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EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SEMILINEAR EQUATIONS ON EXTERIOR DOMAINS

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Abstract. In this article we study radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ where f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, δ), $f \equiv 0$ for $u > \delta$, and where the function $K(r)$ is assumed to be positive and $K(r) \to 0$ as $r \to \infty$. The primitive $F(u) = \int_0^u f(t)\,dt$ has a "hilltop" at $u = \delta$. We prove that if $K(r) \sim r^{-\alpha}$ with $\alpha > 2(N-1)$ and if $R > 0$ is sufficiently small then there are a finite number of solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius R such that $u \to 0$ as $r \to \infty$. We also prove the nonexistence of solutions if R is sufficiently large.

1. Introduction

In this article we study radial solutions of

$$\Delta u + K(r)f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where $x \in \Omega = \mathbb{R}^N \backslash B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin.

We assume there exist β, δ with $0 < \beta < \delta$ such that $f(0) = f(\beta) = f(\delta) = 0$ and $F(u) = \int_0^u f(s)\,ds$ where:

- (H1) f is odd and locally Lipschitz, $f < 0$ on $(0, \beta)$, $f > 0$ on (β, δ), $f \equiv 0$ on (δ, ∞), and $F(\delta) > 0$.

We note it follows that $F(u) = \int_0^u f(s)\,ds$ is even and has a unique positive zero, γ, with $\beta < \gamma < \delta$ such that

- (H2) $F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞), and F is strictly monotone on $(0, \beta)$ and on (β, δ).

In earlier papers [5]–[6] we studied (1.1), (1.3) when $\Omega = \mathbb{R}^N$ and $K(r) \equiv 1$. In [7] we studied (1.1)-(1.3) with $K(r) \equiv 1$ and $\Omega = \mathbb{R}^N \backslash B_R(0)$. In that paper we proved existence of an infinite number of solutions - one with exactly n zeros for each nonnegative integer n such that $u \to 0 \quad \text{as } |x| \to \infty$. Interest in the topic for this paper comes from recent papers [4, 11, 13] about solutions of differential equations on exterior domains.

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When f grows superlinearly at infinity - i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, and $\Omega = \mathbb{R}^N$ then problem (1.1)–(1.3) has been extensively studied [1]-[2], [10, 12, 14]. The type of nonlinearity addressed here has not been studied as extensively [5]-[7].

Since we are interested in radial solutions of (1.1)-(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $x \in \mathbb{R}^N$ and $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that u solves

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R,\infty) \text{ where } R > 0, \tag{1.4}$$

$$u(R) = 0, \quad u'(R) = b > 0. \tag{1.5}$$

We assume that there exist constants $c_1 > 0$, $c_2 > 0$, and $\alpha > 0$ such that

(H3) $c_1 r^{-\alpha} \le K(r) \le c_2 r^{-\alpha}$ for $\alpha > 2(N-1)$ on $[R,\infty)$.

In addition, we assume that

(H4) K, K' are continuous on $[R,\infty)$, $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$, and $\frac{rK'}{K} + 2(N-1) < 0$ on $[R,\infty)$.

Note that (H4)implies $r^{2(N-1)}K(r)$ is nonincreasing. In papers [8]-[9] we have discussed the case when $0 < \alpha < 2(N-1)$.

Theorem 1.1. *Let $N \ge 2$ and $\alpha > 2(N-1)$. Assuming* (H1)–(H4) *then if R is sufficiently large then there are no solutions of* (1.4)-(1.5) *such that* $\lim_{r\to\infty} u(r) = 0$.

