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# EXISTENCE AND BEHAVIOR OF POSITIVE SOLUTIONS TO ELLIPTIC SYSTEM WITH HARDY POTENTIAL

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Abstract. In this article, we study a class of elliptic systems with Hardy potentials. We analyze the possible behavior of radial solutions to the system when $p, t > 1$, $q, s > 0$ and $\lambda, \mu > (N-2)^2/4$, and obtain the existence of positive solutions to the system with the Dirichlet boundary condition under certain conditions. When $\lambda, \mu \le 0$, $p, t > 1$ and $q, s > 0$, we show that any radial positive solution is decreasing in $r$.

## 1. Introduction

We consider the elliptic system with singular potentials

$$\begin{aligned} -\Delta u &= \lambda \frac{u}{|x|^2} - u^p v^q, \quad x \in B_1(0)\backslash\{0\}, \\ -\Delta v &= \mu \frac{v}{|x|^2} - u^s v^t, \quad x \in B_1(0)\backslash\{0\}, \end{aligned} \tag{1.1}$$

where $p > 1$, $t > 1$, and $B_1(0) \subset \mathbb{R}^N$ $(N \ge 3)$ is a unit ball centered at origin. The right hand side in (1.1) contains singular terms, which are usually called the singular inverse square potentials or Hardy potential in the literature. In this study, we investigate system (1.1) with the Dirichlet boundary condition

$$u(x) = v(x) = 0, \quad x \in \partial B_1(0). \tag{1.2}$$

Elliptic problems with Hardy potential have been an interesting topic in the field of singular partial differential equation for a long time. Let us briefly review some results with respect to the single equation with Hardy potential. Guerch and Véron [8] studied the equation

$$-\Delta u = V(x)u - h(u), \tag{1.3}$$

where $V(x) = \lambda |x|^{-2}$ and $h(u)$ has the property like $u^p$. They considered the classification of positive solutions as $\lambda \le \frac{(N-2)^2}{4}$, and showed the behavior of positive solutions as $\lambda > \frac{(N-2)^2}{4}$ under some conditions. In a recent work [1], Cirstea studied

$$-\Delta u = \lambda \frac{u}{|x|^2} - b(x) f(u), \quad \forall x \in \Omega\backslash\{0\}, \tag{1.4}$$

where $\Omega$ is a domain containing the origin, and the function $b(x)$ is a continuous positive function (may vanish at the origin) or a singular function with a singular

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origin. It is well-known that $(N-2)^2/4$ is the Hardy constant corresponding to the Hardy potential. The function $f(u)$ has similar behavior like $u^p$ $(p>1)$. When $\lambda \le \left(\frac{N-2}{2}\right)^2$, the qualitative properties of positive solutions at the origin were presented and a complete classification was provided.

The supercritical case has recently been studied. In [14] the authors considered the singular logistic equation

$$\begin{gathered} -\Delta u = \lambda \frac{u}{|x|^2} - b(x)u^p, \quad x \in \Omega\backslash\{0\}, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.5}$$

where $\lambda > \frac{(N-2)^2}{4}$, $p>1$ and $b(x)$ is a nonnegative continuous function over $\overline{\Omega}$. It shows that (1.5) has a minimal positive solution and a maximal positive solution if $b(x)$ is a positive function. Suppose that $b(x)$ has a vanishing set denoted as $\Omega_0 = \{x \in \Omega : b(x) = 0\}$. If $\Omega_0 \Subset \Omega$, the boundary of $\Omega$ belongs to $C^\mu$, $\Omega_0$ is a connected set and $b(0) \neq 0$, then problem (1.5) has a minimal positive solution and a maximal positive solution. In [14], the authors considered the problem

$$-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\theta u^p, \quad x \in \Omega\backslash\{0\}, \tag{1.6}$$

where $\theta > -2$ and $\lambda > \frac{(N-2)^2}{4}$. The asymptotic estimate of positive solutions to (1.6) was

$$\lim_{|x|\to 0} \frac{u(x)}{|x|^{-\frac{2+\theta}{p-1}}} = \Big[\lambda + \frac{2+\theta}{p-1}\Big(\frac{2+\theta}{p-1} + 2 - N\Big)\Big]^{1/(p-1)}.$$

The uniqueness of positive solutions to (1.6) with $u=0$ for all $x \in \partial\Omega$ was shown. In fact, a direct calculation yields

$$\lambda + \frac{2+\theta}{p-1}(\frac{2+\theta}{p-1} + 2 - N) > 0 \quad \text{when } \lambda > (N-2)^2/4.$$

This conclusion shows that any positive solution of (1.6) blows up at the origin when $\theta > -2$ and $\lambda > \frac{(N-2)^2}{4}$. When $\theta = -2$, positive solutions are uniformly bounded near the origin. When $\theta < -2$, any positive solution vanishes at the origin. So positive solutions of (1.6) have no singularity when $\theta \le -2$. In addition, for problem (1.5), we can prove the uniqueness of positive solutions and give the exact behavior of the positive solution in a similar manner.

