8l(L_2-L_1)}{L_1+L_3-L_2} < \min\{a,b\}$ we first choose $N_4$ satisfying

$$\frac{8l(L_2-L_1)}{L_1+L_3-L_2} < N_4 \le \min\{a,b\}.$$

Once $N_4$ is fixed, we pick $N_2$ satisfying

$$2l < N_2 < \frac{L_1+L_3-L_2}{4(L_2-L_1)} N_4.$$

Once the above constants are fixed, we take $\varepsilon < \frac{l}{8(L^2+1)}$ and $\varepsilon_1 < \frac{1}{4}$ such that

$$N_2(l - L^2\varepsilon - \varepsilon) - \big(l^2 + 2l^2\varepsilon_1\big) > 0.$$

Finally, choosing $N_1$ large enough such that the last coefficient in (4.21) is positive. From the above, we deduce that there exists two positive constants $\alpha_1$ and $\alpha_2$ such that (4.21) becomes

$$\frac{d}{dt}L(t) \le -\alpha_1 E(t) + \alpha_2 \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x. \tag{4.22}$$

On the other hand, by the definition of the functionals $\mathscr{D}(t)$, $\mathscr{F}_1(t)$, $\mathscr{F}_2(t)$, $\mathscr{F}_3(t)$ and $E(t)$, for $N_1$ large enough, there exists a positive constant $\beta$ satisfying

$$|N_2\mathscr{D}(t) + \mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + \mathscr{F}_3(t)| \le \beta E(t),$$

which implies

$$(N_1-\beta)E(t) \le L(t) \le (N_1+\beta)E(t).$$

To finish the proof of the stability estimates, we need to estimate the last term in (4.22). For the convenience of reading, we briefly repeat the process of [13].

Using (A2) and (4.1), we have

$$\begin{aligned}
\xi(t)\int_\Omega \int_0^t g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x &\le \int_\Omega \int_0^t \xi(s)g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\
&\le -\int_\Omega \int_0^t g'(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\
&\le -\int_\Omega \int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\
&\le -2\frac{d}{dt}E(t).
\end{aligned} \tag{4.23}$$

Moreover, (A2) and the definition of $E(t)$ give us

$$\int_\Omega u_x^2(x,t)\mathrm{d}x \le \frac{2}{l}E(t) \le \frac{2}{l}E(0), \quad \forall t \in \mathbb{R}_+.$$

Using (2.1), (2.7), (2.9) and (4.1), we arrive at

$$\begin{aligned}
\xi(t)\int_\Omega |\eta_x^t(x,s)|^2 \mathrm{d}x &= \xi(t)\int_\Omega (u_x(x,t) - u_x(x,t-s))^2 \mathrm{d}x \\
&\le 2\xi(t)\int_\Omega u_x^2(x,t)\mathrm{d}x + 2\xi(t)\int_\Omega u_x^2(x,t-s)\mathrm{d}x \\
&\le \frac{8}{l}E(0)\xi(t) + 2m_0\xi(t), \quad \forall t,s \in \mathbb{R}_+.
\end{aligned}$$

Then, we infer that for all $t \in \mathbb{R}_+$,

$$\xi(t) \int_\Omega \int_t^{+\infty} g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \le \big(\frac{8}{l}E(0) + 2m_0\big)\xi(t) \int_t^{+\infty} g(s)\mathrm{d}s. \tag{4.24}$$

Multiplying (4.22) by $\xi(t)$ and using (4.23) and (4.24), we obtain

$$\xi(t)\frac{d}{dt}L(t) + \beta_1 \frac{d}{dt}E(t) \le -\alpha_1 \xi(t)E(t) + \beta_2 \xi(t) \int_t^{+\infty} g(s)\mathrm{d}s, \tag{4.25}$$

where $\beta_1 = 2\alpha_2$ and $\beta_2 = \alpha_2\big(\frac{8}{l}E(0) + 2m_0\big)$.