Theorem 1.2. *Let $N > 2$ and $\alpha > 2(N-1)$. Assuming* (H1)–(H4) *and given a nonnegative integer n then if $R > 0$ is sufficiently small then there are constants $b_i > 0$ and solutions u_i with $0 \le i \le n$ of* (1.4)-(1.5) *with $b = b_i$ such that* $\lim_{r\to\infty} u_i(r) = 0$ *and u_i has i zeros on (R,∞).*

An important step in proving this result is showing that solutions can be obtained with more and more zeros by choosing b appropriately. Intuitively it can be of help to interpret (1.4) as an equation of motion for a point $u(r)$ moving in a double-well potential $F(u)$ subject to a damping force $-\frac{N-1}{r}u'$. This potential however becomes flat at $u = \pm\delta$. According to (1.5) the system has initial position zero and initial velocity $b > 0$. We will see that if $b > 0$ is sufficiently small then the solution will "fall" into the well at $u = \beta$ and - due to damping - it will be unable to leave the well whereas if $b > 0$ is sufficiently large the solution will reach the top of the hill at $u = \delta$ and will continue to move to the right indefinitely. For an appropriate value of b - which we denote b^{**} - the solution will reach the top of the hill at $u = \delta$ as $r \to \infty$. For values of b slightly less than b^{**} the solutions will not make it to the top of the hill at $u = \delta$ and they will nearly stop moving. Thus the solution "loiters" near the hilltop at $F(\delta)$ on a sufficiently long interval and will usually "fall" into the positive well at $u = \beta$ or the negative well at $u = -\beta$ after passing the origin a finite number of times, say n. For the right value of b - which we denote as b_n - the solution comes to rest at the local maximum of the function $F(u)$ at the origin as $r \to \infty$ after passing the origin n times.

In contrast to this is a double-well potential that goes off to infinity as $|u| \to \infty$ - for example $F(u) = u^2(u^2 - 4)$. Here the solutions of (1.4)-(1.5) behave quite differently. As b increases the number of zeros of u increases as $b \to \infty$. Thus the number of times that u reaches the local maximum of $F(u)$ at the origin increases as the parameter b increases. See for example [10, 12, 14].

2. Preliminaries and Proof of Theorem 1.1

Proof of Theorem 1.1. We observe since $\alpha > 2(N-1)$, by (1.4) and (H4)

$$\Big(\frac{1}{2}\frac{u'^2}{K} + F(u)\Big)' = -\frac{u'^2}{2rK}\Big(2(N-1) + \frac{rK'}{K}\Big) \geq 0. \tag{2.1}$$

Hence $\frac{1}{2}\frac{u'^2}{K} + F(u)$ is nondecreasing. Now suppose there is a solution of (1.4)-(1.5) such that $\lim_{r\to\infty} u(r) = 0$. Then u must have a first local maximum, M, such that $u' > 0$ on $[R, M)$. Then since $\frac{1}{2}\frac{u'^2}{K} + F(u)$ is nondecreasing we see that

$$\frac{1}{2}\frac{u'^2}{K} + F(u) \leq F(u(M)) \quad \text{on } (R, M).$$

Rewriting this and using (H3) we see that

$$\frac{|u'|}{\sqrt{2}\sqrt{F(u(M)) - F(u)}} \leq \sqrt{K} \leq \sqrt{c_2}\, r^{-\alpha/2} \quad \text{on } (R, M).$$

Integrating on (R, M) and noting that $\alpha > 2$ (since $\alpha > 2(N-1)$ and $N \geq 2$) gives

$$\int_0^{u(M)} \frac{dt}{\sqrt{2}\sqrt{F(u(M)) - F(t)}} \leq \frac{\sqrt{c_2}}{\frac{\alpha}{2} - 1}(R^{1-\frac{\alpha}{2}} - M^{1-\frac{\alpha}{2}}) \leq \frac{\sqrt{c_2}}{\frac{\alpha}{2} - 1} R^{1-\frac{\alpha}{2}}. \tag{2.2}$$

In addition, since $\frac{1}{2}\frac{u'^2}{K} + F(u)$ is nondecreasing we see that $0 < \frac{1}{2}\frac{b^2}{K(R)} \leq F(u(M))$ so $u(M) > \gamma$. Further it follows from (H1)-(H2) that $F(u(M)) \leq F(\delta)$ and $F(t) \geq -F_0$ for all $t \geq 0$ where $F_0 > 0$ and therefore $F(u(M)) - F(t) \leq F(\delta) + F_0$. Therefore (2.2) implies

$$\frac{\gamma}{\sqrt{2}\sqrt{F(\delta) + F_0}} \leq \frac{\sqrt{c_2}}{\frac{\alpha}{2} - 1} R^{1-\frac{\alpha}{2}}. \tag{2.3}$$