García-Melián and Rossi [7] considered the competitive type system

$$\begin{gathered} \Delta u = u^p v^q, \quad x \in \Omega, \\ \Delta v = u^s v^t, \quad x \in \Omega, \\ u = v = \infty, \quad x \in \partial\Omega, \end{gathered} \tag{1.7}$$

where $\Omega$ is a bounded smooth domain, $p,t>1$ and $q,s>0$. The existence, uniqueness, blow-up rate and nonexistence of positive solutions were established under certain conditions. Li and Wang [9] considered an elliptic system in a smooth bounded domain,

$$\begin{gathered} -\Delta u = u(a_1 - b_1 u^m - c_1 v^n) \quad x \in \Omega, \\ -\Delta v = v(a_2 - b_2 u^p - c_2 v^q), \quad x \in \Omega, \\ u = v = +\infty, \quad x \in \partial\Omega, \end{gathered} \tag{1.8}$$

where $a_i \ge 0$, $b_i, c_i$ $(i=1,2)$ are positive constants, $m,q>0$ and $n,p \ge 0$. Based on the construction of certain sub-solution and upper-solution, some conditions on the parameters and the exponents to ensure the existence of positive solutions were explored. García-Melián [6] extended the results in [7] to study the existence and uniqueness of positive solutions of

$$\begin{gathered} \Delta u = a(x)u^p v^q, \quad x \in \Omega, \\ \Delta v = b(x)u^r v^s, \quad x \in \Omega, \\ u = v = \infty, \quad x \in \partial\Omega, \end{gathered} \tag{1.9}$$

where $a(x)$ and $b(x)$ satisfy

$$\begin{gathered} C_2 d(x, \partial\Omega)^{\gamma_1} \le a(x) \le C_1 d(x, \partial\Omega)^{\gamma_1} \text{ for } x \in \Omega, \\ C_2 d(x, \partial\Omega)^{\gamma_2} \le b(x) \le C_1 d(x, \partial\Omega)^{\gamma_2} \text{ for } x \in \Omega, \end{gathered}$$

for $\gamma_1 > -2$, $\gamma_2 > -2$.

This article is mainly devoted to the behavior analysis of positive solutions to (1.1) with the boundary condition (1.2). In our discussions, we apply some arguments and techniques with respect to boundary blow-up problems, which can be found in [2, 3, 4, 5, 10, 11, 12]. We also use the methods of subsolution and supersolution [13].

Let us summarize our main results into the following three theorems. For any radial positive solution $(u(x), v(x))$, the first theorem reveals the behavior of $u(x) + v(x)$ at the origin.

**Theorem 1.1.** *Suppose that $\lambda, \mu > \frac{(N-2)^2}{4}$, $p, t \ge 1$, $q, s > 0$ and $(u, v)$ is an arbitrary radial positive solution of (1.1). Then*

$$\lim_{|x| \to 0} \{u(x) + v(x)\} = \infty.$$

When the positive solution $(u, v)$ satisfies $u \to \infty$ and $v \to \infty$ as $|x| \to 0$, $(u, v)$ is said to be a blow-up solution. The following theorem is regarding the existence of blow-up solution and an estimate of blow-up solution near the origin.

**Theorem 1.2.** *Suppose that $\lambda, \mu > \frac{(N-2)^2}{4}$, $p - 1 > s > 0$ and $t - 1 > q > 0$. Then the following two statements are true.*

- (i) *Problem (1.1) with condition (1.2) has at least one positive blow-up solution $(U, V)$.*
- (ii) *Assume that $\lambda > \max\{4\lambda_1[B_1(0)], (N-2)^2/4\}$ and $(\hat{u}, \hat{v})$ is a blow-up solution of the problem (1.1) with condition (1.2). Then for any $\tau > 0$ there exists $\delta := \delta(\tau) > 0$ and $C_1, C_2 > 0$, such that*

$$C_1 |x|^{-\frac{2(t-1-q)}{(p-1)(t-1)-qs} + \tau} \le \hat{u}(x) \le C_2 |x|^{-\frac{2(t-1-q)}{(p-1)(t-1)-qs} - \tau}, \quad \forall x \in B_\delta 0) \backslash \{0\}, \tag{1.10}$$

$$C_1 |x|^{-\frac{2(p-1-s)}{(p-1)(t-1)-qs} + \tau} \le \hat{v}(x) \le C_2 |x|^{-\frac{2(p-1-s)}{(p-1)(t-1)-qs} - \tau}, \quad \forall x \in B_\delta(0) \backslash \{0\}. \tag{1.11}$$

For the parameters $\lambda, \mu \le 0$, we may obtain the behavior of positive solutions.

**Theorem 1.3.** *Suppose that $\lambda, \mu \le 0$, $p, t > 1$, $q, s > 0$ and $(u, v)$ is an arbitrary positive radial solution of (1.1) with condition (1.2). Then we have*

- (i) *$u'(r), v'(r) < 0$ for $r \in (0, 1]$. Moreover, $u(r)$ and $v(r)$ are convex functions; and*
- (ii) *$u(r) \to \infty$ and $v(r) \to \infty$ when $r \to 0$.*

The rest of this article is organized as follows. In Section 2, we introduce some preliminary results which will be used in the proofs of our main results. In Section 3, some rough estimates of positive solution of the problem (1.1) are established. Section 4 is dedicated to the existence of positive solution to problem (1.1) with boundary condition (1.2) and the estimate of blow-up rate of positive solution at the origin, i.e., the proof of Theorem 1.2. Finally, in Section 5 we show the proof of Theorem 1.3.