Now, we define functionals $\mathscr{L}(t)$ and $h(t)$ as

$$\mathscr{L}(t) = \xi(t)L(t) + \beta_1 E(t) \quad \text{and} \quad h(t) = \xi(t) \int_t^{+\infty} g(s)\mathrm{d}s.$$

The fact that $L(t)$ and $E(t)$ are equivalent and (A2) imply that for some positive constants $\eta_1$ and $\eta_2$,

$$\eta_1 E(t) \le \mathscr{L}(t) \le \eta_2 E(t)\,. \tag{4.26}$$

Using (4.25), (4.26) and (A2), we obtain

$$\frac{d}{dt}\mathscr{L}(t) \le -\gamma_0 \xi(t)\mathscr{L}(t) + \beta_2 h(t),$$

where $\gamma_0 = \alpha_1/\eta_2$. We conclude that, for any $\gamma_1 \in (0, \gamma_0)$,

$$\frac{d}{dt}\mathscr{L}(t) \le -\gamma_1 \xi(t)\mathscr{L}(t) + \beta_2 h(t).$$

By integrating over $[0, T]$ with $T \ge 0$, we obtain

$$\mathscr{L}(T) \le e^{-\gamma_1 \int_0^T \xi(s)\mathrm{d}s}\Big(\mathscr{L}(0) + \beta_2 \int_0^T e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} h(t)\mathrm{d}t\Big).$$

Using (4.26), we have

$$E(T) \le \frac{1}{\eta_1} e^{-\gamma_1 \int_0^T \xi(s)\mathrm{d}s}\Big(\mathscr{L}(0) + \beta_2 \int_0^T e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} h(t)\mathrm{d}t\Big). \tag{4.27}$$

We notice that

$$e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} h(t) = \frac{1}{\gamma_1}\Big(e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s}\Big)' \int_t^{+\infty} g(s)\mathrm{d}s.$$

So integration by parts gives us

$$\begin{aligned}
&\int_0^T e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} h(t)\mathrm{d}t \\
&= \frac{1}{\gamma_1}\Big(e^{\gamma_1 \int_0^T \xi(s)\mathrm{d}s} \int_T^{+\infty} g(s)\mathrm{d}s - \int_0^{+\infty} g(s)\mathrm{d}s + \int_0^T e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} g(t)\mathrm{d}t\Big).
\end{aligned}$$

Consequently, combining with (4.27), we have

$$\begin{aligned}
E(T) \le \frac{1}{\eta_1}\Big(&\mathscr{L}(0)e^{-\gamma_1 \int_0^T \xi(s)\mathrm{d}s} + \frac{\beta_2}{\gamma_1} \int_T^{+\infty} g(s)\mathrm{d}s\Big) \\
&+ \frac{\beta_2}{\eta_2\gamma_1} e^{-\gamma_1 \int_0^T \xi(s)\mathrm{d}s} \int_0^T e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} g(t)\mathrm{d}t.
\end{aligned} \tag{4.28}$$

On the other hand, thanks to (A2), we have

$$\Big(e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s}(g(t))^{\gamma_1}\Big)' \le 0, \quad \forall t \in \mathbb{R}_+;$$

then

$$e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s}(g(t))^{\gamma_1} \le (g(0))^{\gamma_1}.$$

Thus

$$\int_0^T e^{\gamma_1 \int_0^t \xi(s)\mathrm{d}s} g(t)\mathrm{d}t \le (g(0))^{\gamma_1} \int_0^T (g(t))^{1-\gamma_1}\mathrm{d}t. \tag{4.29}$$

Finally, (4.28) and (4.29) imply that for the solution of (1.1)-(1.3) with

$$\gamma_2 = \frac{1}{\eta_1} \max\Big\{\mathscr{L}(0), \frac{\beta_2}{\gamma_1}, \frac{\beta_2}{\gamma_1}(g(0))^{\gamma_1}\Big\},$$

(2.11) holds. The proof is complete. □

4.2. **Case $\mu_2 = \mu_1$.** In this subsection, we assume that $\mu_1 = \mu_2 = \mu$ and prove the decay result of problem (1.1)-(1.3). By (2.8), we choose $\zeta = \tau\mu$, then we obtain the following consequence of Lemma 4.1.

**Lemma 4.5.** *Let $(u, v, z)$ be the solution of (2.2)-(2.3). Assume that $\mu_1 = \mu_2 = \mu$ and $E(t)$ satisfies (2.7). Then we have the inequality*

$$\frac{d}{dt}E(t) \le \frac{1}{2} \int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x. \tag{4.30}$$

When $\mu_1 = \mu_2 = \mu$, we need negative term $-\int_\Omega u_t^2\mathrm{d}x$ to get $-cE(t)$. For this purpose, we define the functional

$$\mathscr{F}_4(t) = -\int_\Omega u_t \int_0^\infty g(s)(u(t) - u(t-s))\mathrm{d}s\mathrm{d}x.$$

Then we have the following estimate.