We note that the left-hand side of (2.3) is positive and independent of R but that the right-hand side goes to zero as $R \to \infty$ since $\alpha > 2$. Thus we see that if R is sufficiently large then (2.3) is violated hence there are no solutions u of (1.4)-(1.5) such that $\lim_{r\to\infty} u(r) = 0$ if R is sufficiently large. This completes the proof. $\square$

For the remainder of this paper we assume $\alpha > 2(N-1)$ and $N > 2$. Now we make the change of variables

$$u(r) = w(r^{2-N}).$$

Then (1.4)-(1.5) becomes

$$w'' + h(t) f(w) = 0, \tag{2.4}$$

and

$$w(R^{2-N}) = 0, \quad w'(R^{2-N}) = -\frac{bR^{N-1}}{N-2} < 0 \tag{2.5}$$

where $h(t) = T(t^{\frac{1}{2-N}})$ and $T(r) = \frac{r^{2(N-1)}K(r)}{(N-2)^2}$. Then from (H3) and (H4) we see:

$$h(t) = T(t^{\frac{1}{2-N}}) \sim \frac{t^q}{(N-2)^2} \quad \text{for } 0 < t \leq R^{2-N}, \tag{2.6}$$

where

$$q = \frac{\alpha - 2(N-1)}{N-2} > 0, \quad \lim_{t\to 0^+} \frac{t h'(t)}{h(t)} = q.$$

In addition, it follows from (H3)-(H4) that

$$\frac{c_1}{(N-2)^2}t^q \le h(t) \le \frac{c_2}{(N-2)^2}t^q \text{ and } h' > 0 \quad \text{for } 0 < t \le R^{2-N}. \tag{2.7}$$

Since we are seeking solutions of (1.4)-(1.5) with $\lim_{r\to\infty} u(r) = 0$ we see that this is equivalent to seeking solutions of (2.4)-(2.5) with $\lim_{t\to 0^+} w(t) = 0$. Instead though we now attempt to solve (2.4) with initial conditions at $t = 0$ instead of $t = R^{2-N}$,

$$w(0) = 0, \quad w'(0) = a > 0. \tag{2.8}$$

(We note that we will occasionally write $w(t) = w(t, a)$ to emphasize the dependence of w on a).

We attempt now to show that if $R > 0$ is sufficiently small and n is a nonnegative integer then there are $a_i > 0$ with $a_0 < a_1 < \cdots < a_n$ such that $w(R^{2-N}, a_i) = 0$ and $w(t, a_i)$ has i zeros on $(0, R^{2-N})$.

To proceed we temporarily extend the definition of the function h so that

$$h(t) = h(R^{2-N}) + \frac{h'(R^{2-N})}{qR^{(2-N)(q-1)}}[t^q - R^{(2-N)q}] \text{ for } t > R^{2-N}.$$

Note then that (2.7) holds on $(0, \infty)$.

A useful function in the analysis of (2.4)-(2.5) is

$$E(t) = \frac{1}{2}\frac{w'^2(t)}{h(t)} + F(w(t)) \quad \text{for } t > 0. \tag{2.9}$$

Using (2.4), we obtain

$$E'(t) = -\frac{w'^2 h'}{2h^2} \le 0 \quad \text{since } h' > 0 \text{ for } t > 0. \tag{2.10}$$

Thus E is nonincreasing. Also note that $\lim_{t\to 0^+} E(t) = +\infty$. We also observe using (2.4),

$$\frac{1}{2}w'^2 + h(t)F(w) = \frac{1}{2}a^2 + \int_0^t h'(s)F(w)\,ds. \tag{2.11}$$