## 2. Preliminary results

To make our discussions in a straightforward manner, we need the following three technical lemmas. The first lemma is regarding the comparison principle which can be found in [3, 5].

**Lemma 2.1.** *Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\alpha(x)$ and $\beta(x)$ are continuous functions in $\Omega$ with $\|\alpha\|_\infty < \infty$, and $\beta(x)$ is nonnegative and not identically zero. Let $u_1, u_2 \in C^1(\Omega)$ be positive in $\Omega$ and in the weak sense satisfy*

$$\Delta u_1 + \alpha(x)u_1 - \beta(x)g(u_1) \le 0 \le \Delta u_2 + \alpha(x)u_2 - \beta(x)g(u_2), \quad x \in \Omega,$$
$$\limsup_{x\to\partial\Omega}(u_2 - u_1) \le 0,$$

*where $g(u)$ is continuous and such that $\frac{g(u)}{u}$ is strictly increasing with respect to $u$ in the range of $\min\{u_1, u_2\} < u < \max\{u_1, u_2\}$. Then $u_2 \le u_1$ holds.*

For the Hardy potential, it is well-known that the Hardy constant $H = \frac{(N-2)^2}{4}$ and the Hardy inequality

$$\frac{(N-2)^2}{4}\int_\Omega \frac{\phi^2}{|x|^2}dx \le \int_\Omega |\nabla\phi|^2 dx, \quad \forall \phi \in W_0^{1,2}(\Omega).$$

In addition, $\frac{(N-2)^2}{4}$ can be expressed as

$$\frac{(N-2)^2}{4} = \inf_{\phi\in W_0^{1,2}(\Omega)/\{0\}} \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega \frac{\phi^2}{|x|^2}dx}, \tag{2.1}$$

but it is not attained. Let $\lambda_1[a(x), \Omega]$ denote the first eigenvalue of

$$-\Delta u = \lambda a(x)u, \quad x \in \Omega,$$
$$u = 0, \quad x \in \partial\Omega.$$

A close relation between the Hardy constant and the first eigenvalue of the Dirichlet eigenvalue problem is given in [14].

**Lemma 2.2.** *Suppose that $\Omega$ is a bounded smooth domain and $0 \in \Omega$. Then we have*

$$\lim_{\delta\to 0}\lambda_1\big[\frac{1}{|x|^2}, \Omega_\delta\big] = \frac{(N-2)^2}{4};$$
$$\lim_{\epsilon\to 0}\lambda_1\big[\frac{1}{|x|^2+\epsilon}, \Omega\big] = \frac{(N-2)^2}{4},$$

*where $\Omega_\delta = \{x \in \Omega : |x| > \delta\}$.*

By the uniqueness and blow-up rate of positive solutions for a single equation with the Hardy potential [14], we can obtain the following conclusion.

**Lemma 2.3.** *Suppose that $p > 1$, $\sigma > -2$, $\lambda > (N-2)^2/4$, $\Omega$ is a bounded smooth domain and $0 \in \Omega$.*

(i) *Assume that $u$ is any positive solution of*
$$-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, \ x \in \Omega\backslash\{0\}.$$
*Then $u$ satisfies*
$$\lim_{|x|\to 0} |x|^{\frac{2+\sigma}{p-1}} u(x) = \Big[\lambda + \frac{2+\sigma}{p-1}\Big(\frac{2+\sigma}{p-1} + 2 - N\Big)\Big]^{1/(p-1)}.$$

(ii) *Assume that $c$ is a nonnegative constant ($c = +\infty$ is also allowed). Then the problem*
$$\begin{gathered} -\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in \Omega, \\ u = c, \quad x \in \partial\Omega \end{gathered} \tag{2.2}$$
*has a unique positive solution $U_{p,\sigma}$.*

*Proof.* Firstly, Part (i) is the direct result of [14]. Now, we simply sketch the proof of Part (ii). Suppose that $c = 0$. Since $\lambda > \frac{(N-2)^2}{4}$, for any sufficiently small $\delta > 0$, by Lemma 2.2 and the standard arguments of logistic equations, it follows that
$$\begin{gathered} -\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in \Omega_\delta, \\ u = 0, \quad x \in \partial\Omega_\delta \end{gathered}$$
has a unique positive solution $u_\delta$. By the comparison principle and regularity arguments of elliptic equations, $w = \lim_{\delta\to 0} u_\delta$ is a positive solution of the problem (2.2). Suppose that $c > 0$ $(= \infty)$, then a similar argument shows that (2.2) has a positive solution $w$. Suppose that $\omega$ is any positive solution of (2.2). By Part (i), we have
$$\lim_{|x|\to 0} |x|^{\frac{2+\sigma}{p-1}} \omega(x) = \Big[\lambda + \frac{2+\sigma}{p-1}\Big(\frac{2+\sigma}{p-1} + 2 - N\Big)\Big]^{1/(p-1)}.$$
If $c = \infty$, by the standard arguments of boundary blow-up problems, we can find the exact behavior near boundary for positive solutions. By the standard method, one can see the uniqueness of positive solutions. □

## 3. Blow-up behavior of positive solutions

In this section, we present some behavior analysis of positive solutions of (1.1) and prove Theorem 1.1.