**Lemma 4.6.** *The functional $\mathscr{F}_4(t)$ satisfies*

$$\begin{aligned}
\frac{d}{dt}\mathscr{F}_4(t) &\le -(g_0 - \delta_2 - \delta_2\mu)\int_\Omega u_t^2\mathrm{d}x + \delta_2 l^2 \int_\Omega u_x^2\mathrm{d}x + \delta_2\mu \int_\Omega z^2(x,1,t)\mathrm{d}x \\
&\quad + \Big(g_0 + \frac{g_0}{4\delta_2} + \frac{\mu g_0 L^2}{2\delta_2}\Big) \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x \\
&\quad - \frac{g(0)L^2}{\delta_2} \int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{4.31}$$

*Proof.* Taking the derivative of $\mathscr{F}_4(t)$ with respect to $t$ and using (2.2), we have

$$\begin{aligned}
&\frac{d}{dt}\mathscr{F}_4(t) \\
&= -\int_\Omega \Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s - \mu u_t - \mu z(x,1,t)\Big) \\
&\quad \times \int_0^\infty g(s)(u(t) - u(t-s))\mathrm{d}s\mathrm{d}x - \int_\Omega u_t \int_0^\infty g(s)(u_t(t) - u_t(t-s))\mathrm{d}s\mathrm{d}x \\
&= \int_\Omega lu_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))\mathrm{d}s\mathrm{d}x - g_0 \int_\Omega u_t^2\mathrm{d}x \\
&\quad + \int_\Omega u_t \int_0^\infty g(s)\eta_s^t(s)\mathrm{d}s\mathrm{d}x + \int_\Omega \Big(\int_0^\infty g(s)(u_x(t) - u_x(t-s))\mathrm{d}s\Big)^2\mathrm{d}x \\
&\quad + \int_\Omega \mu u_t \int_0^\infty g(s)(u(t) - u(t-s))\mathrm{d}s\mathrm{d}x
\end{aligned}$$

$$+\int_\Omega \mu z(x,1,t)\int_0^\infty g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x. \tag{4.32}$$

Using Young's inequality and (4.7), we obtain for any $\delta_2>0$,

$$\begin{aligned} &\int_\Omega l u_x \int_0^\infty g(s)(u_x(t)-u_x(t-s))\mathrm{d}s\mathrm{d}x \\ &\le \delta_2 l^2\int_\Omega u_x^2\mathrm{d}x + \frac{g_0}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\int_\Omega \mu u_t \int_0^\infty g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x \\ &\le \delta_2 \mu\int_\Omega u_t^2\mathrm{d}x + \frac{\mu g_0 L^2}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{4.34}$$

$$\begin{aligned} &\int_\Omega \mu z(x,1,t) \int_0^\infty g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x \\ &\le \delta_2 \mu\int_\Omega z^2(x,1,t)\mathrm{d}x + \frac{\mu g_0 L^2}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{4.35}$$

We notice that

$$\begin{aligned} &\int_\Omega \Big(\int_0^\infty g(s)(u_x(t)-u_x(t-s))\mathrm{d}s\Big)^2\mathrm{d}x \\ &\le \int_\Omega\int_0^\infty g(s)\mathrm{d}s\Big(\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\Big)\mathrm{d}x \\ &\le g_0\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} \int_\Omega u_t\int_0^\infty g(s)\eta_s^t(s)\mathrm{d}s\mathrm{d}x &= -\int_\Omega u_t\int_0^\infty g'(s)\eta^t(s)\mathrm{d}s\mathrm{d}x \\ &\le \delta_2\int_\Omega u_t\mathrm{d}x - \frac{g(0)L^2}{4\delta_2}\int_0^\infty g'(s)\eta_x^t(s)\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{4.37}$$

Inserting the estimates (4.33)-(4.37) into (4.32), we obtain (4.31). The proof is complete. □

Now, we define the Lyapunov functional

$$G(t)=N_1E(t)+N_2\mathscr{D}(t)+\mathscr{F}_1(t)+N_4\mathscr{F}_2(t)+N_5\mathscr{F}_3(t)+N_6\mathscr{F}_4(t), \tag{4.38}$$

where $N_1,N_2,N_4,N_5$ and $N_6$ are positive constants that will be fixed later.