Another useful equation is obtained by integrating (2.4) on $(0, t)$ and using (2.8) which gives

$$w'(t) = a - \int_0^t h(x)f(w(x))\,dx. \tag{2.12}$$

Integrating again on $(0, t)$ gives

$$w(t) = at - \int_0^t \int_0^s h(x)f(w(x))\,dx\,ds. \tag{2.13}$$

3. Proof of Theorem 1.2

From the standard theory of ordinary differential equations there exists a unique solution of (2.4), (2.8) on $[0, 2\epsilon)$ for some $\epsilon > 0$. Since E is nonincreasing then $\frac{1}{2}\frac{w'^2(t)}{h(t)} + F(w(t)) = E(t) \le E(\epsilon)$ for $t > \epsilon$ from which it follows that w and w' are uniformly bounded on compact subsets of $[0, \infty)$ and thus the solution $w(t)$ of (2.4), (2.8) exists on all of $[0, \infty)$ and varies continuously with respect to a on compact subsets of $[0, \infty)$.

Lemma 3.1. *Let $\alpha > 2(N-1)$, $N > 2$, and let w satisfy* (2.4), (2.8)*. Suppose* (H1)–(H4) *hold. Then there exists an $r_a > 0$ such that $w(r_a) = \beta$ and $0 < w < \beta$ on $(0, r_a)$. Also, $r_a \to \infty$ as $a \to 0^+$. In addition, $|w(t,a)| < \delta$ if $a > 0$ is sufficiently small.*

Proof. By (2.8) we have $w'(0) = a > 0$ so it follows that w is initially increasing. If $0 < w < \beta$ for all $t > 0$ then $f(w) < 0$ by (H1) and we see from (2.13) that $w(t) > at$. Thus $w(t)$ exceeds β for large enough t contradicting that $0 < w < \beta$. Thus there is an $r_a > 0$ such that $w(r_a) = \beta$ and $0 < w < \beta$ on $(0, r_a)$.

For the next part of the lemma we note first that if $|w(t,a)| < \gamma$ for all $t \geq 0$ then there is nothing to prove since $\gamma < \delta$. So suppose now that there exists $s_a > 0$ such that $|w(s_a)| = \gamma$ and $|w| < \gamma$ on $(0, s_a)$. Evaluating (2.11) at $t = s_a$ gives

$$\frac{1}{2}w'^2(s_a) \leq \frac{1}{2}a^2 \tag{3.1}$$

since $F(w(s_a)) = F(\gamma) = 0$ and $F(w) \leq 0$ on $(0, s_a)$. Using (3.1) and the fact that E is nonincreasing gives

$$F(w) \leq \frac{1}{2}\frac{w'^2}{h(t)} + F(w) = E(t) \leq E(s_a) = \frac{1}{2}w'^2(s_a) \leq \frac{1}{2}a^2 \text{ for } t \geq s_a. \tag{3.2}$$

Thus if $\epsilon > 0$ and $a > 0$ is sufficiently small then we see from (H2) and (3.2) that $|w| < \gamma + \epsilon < \delta$ for $t \geq 0$. This proves the last statement in Lemma 3.1.

Next observe from (H1) that $|f(w)| \leq C_1|w|$ for all w for some $C_1 > 0$. Using this along with (2.7) in (2.13) and estimating gives

$$|w(t)| \leq at + \frac{C_1 c_2}{(N-2)^2} t^{q+1} \int_0^t |w(s)|\, ds.$$

Applying the Gronwall inequality [3] we then obtain

$$|w| \leq a\Big(t + p(t)\int_0^t s e^{P(t)-P(s)}\, ds\Big) \tag{3.3}$$

where:

$$P(t) = \int_0^t p(s)\, ds = \int_0^t \frac{C_1 c_2 s^{q+1}}{(N-2)^2}\, ds = \frac{C_1 c_2 t^{q+2}}{(q+2)(N-2)^2}.$$

Evaluating (3.3) at $t = r_a$ gives

$$\beta \leq a\Big(r_a + p(r_a)\int_0^{r_a} s e^{P(r_a)-P(s)}\, ds\Big). \tag{3.4}$$

It follows from (3.4) and since $p(t)$, $P(t)$ are continuous that $r_a \to \infty$ as $a \to 0^+$. This completes the proof. $\square$

Lemma 3.2. *Let $\alpha > 2(N-1)$, $N > 2$, and let w satisfy* (2.4), (2.8)*. Suppose* (H1)–(H4) *hold. If $a > 0$ is sufficiently large then there exists a $t_a > 0$ such that $w(t_a) = \delta$ and $w(t) < \delta$ on $[0, t_a)$.*