**Lemma 3.1.** *Suppose that $p, t \geq 1$, $q, s > 0$, $\lambda, \mu > \frac{(N-2)^2}{4}$ and $(u(x), v(x))$ is an arbitrary solution of (1.1). Then*
$$\limsup_{|x|\to 0} \{u(x) + v(x)\} = \infty.$$
*That is,*
$$\limsup_{|x|\to 0} u(x) = \infty \quad \text{or} \quad \limsup_{|x|\to 0} v(x) = \infty.$$

*Proof.* By the way of contradiction, we suppose that $\limsup_{|x|\to 0^+}\{u(x)+v(x)\} = M$, where $M \in [0,\infty)$. Then there is a $\delta > 0$ such that $0 < v(x) \le M+1$ for all $x \in B_\delta(0)\backslash\{0\}$.

**Case 1: $p > 1$.** From the first equation of (1.1), we have

$$-\Delta u \ge \lambda \frac{u}{|x|^2} - (M+1)^q u^p \text{ in } B_\delta(0)\backslash\{0\}. \tag{3.1}$$

Let $w(x) = (M+1)^{\frac{q}{p-1}} u$, then we obtain

$$-\Delta w \ge \lambda \frac{w}{|x|^2} - w^p \text{ in } B_\delta(0)\backslash\{0\}.$$

Since $\lambda > \frac{(N-2)^2}{4}$, by Lemma 2.2, for any sufficiently small $\epsilon > 0$, the problem

$$\begin{gathered} -\Delta u = \lambda \frac{u}{|x|^2} - u^p, \quad x \in B_\delta\backslash \overline{B}_\epsilon(0), \\ u = 0, \quad |x| = \delta \text{or } |x| = \epsilon \end{gathered}$$

has a unique positive solution $u_\epsilon$. By the comparison principle, the function $u_*(x) := \lim_{\epsilon\to 0} u_\epsilon(x)$ is well defined in $B_\delta(0)\backslash\{0\}$, and hence by the regularity arguments, $u_*$ satisfies

$$-\Delta u_* = \lambda \frac{u_*}{|x|^2} - u_*^p \text{ in } B_\delta(0)\backslash\{0\}.$$

By the comparison principle,

$$w(x) \ge u_\epsilon(x) \quad \text{for all } x \in B_\delta(0)\backslash \overline{B}_\epsilon(0).$$

Letting $\epsilon \to 0$, we find

$$w(x) \ge u_*(x) \quad \text{for all } x \in B_\delta(0)\backslash\{0\}.$$

By Lemma 2.3, $\lim_{|x|\to 0^+} u(x) = \infty$, which contradicts $\limsup_{|x|\to 0^+}\{u(x)+v(x)\} = M$.

**Case 2: $p = 1$.** Since $\lambda > \frac{(N-2)^2}{4}$ and using (3.1), there is $\tau \in (0,\delta)$ such that

$$-\Delta u \ge \frac{1}{2}(\lambda + \frac{(N-2)^2}{4})\frac{u}{|x|^2} \quad \text{for } x \in B_\tau(0)\backslash\{0\}.$$

For each $\eta \in (0,\tau)$, $u$ is a positive supersolution of

$$\begin{gathered} -\Delta \phi = \frac{1}{2}(\lambda + \frac{(N-2)^2}{4})\frac{\phi}{|x|^2}, \quad x \in B_\tau(0)\backslash \overline{B}_\eta(0), \\ \phi(x) = 0, \quad x \in \partial B_\tau(0) \cup \partial B_\eta(0). \end{gathered}$$

So, we derive that

$$\lambda_1\big[\frac{1}{|x|^2}, B_\tau(0)\backslash \overline{B}_\eta(0)\big] > \frac{1}{2}(\lambda + \frac{(N-2)^2}{4}),$$

which contradicts Lemma 2.2, namely

$$\lim_{\eta\to 0} \lambda_1\big[\frac{1}{|x|^2}, B_\tau(0)\backslash \overline{B}_\eta(0)\big] = \frac{(N-2)^2}{4}.$$

□

**Remark 3.2.** Suppose that $p,t \ge 1$, $q,s > 0$ and $\lambda,\mu > (N-2)^2/4$. From the proof of Lemma 3.1, it follows that $\lim_{|x|\to 0} v(x) = \infty$ if $u$ is bounded near the origin, and $\lim_{|x|\to 0} u(x) = \infty$ if $v$ is bounded near the origin.