Taking the derivative of (4.38) with respect to $t$ and taking advantage of the above lemmas, we have

$$
\begin{aligned}
\frac{d}{dt}G(t) \le & -\Big\{N_6(g_0-\delta_2-\delta_2\mu)-N_2-\Big(\frac{l+g_0}{2}+\frac{M^2\mu^2}{4\varepsilon_1}+\varepsilon_1 M^2\Big) \\
&\quad - N_5\Big\}\int_\Omega u_t^2\mathrm{d}x \\
&\quad -\Big\{N_5c_2-\frac{N_2\mu^2}{4\varepsilon}-\frac{M^2\mu^2}{4\varepsilon_1}-N_6\delta_2\mu\Big\}\int_\Omega z^2(x,1,t)\mathrm{d}x \\
&\quad -\big\{N_2(l-L^2\varepsilon-\varepsilon)-\big(l^2+2l^2\varepsilon_1\big)-N_6\delta_2 l^2\big\}\int_\Omega u_x^2\mathrm{d}x \\
&\quad -\big\{\frac{b(L_1+L_3-L_2)}{4(L_2-L_1)}N_4+N_2b\big\}\int_{L_1}^{L_2}v_x^2\mathrm{d}x \\
&\quad -\Big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4-N_2\Big\}\int_{L_1}^{L_2}v_t^2\mathrm{d}x \\
&\quad -(b-N_4)\frac{b}{4}\big((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\big) \\
&\quad -(a-N_4)\big[\frac{L_1}{4}v_t^2(L_1,t)+\frac{L_3-L_2}{4}v_t^2(L_2,t)\big] \\
&\quad +c(N_2,N_6)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x \\
&\quad +\Big(\frac{N_1}{2}-\frac{g(0)}{4\varepsilon_1}-\frac{N_6g(0)L^2}{4\delta_2}\Big)\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}
\tag{4.39}
$$

At this moment, we wish all coefficients except the last two in (4.39) will be negative. We want to choose $N_2$ and $N_4$ to ensure that

$$
\begin{gathered}
a-N_4\ge 0,\quad b-N_4\ge 0,\\
\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4-N_2>0.
\end{gathered}
$$

. For this purpose, since $\frac{8l(L_2-L_1)}{L_1+L_3-L_2}<\min\{a,b\}$ we first choose $N_4$ satisfying

$$
\frac{8l(L_2-L_1)}{L_1+L_3-L_2}<N_4\le\min\{a,b\}.
$$

Once $N_4$ is fixed, we pick $N_2$ satisfying

$$
2l<N_2<\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4.
$$

Then we take $\varepsilon<\frac{l}{8(L^2+1)}$ and $\varepsilon_1<\frac{1}{8}$ such that

$$
N_2(l-L^2\varepsilon-\varepsilon)-2l^2\varepsilon_1>\frac{3}{2}l^2.
$$

Once $\varepsilon$ and $\varepsilon_1$ are fixed, we take $N_5$ satisfying

$$
N_5>\max\Big\{\frac{2N_2\mu^2}{\varepsilon c_2},\frac{2M^2\mu^2}{\varepsilon_1c_2}\Big\}
$$

such that

$$
N_5c_2-\frac{N_2\mu^2}{4\varepsilon}-\frac{M^2\mu^2}{4\varepsilon_1}>\frac{3}{4}N_5c_2.
$$

Further, we choose $N_6$ satisfying

$$N_6 > \max\Big\{\frac{8N_2}{g_0}, \frac{4(l+g_0)}{g_0} + \frac{2M^2\mu^2}{\varepsilon_1 g_0} + \frac{8\varepsilon_1 M^2}{g_0}, \frac{8N_5}{g_0}\Big\}$$

such that

$$\frac{7}{8}N_6 g_0 - N_2 - \Big(\frac{l+g_0}{2} + \frac{M^2\mu^2}{4\varepsilon_1} + \varepsilon_1 M^2\Big) - N_5 > \frac{N_6 g_0}{2}.$$

Then, we pick $\delta_2$ satisfying

$$\delta_2 < \min\Big\{\frac{g_0}{8(1+\mu)}, \frac{N_5 c_2}{8N_6\mu}, \frac{1}{4N_6}\Big\}$$

such that

$$\frac{3}{4}N_5 c_2 - N_6\delta_2\mu > \frac{5}{8}N_5 c_2, \quad \frac{1}{8}N_6 g_0 - \delta_2 - \delta_2\mu > 0, \quad \frac{l^2}{2} - N_6\delta_2 l^2 > \frac{l^2}{4}.$$

Finally, choosing $N_1$ large enough such that the first two coefficients in (4.39) are negative and the last coefficient in (4.39) is positive. From the above, we deduce that there exists two positive constants $\alpha_3$ and $\alpha_4$ such that (4.39) becomes

$$\frac{d}{dt}G(t) \le -\alpha_3 E(t) + \alpha_4 \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x. \tag{4.40}$$

The remaining part of the proof of Theorem 2.2 can be finished, following the same steps as in the previous proof. □

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