Proof. It follows from (H1) that $|f(w)| \leq C_2$ for some $C_2 > 0$ so by (2.7) and (2.12):

$$w' \geq a - \frac{C_2 c_2 t^{q+1}}{(q+1)(N-2)^2} \quad \text{for } t \geq 0.$$

Integrating on $(0,t)$ gives

$$w(t) \geq at - \frac{C_2 c_2 t^{q+2}}{(q+2)(q+1)(N-2)^2} \quad \text{for } t \geq 0.$$

Thus for large enough a we have

$$w(1) \geq a - \frac{C_2 c_2}{(q+2)(q+1)(N-2)^2} \geq \delta.$$

Therefore $w(t)$ exceeds δ if $a > 0$ is sufficiently large. This completes the proof. $\square$

Let

$$S = \{a > 0: \text{ there is a } t_a > 0 \text{ such that } w(t_a, a) = \delta \text{ and } 0 < w < \delta \text{ on } (0, t_a)\}.$$

By Lemma 3.2 the set S is nonempty and from Lemma 3.1 the set S is bounded from below by a positive constant. Now we let:

$$0 < a^* = \inf S.$$

Lemma 3.3. *Let $\alpha > 2(N-1)$, $N > 2$, and let w satisfy* (2.4), (2.8). *Suppose* (H1)–(H4) *hold. Then $w(t,a^*) \to \delta$ as $t \to \infty$ and $w'(t,a^*) > 0$ on $[0,\infty)$.*

Proof. We first show $w(t,a^*) < \delta$ on $[0,\infty)$. If not then there is a $t_{a^*} > 0$ such that $w(t_{a^*},a^*) = \delta$ and $w(t,a^*) < \delta$ on $[0,t_{a^*})$. Thus $w'(t_{a^*},a^*) \geq 0$. In fact $w'(t_{a^*},a^*) > 0$ for if $w'(t_{a^*},a^*) = 0$ then by uniqueness of solutions of initial value problems $w(t,a^*) \equiv \delta$ contradicting that $w(0,a^*) = 0$. So since $w'(t_{a^*},a^*) > 0$ and $w(t_{a^*},a^*) = \delta$ then there is an $x_{a^*} > t_{a^*}$ such that $w(x_{a^*},a^*) > \delta + \epsilon$ for some $\epsilon > 0$. Now for $a < a^*$ but a close to a^* then by continuity with respect to initial conditions we have $w(x_{a^*},a) > \delta$ contradicting the definition of a^*. Thus $w(t,a^*) < \delta$ on $[0,\infty)$. Next we show

$$E(t,a^*) \geq F(\delta) \quad \text{for all } t > 0. \tag{3.5}$$

So suppose not. Then there is a $t_0 > 0$ such that $E(t_0,a^*) < F(\delta)$. By continuity with respect to initial conditions $E(t_0,a) < F(\delta)$ for $a > a^*$ and a close to a^*. However, for $a > a^*$ there is a $t_a > 0$ such that $w(t_a,a) = \delta$ and $w'(t_a,a) > 0$ so therefore since $f(w) \equiv 0$ for $w > \delta$ (by (H1)) then by (2.4) it follows that $w(t,a) = w'(t_a,a)(t-t_a) + \delta \geq \delta$ for $t \geq t_a$ and thus $E(t,a) \geq F(\delta)$ for all $t > t_a$. Then since E is nonincreasing (by (2.10)) it follows that $E(t,a) \geq F(\delta)$ for all $t > 0$ contradicting that $E(t_0,a) < F(\delta)$. Thus $E(t,a^*) \geq F(\delta)$ for $t > 0$.