*Proof of Theorem 1.1.* Suppose that $(u(x), v(x))$ is an arbitrary positive radial solution of (1.1). By Lemma 3.1, we have

$$\limsup_{|x|\to 0}\{u(x)+v(x)\}=\infty.$$

It suffices to show that

$$\liminf_{|x|\to 0}\{u(x)+v(x)\}=\infty.$$

By the way of contradiction, we suppose $\liminf_{|x|\to 0}\{u(x)+v(x)\}<\infty$. For convenience of our statement, we denote $u(r)=u(x)$ and $v(r)=v(x)$ when $|x|=r$. In view of

$$\liminf_{|x|\to 0}\{u(x)+v(x)\}<\limsup_{|x|\to 0}\{u(x)+v(x)\},$$

there is $\{r_n\}$ with $r_n\to 0$ such that

$$\begin{gathered} u''(r_n)+v''(r_n)\ge 0,\quad u'(r_n)+v'(r_n)=0,\\ \lim_{n\to\infty}\{u(r_n)+v(r_n)\}=\liminf_{|x|\to 0}\{u(x)+v(x)\}. \end{gathered}$$

Without loss of generality, we assume that $\lambda\le\mu$. It follows from (1.1) that

$$\frac{\lambda}{r_n^2}\le\frac{u(r_n)^p v(r_n)^q+u(r_n)^s v(r_n)^t}{u(r_n)+v(r_n)}, \tag{3.2}$$

which implies

$$\frac{\lambda}{r_n^2}<u(r_n)^{p-1}v(r_n)^q+u(r_n)^s v(r_n)^{t-1}.$$

So, we have

$$u(r_n)^{p-1}v(r_n)^q+u(r_n)^s v(r_n)^{t-1}\to\infty\quad (n\to\infty).$$

This indicates that $\{u(r_n)+v(r_n)\}$ is an unbounded set. This contradicts the assumption of $\lim_{n\to\infty}(u(r_n)+v(r_n))<\infty$. □

We now show some analysis on the behavior of positive solutions to (1.1) near the origin.

**Proposition 3.3.** *Suppose that $\lambda,\mu>(N-2)^2/4$, $t-1>q>0$, $p-1>s>0$, and $(u(x), v(x))$ is an arbitrary positive solution of the system* (1.1) *such that both $\lim_{|x|\to 0}u(x)$ and $\lim_{|x|\to 0}v(x)$ exist. Then at least one of following statements holds: (i) both $u(x)$ and $v(x)$ blow up at the origin; (ii) $u(x)$ blows up and $v(x)$ vanishes at the origin; and (iii) $u(x)$ vanishes and $v(x)$ blows up at the origin.*

*Proof.* We claim that either $\lim_{|x|\to 0}u(x)=\infty$ or $\lim_{|x|\to 0}u(x)=0$ holds. Otherwise, we suppose $\lim_{|x|\to 0}u(x)=m\in(0,\infty)$. So there exists a $\delta>0$ such that

$$m/2\le u(x)\le 2m\quad\text{for } x\in B_\delta(0)\backslash\{0\}.$$

Then, we see that

$$-\Delta v\le\mu\frac{v}{|x|^2}-(m/2)^s v^t\quad\text{for } x\in B_\delta(0)\backslash\{0\}.$$

Let $w_1=(m/2)^{\frac{s}{t-1}}v$. Then

$$-\Delta w_1\le\mu\frac{w_1}{|x|^2}-w_1^t\quad\text{in } B_\delta(0)\backslash\{0\}.$$

By Lemma 2.3 and an analogous argument described in [14], it follows that

$$v(x) \le C|x|^{-\frac{2}{t-1}} \text{ for } x \in B_\delta(0)\backslash\{0\}.$$

From the first equation of (1.1), we have

$$-\Delta u \ge \lambda \frac{u}{|x|^2} - C^q |x|^{-\frac{2q}{t-1}} u^p \quad \text{in } B_\delta(0)\backslash\{0\}.$$

Let $w_2 = C^{\frac{q}{p-1}} u$. Then

$$-\Delta w_2 \ge \lambda \frac{w_2}{|x|^2} - |x|^{-\frac{2q}{t-1}} w_2^p \quad \text{in } B_\delta(0)\backslash\{0\}.$$

Hence, by $t-1 > q$ and Lemma 2.3, we deduce that

$$u(x) \ge C_1 |x|^{-\frac{2-\frac{2q}{t-1}}{p-1}} = C_1 |x|^{-\frac{2(t-1-q)}{(p-1)(t-1)}} \quad \text{on } B_\delta(0)\backslash\{0\},$$

which contradicts the assumption $\lim_{|x|\to 0} u(x) = m \in (0,\infty)$.