Next we show $w'(t,a^*) > 0$ for $t \geq 0$. First, since $w'(0,a) = a > 0$ we see that $w'(t,a) > 0$ for small $t > 0$. Suppose then there is an $M > 0$ such that $w'(M,a^*) = 0$ and $w'(t,a^*) > 0$ on $[0,M)$. Then from (2.4) we have $w''(M,a^*) \leq 0$ and so $f(w(M,a^*)) \geq 0$. Thus $w(M,a^*) \geq \beta$. Also since we showed at the beginning of the proof that $w(t,a^*) < \delta$ for $t \geq 0$ it follows that $\beta \leq w(M,a^*) < \delta$ and since F is increasing on (β,δ) (by (H2)) then $E(M,a^*) = F(w(M,a^*)) < F(\delta)$. On the other hand it follows from (3.5) that $E(M,a^*) \geq F(\delta)$ and so we obtain a contradiction. Thus, $w'(t,a^*) > 0$ on $[0,\infty)$.

It now follows from Lemmas 3.1 and 3.2 that there is an L with $\beta < L \leq \delta$ such that $\lim_{t\to\infty} w(t,a^*) = L$. From (2.4) we see that $\frac{w''(t,a^*)}{h(t)} \to -f(L)$ as $t \to \infty$. If $f(L) \neq 0$ then $|w''| \geq \epsilon_0 h(t) > 0$ for large $t > 0$ and for some $\epsilon_0 > 0$. Since $h(t) \sim t^q$ with $q > 0$ then integrating the inequality $|w''| \geq \epsilon_0 h(t) > 0$ twice on (t_0,t) where t_0 is large we see that $|w| \to \infty$ contradicting that $w(t,a^*) \to L$. Thus

$f(L) = 0$ and since $\beta < L \le \delta$ it follows from (H1) that $L = \delta$. This completes the proof. □

Next we let

$$a^{**} = \inf\{a : w'(t, a) > 0 \text{ for } t \ge 0 \text{ and } \lim_{t\to\infty} w(t, a) = \delta\}. \tag{3.6}$$

By Lemma 3.3 we see that

$$a^* \in \{a : w'(t, a) > 0 \text{ for } t \ge 0 \text{ and } \lim_{t\to\infty} w(t, a) = \delta\}.$$

Thus the set on the right-hand side of (3.6) is nonempty and by Lemma 3.1 it is bounded from below by a positive constant. Thus $0 < a^{**} \le a^*$ and a similar argument as in Lemma 3.3 shows that $w(t, a^{**}) \to \delta$ as $t \to \infty$ and $w'(t, a^{**}) > 0$ for $t \ge 0$.

Lemma 3.4. *Let $\alpha > 2(N-1)$, $N > 2$, and let w satisfy* (2.4), (2.8). *Suppose* (H1)–(H4) *hold. If $0 < a < a^{**}$ then $w(t, a)$ has a local maximum, $M_a > 0$, and $M_a \to \infty$ as $a \to (a^{**})^-$. In addition, $w(M_a, a) < \delta$ and $w(M_a, a) \to \delta$ as $a \to (a^{**})^-$.*

Proof. If $a < a^{**}$ and $w'(t, a) > 0$ for $t \ge 0$ then we see as in Lemma 3.3 that $w(t, a) \to \delta$ contradicting the definition of a^{**}. Thus there exists $M_a > 0$ such that $w'(t, a) > 0$ on $[0, M_a)$ and $w'(M_a, a) = 0$. Then $w''(M_a, a) \le 0$ and so $f(w(M_a, a)) \ge 0$. Thus $w(M_a, a) \ge \beta$. Since we know $w(t, a)$ does not attain the value δ because $a < a^{**} \le a^*$ we therefore have $\beta \le w(M_a, a) < \delta$. Now if the $\{M_a\}$ were bounded then a subsequence would converge to some $M_{a^{**}}$ and so by the Arzela-Ascoli theorem a subsequence of $w(t, a)$ and $w'(t, a)$ would converge uniformly to $w(t, a^{**})$ and $w'(t, a^{**})$ on $[0, M_{a^{**}} + 1]$ as $a \to (a^{**})^-$ and $w'(M_{a^{**}}, a^{**}) = 0$ contradicting $w'(t, a^{**}) > 0$ from the remarks after Lemma 3.3. Thus $M_a \to \infty$ as $a \to (a^{**})^-$.