Similarly, we can prove that either $\lim_{|x|\to 0} v(x) = \infty$ or $\lim_{|x|\to 0} v(x) = 0$ holds. □

**Proposition 3.4.** *Suppose that $p,t \ge 1$, $q,s > 0$ and $\lambda,\mu > (N-2)^2/4$ and $(u(x),v(x))$ is an arbitrary positive solution of (1.1). Then both $u^{p-1}v^q$ and $u^s v^{t-1}$ are unbounded near the origin. In particular, when $p=1$ and $t=1$, both $u$ and $v$ are unbounded near the origin.*

*Proof.* Suppose that $u^{p-1}v^q$ is bounded near the origin. Using the first equation in (1.1), we have

$$-\Delta u + (u^{p-1}v^q)u = \lambda \frac{u}{|x|^2} \quad \text{in } B_1(0)\backslash\{0\}.$$

Since $u^{p-1}v^q$ is bounded near the origin, there is a sufficiently small $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0]$,

$$-\Delta \ge \frac{1}{2}\big[\lambda + \frac{(N-2)^2}{4}\big]\frac{u}{|x|^2} \quad \text{in } B_{\delta_0}(0).$$

This implies

$$\lambda_1\big[\frac{1}{|x|^2}, B_{\delta_0}(0)\backslash B_\delta(0)\big] \ge \frac{1}{2}\big[\lambda + \frac{(N-2)^2}{4}\big] \quad \text{for } \delta \in (0,\delta_0),$$

which is a contradiction to

$$\lim_{\delta\to 0} \lambda_1\big[\frac{1}{|x|^2}, B_{\delta_0}(0)\backslash B_\delta(0)\big] = \frac{(N-2)^2}{4}.$$

□

## 4. Existence and estimates of blow-up solutions

In this section, we deal with the existence of positive blow-up solutions of the problem (1.1) with the boundary condition (1.2) and establish estimates of positive blow-up solutions near the origin.

Suppose that $\lambda > 4\lambda_1[B_1(0)]$ and $\alpha \in (0,1)$ satisfies $\lambda\frac{\alpha^2}{(1+\alpha)^2} > \lambda_1[B_1(0)]$. For the case $\sigma \ge 0$, let $u_{0,\alpha}$ denote the unique positive solution of

$$\begin{gathered} -\Delta u = \lambda \frac{\alpha^2}{(1+\alpha)^2} u - \alpha^2 (1+\alpha)^\sigma u^p, \quad x \in B_1(0) \\ u(x) = 0, \quad x \in \partial B_1(0). \end{gathered}$$

For convenience, for the case of $-2 < \sigma < 0$, by $u_{0,\alpha}$ we denote the unique positive solution of

$$-\Delta u = \lambda \frac{\alpha^2}{(1+\alpha)^2} u - \alpha^2 (1-\alpha)^\sigma u^p, \quad x \in B_1(0)$$
$$u(x) = 0, \quad x \in \partial B_1(0).$$

When $\sigma \geq 0$, let $u_{\infty,\alpha}$ denote the unique positive solution of

$$-\Delta u = \lambda \frac{\alpha^2}{(1-\alpha)^2} u - \alpha^2 (1-\alpha)^\sigma u^p, \quad x \in B_1(0)$$
$$u(x) = \infty, \quad x \in \partial B_1(0).$$

For the case of $-2 < \sigma < 0$, let $u_{\infty,\alpha}$ denote the unique positive solution of

$$-\Delta u = \lambda \frac{\alpha^2}{(1-\alpha)^2} u - \alpha^2 (1+\alpha)^\sigma u^p, \quad x \in B_1(0)$$
$$u(x) = \infty, \quad x \in \partial B_1(0).$$

In fact, there is a relation between $u_{0,\alpha}, u_{\infty,\alpha}$ and $\alpha$. For our convenience, we denote $u_{0,\alpha}$ and $u_{\infty,\alpha}$ by $u_0$ and $u_\infty$, respectively. In some places, we will write $u_\infty(x;\lambda)$ and $u_0(x;\lambda)$ instead of $u_\infty$ and $u_0$, to clearly indicate the relation depending on the parameter $\lambda$.

**Lemma 4.1.** *Let $p > 1$, $\sigma > -2$, $\lambda > 4\lambda_1[B_1(0)]$ and $U$ be a positive solution of*

$$-\Delta U = \lambda \frac{U}{|x|^2} - |x|^\sigma U^p \quad \text{on } B_R(0)\backslash\{0\}.$$

*Then we have*

$$u_0(0)|x|^{-\frac{2+\sigma}{p-1}} \leq U(x) \leq u_\infty(0)|x|^{-\frac{2+\sigma}{p-1}} \quad \text{for } x \in B_{R/2}(0)\backslash\{0\}. \tag{4.1}$$

*Proof.* Suppose that $x_0 \in B_{R/2}(0)\backslash\{0\}$ is an arbitrary point. Since $\lambda > 4\lambda_1[B_1(0)]$, we can choose $\alpha \in (0,1)$ close to 1 such that $\lambda \frac{\alpha^2}{(1+\alpha)^2} > \lambda_1[B_1(0)]$. Let

$$\tilde{U}(x) = |x_0|^{\frac{2+\sigma}{p-1}} U(x_0 + \alpha|x_0|x), \quad x \in B_1(0).$$

So, for all $x \in B_1(0)$, we obtain

$$(1-\alpha)|x_0| \leq |(x_0 + \alpha|x_0|x)| \leq (1+\alpha)|x_0|.$$

When $\sigma \geq 0$, a straightforward calculation leads to

$$-\Delta \tilde{U} \leq \lambda \frac{\alpha^2}{(1-\alpha)^2} \tilde{U} - \alpha^2 (1-\alpha)^\sigma \tilde{U}^p \quad \text{in } x \in B_1(0),$$
$$-\Delta \tilde{U} \geq \lambda \frac{\alpha^2}{(1+\alpha)^2} \tilde{U} - \alpha^2 (1+\alpha)^\sigma \tilde{U}^p \quad \text{in } x \in B_1(0).$$

We find that if $0 > \sigma > -2$, then

$$-\Delta \tilde{U} \leq \lambda \frac{\alpha^2}{(1-\alpha)^2} \tilde{U} - \alpha^2 (1+\alpha)^\sigma \tilde{U}^p \quad x \in B_1(0),$$
$$-\Delta \tilde{U} \geq \lambda \frac{\alpha^2}{(1+\alpha)^2} \tilde{U} - \alpha^2 (1-\alpha)^\sigma \tilde{U}^p \quad x \in B_1(0).$$

By the comparison principle, we obtain

$$u_0(x) \leq \tilde{U}(x) \leq u_\infty(x) \quad \text{for all } x \in B_1(0).$$

Particularly, choosing $x = 0$ and by the arbitrariness of $x_0$, we arrive at (4.1). $\square$

**Lemma 4.2.** *Suppose that $C > 0$, $R > 0$ and $\lambda > \max\{4\lambda_1[B_1(0)],\ (N-2)^2/4\}$.*

(i) *If $\mathcal{U} > 0$ satisfies*
$$-\Delta\mathcal{U} \geq \lambda\frac{\mathcal{U}}{|x|^2} - C|x|^\sigma\mathcal{U}^p, \quad x \in B_R(0)\backslash\{0\},$$
*then*
$$\mathcal{U}(x) \geq C^{-\frac{1}{p-1}}u_0(0)|x|^{-\frac{2+\sigma}{p-1}} \quad \text{for } x \in B_{R/2}(0)\backslash\{0\}. \tag{4.2}$$

(ii) *If $\mathcal{U} > 0$ satisfies*
$$-\Delta\mathcal{U} \leq \lambda\frac{\mathcal{U}}{|x|^2} - C|x|^\sigma\mathcal{U}^p, \quad x \in B_R(0)\backslash\{0\}, \tag{4.3}$$
*then*
$$\mathcal{U}(x) \leq C^{-\frac{1}{p-1}}u_\infty(0)|x|^{-\frac{2+\sigma}{p-1}} \quad \text{for } x \in B_{R/2}(0)\backslash\{0\}. \tag{4.4}$$

*Proof.* From the condition in case (i), it follows that $C^{\frac{1}{p-1}}\mathcal{U}$ is a supersolution of
$$\begin{gathered} -\Delta u = \lambda\frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in B_R(0)\backslash\{0\}, \\ u = 0, \quad x \in \partial B_R(0). \end{gathered} \tag{4.5}$$
Let $U$ be the unique positive solution of (4.5), then
$$U(x) = \lim_{\delta\to 0} U_\delta(x),$$
where $U_\delta$ is the unique positive solution of
$$\begin{gathered} -\Delta u = \lambda\frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in B_R(0)\backslash\overline{B}_\delta(0), \\ u = 0, \quad x \in \partial B_R(0) \cup \partial B_\delta(0). \end{gathered} \tag{4.6}$$
So, the comparison principle yields
$$U_\delta(x) \leq C^{\frac{1}{p-1}}\mathcal{U}(x) \quad \text{for all } x \in B_R(0)\backslash B_\delta(0).$$
Further, letting $\delta \to 0$, we have
$$U(x) \leq C^{\frac{1}{p-1}}\mathcal{U}(x) \quad \text{for all } x \in B_R(0).$$
In view of Lemma 4.1, we arrive at (4.2).

Now, we prove conclusion (ii). Suppose that $\mathcal{U} > 0$ satisfies (4.3). Let $V_\delta$ be the unique positive solution of
$$\begin{gathered} -\Delta u = \lambda\frac{u}{|x|^2} - |x|^\sigma u^p, \quad x \in B_R(0)\backslash\overline{B}_\delta(0), \\ u = \infty, \quad x \in \partial B_R(0) \cup \partial B_\delta(0). \end{gathered} \tag{4.7}$$
From [14], we know that
$$\hat{U}(x) := \lim_{\delta\to 0} V_\delta(x) \quad \text{for all } x \in B_R(0)\backslash\{0\} \text{ is well defined},$$
and that $\hat{U}$ satisfies
$$\begin{gathered} -\Delta\hat{U} = \lambda\frac{\hat{U}}{|x|^2} - |x|^\sigma\hat{U}^p, \quad x \in B_R(0)\backslash\{0\}, \\ \hat{U} = \infty, \quad x \in \partial B_R(0). \end{gathered}$$

So by the comparison principle, it is easy to see

$$C^{\frac{1}{p-1}}\mathcal{U}(x) \le \hat{U}(x) \quad \text{for } x \in B_R(0)\backslash\{0\}.$$

Using Lemma 4.1 again, we arrive at (4.4). □

Next, we give the proof of Theorem 1.2, i.e., we explore the existence of blow-up solution of the problem (1.1) with the boundary condition (1.2), and give the estimate of positive solutions near the origin.