Also, as $a \to (a^{**})^-$ with $a < a^{**}$ we know $w(t, a)$ must get arbitrarily close to δ by continuity with respect to initial conditions and so $w(M_a, a) \to \delta$ as $a \to (a^{**})^-$. This completes the proof. □

Lemma 3.5. *Let $\alpha > 2(N-1)$, $N > 2$, and let w satisfy* (2.4), (2.8). *Suppose* (H1)–(H4) *hold. Given a positive integer n if $0 < a < a^{**}$ and a is sufficiently close to a^{**} then $w(t, a)$ has at least n zeros on $(0, \infty)$. In addition denoting the nth zero as $z_n(a)$ then $z_n(a) < R^{2-N}$ if R is sufficiently small and a is sufficiently close to a^{**} with $a < a^{**}$.*

Proof. From Lemma 3.4 we know that for a sufficiently close to a^{**} with $a < a^{**}$ then w has a local maximum M_a and $w(M_a) > \gamma > \beta$. From (2.4) it follows that $w'' < 0$ while $w > \beta$ and since $w'(M_a) = 0$ it follows that there exists $y_a > M_a$ such that $w(y_a) = \beta$. Thus there is an x_a with $M_a < x_a < y_a$ such that $w(x_a) = \gamma$.

From (2.10) we have

$$\frac{1}{2}\frac{w'^2}{h(t)} + F(w) = E(t) \le E(M_a) = F(w(M_a, a)) \quad \text{for } t \ge M_a.$$

Rewriting this gives

$$\frac{|w'|}{\sqrt{h}} \le \sqrt{2}\sqrt{F(w(M_a, a)) - F(w)}. \tag{3.7}$$

Now it follows from (2.6) that $0 < \frac{th'}{h} \le c_3$ for some $c_3 > 0$ and $t > 0$. Then from this and (2.7) we see that

$$0 < \frac{h'}{h^{3/2}} = \frac{th'}{h} \frac{1}{th^{1/2}} \le \frac{c_3(N-2)}{\sqrt{c_1}} \frac{1}{t^{\frac{q}{2}+1}}. \tag{3.8}$$

Thus from (2.10), (3.7)-(3.8), and (H3)

$$\begin{aligned} -E' = \frac{w'^2 h'}{2h^2} &= \frac{|w'|}{2\sqrt{h}} \; \frac{h'}{h^{3/2}} |w'| \\ &\le \frac{c_3(N-2)}{\sqrt{2c_1}} \sqrt{F(w(M_a,a)) - F(w)} \frac{1}{t^{\frac{q}{2}+1}} |w'|. \end{aligned} \tag{3.9}$$

Suppose now that $M_a < s < t$ and that $w' < 0$ on (M_a, t). Then integrating (3.9) on (M_a, t) and estimating we obtain

$$E(M_a,a) - E(t,a) \le \frac{c_3}{\sqrt{2c_1}} \frac{(N-2)}{M_a^{\frac{q}{2}+1}} \int_{w(t,a)}^{w(M_a,a)} \sqrt{F(w(M_a,a)) - F(y)}\, dy. \tag{3.10}$$

Let us assume $w(t,a) > 0$ and $w'(t,a) < 0$ for $t > M_a$. Then $[w(t,a), w(M_a,a)] \subset [0,\delta]$ and the integrand in (3.10) is bounded hence the integral in (3.10) is bounded independent of a. Thus the right-hand side of (3.10) goes to 0 as $a \to (a^{**})^-$ because $M_a \to \infty$ from Lemma 3.4 and the integral is uniformly bounded. Thus since $E(M_a,a) = F(u(M_a,a)) \to F(\delta)$ as $a \to (a^{**})^-$ by Lemma 3.4 it follows from (3.10) that $E(t,a) \to F(\delta)$ as $a \to (a^{**})^-$. Thus $E(t,a) \ge \frac{1}{2}F(\delta)$ for a close to a^{**} and $a < a^{**}$. In particular on (x_a, t) where $0 < w(t,a) < \gamma$ it follows that $F(w) \le 0$ so

$$\frac{1}{2} \frac{w'^2(t,a)}{h(t)} \ge \frac{1}{2} \frac{w'^2(t,a)}{h(t)} + F(w(t,a)) = E(t,a) \ge \frac{1}{2} F(\delta) \quad \text{on } (x_a, t) \tag{3.11}$$

hence from (2.6) and (H3)-(H4),

$$-w'(t,a) \ge \frac{\sqrt{c_1 F(\delta)}}{N-2} t^{q/2} \quad \text{on } (x_a, t)$$

and so integrating on (x_a, t) gives

$$w(t,a) \le \gamma - \frac{\sqrt{c_1 F(\delta)}}{(N-2)(\frac{q}{2}+1)} \big(t^{\frac{q}{2}+1} - x_a^{\frac{q}{2}+1}\big) \to -\infty \quad \text{as } t \to \infty$$

which contradicts that $w > 0$. Thus there exists $z_a > x_a$ such that $w(z_a,a) = 0$ and $w(t,a) > 0$ on $(0, z_a)$. By uniqueness of solutions of initial value problems we have $w'(z_a,a) < 0$ and so while $-\beta < w(t,a) < 0$ then $w'' < 0$ by (2.4) and so we see that there is a $Y_a > z_a$ such that $w(Y_a,a) = -\beta$. Now if $w(t,a)$ does not have a local minimum for $t > Y_a$ then we can show in a similar way as we did in Lemma 3.3 that $w \to L$ but now where $L < -\beta$ and $f(L) = 0$ implying $L = -\delta$. But since E is nonincreasing and F is even this would imply $F(\delta) = F(-\delta) \le \lim_{t\to\infty} E(t,a) \le E(M_a,a) = F(w(M_a,a))$ and hence by (H2) we have $w(M_a,a) \ge \delta$. But recall from Lemma 3.4 that since $a < a^{**}$ then $w(M_a,a) < \delta$ thus we obtain a contradiction. Therefore it must be the case that $w(t,a)$ has a local minimum, $m_a > z_a$, and in a similar way as in Lemma 3.4 it is possible to show $m_a \to \infty$ and $w(m_a,a) \to -\delta$ as $a \to (a^{**})^-$. Also as we did at the beginning of this lemma we can show that $w(t,a)$ has a second zero $z_{2,a} > z_a$ if a is sufficiently close to a^{**} and $a < a^{**}$. Similarly we can show that $w(t,a)$ has any desired (finite) number of zeros by choosing a sufficiently close to a^{**} with $a < a^{**}$. This completes the proof. □

Thus we see that $z_k(a)$ the kth zero of $w(t,a)$ on $(0,\infty)$ is defined as long as a is sufficiently close to a^{**} with $a < a^{**}$. It follows from continuous dependence of solutions on initial conditions that $z_k(a)$ is a continuous function of a. In addition $\lim_{a\to(a^{**})^-} z_k(a) = \infty$. This follows for if the $z_k(a)$ were bounded then for a subsequence (again labeled a) we would have $z_k(a) \to z^{**}$ and by the Arzela-Ascoli theorem $w(z^{**},a^{**}) = 0$ contradicting that $w(t,a^{**}) > 0$ on $(0,\infty)$.

Finally suppose R is sufficiently small and $a < a^{**}$ is sufficiently close to a^{**} so that $z_k(a) < R^{2-N}$. Then since we know $z_k(a)$ is continuous with $z_k(a) < R^{2-N} < \infty$ and $\lim_{a\to(a^{**})^-} z_k(a) = \infty$ then it follows from the intermediate value theorem that there is a smallest value of a denoted a_k such that $z_k(a_k) = R^{2-N}$. Thus $w(t,a_k)$ is a solution of (2.4) with k zeros on $(0,R^{2-N}]$. Now we let $b_k = (2-N)R^{1-N}w'(R^{2-N},a_k)$ and then finally if we let $u_k(r,b_k) = (-1)^k w(r^{2-N},a_{k+1})$ then $u_k(r,b_k)$ is a solution of (1.4)-(1.5) with $b = b_k$, with k zeros on (R,∞), and $\lim_{r\to\infty} u_k(r,b_k) = 0$. This completes the proof.

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