*Proof of Theorem 1.2.* Since $s < p-1$ and $q < t-1$, we may choose $\gamma, \sigma \in (-2, 0)$ such that

$$-\frac{\gamma}{q} = \frac{2+\sigma}{t-1} \quad \text{and} \quad -\frac{\sigma}{s} = \frac{2+\gamma}{p-1}. \tag{4.8}$$

Denote by $U_{p,\gamma}$ the unique positive solution of (4.5) with $\sigma$ being replaced by $\gamma$ and $R = 1$, and denote by $U_{t,\sigma}$ the unique positive solution of (4.5) with $p$ being replaced by $t$ and $R = 1$. Choose $\eta \in (s/(t-1), (p-1)/q)$. A simple calculation shows that for sufficiently small $\epsilon > 0$ and sufficiently large $M > 0$, we have

$$\begin{gathered}
-\Delta(\epsilon U_{p,\gamma}) \le \lambda\frac{\epsilon U_{p,\gamma}}{|x|^2} - (\epsilon U_{p,\gamma})^p(\epsilon^{-\eta}U_{t,\sigma})^q \quad \text{in } B_1(0)\backslash\{0\},\\
-\Delta(\epsilon^{-\eta} U_{t,\sigma}) \ge \lambda\frac{\epsilon^{-\eta} U_{t,\sigma}}{|x|^2} - (\epsilon U_{p,\gamma})^s(\epsilon^{-\eta}U_{t,\sigma})^t \quad \text{in } B_1(0)\backslash\{0\},\\
-\Delta(M U_{p,\gamma}) \ge \lambda\frac{M U_{p,\gamma}}{|x|^2} - (M U_{p,\gamma})^p(M^{-\eta}U_{t,\sigma})^q \text{ in } B_1(0)\backslash\{0\},\\
-\Delta(M^{-\eta} U_{t,\sigma}) \le \lambda\frac{M^{-\eta} U_{t,\sigma}}{|x|^2} - (M U_{p,\gamma})^s(M^{-\eta}U_{t,\sigma})^t \text{ in } B_1(0)\backslash\{0\}.
\end{gathered}$$

So, $(\epsilon U_{p,\gamma}, M^{-\eta}U_{t,\sigma})$ and $(MU_{p,\gamma}, \epsilon^{-\eta}U_{t,\sigma})$ are a pair of subsolution and supersolution of

$$\begin{gathered}
-\Delta u = \lambda\frac{u}{|x|^2} - u^p v^q, \quad x \in B_1(0)\backslash\overline{B}_{1/n}(0),\\
-\Delta v = \mu\frac{v}{|x|^2} - u^s v^t, \quad x \in B_1(0)\backslash\overline{B}_{1/n}(0),\\
u = v = 0, \quad x \in \partial B_1(0),\\
u = \epsilon U_{p,\gamma}, \quad v = M^{-\eta}U_{t,\sigma}, \quad x \in \partial B_{1/n}(0).
\end{gathered} \tag{4.9}$$

By the supersolution and subsolution method, we know that (4.9) has at least one solution $(U_n(x), V_n(x))$ satisfying

$$\begin{gathered}
\epsilon U_{p,\gamma} \le U_n(x) \le MU_{p,\gamma}(x) \quad \text{for } x \in B_1(0)\backslash\overline{B}_{1/n}(0),\\
M^{-\eta}U_{t,\sigma}(x) \le V_n(x) \le \epsilon^{-\eta}U_{t,\sigma}(x) \quad \text{for } x \in B_1(0)\backslash\overline{B}_{1/n}(0).
\end{gathered}$$

In view of the regularity arguments of elliptic equations, $(U_n, V_n)$ has a convergent subsequence in $C^2_{\rm loc}(\overline{B}_1(0)\backslash\{0\}) \times C^2_{\rm loc}(\overline{B}_1(0)\backslash\{0\})$. We by $(u, v)$ denote the limit functions, and hence $(u, v)$ is a solution of the problem (1.1). Clearly, since both $U_{p,\gamma}$ and $U_{t,\sigma}$ blow up at the origin, we can see that both $u$ and $v$ blow up at the origin. The proof of (i) is complete.

Now, we prove the conclusion (ii). By hypothesis, both $\hat{u}(x)$ and $\hat{v}(x)$ have positive bound from below in $B_{1/2}(0)$. Denote $m_0 = \inf\{\hat{v}(x) : x \in B_{1/2}(0)\}$, and hence $\hat{u}$ satisfies

$$-\Delta\hat{u} \le \lambda\frac{\hat{u}}{|x|^2} - m_0^q\hat{u}^p, \quad x \in B_{1/2}(0)\backslash\{0\}.$$

In view of Lemma 4.2, we have

$$\hat{u}(x) \le m_0^{-\frac{q}{p-1}} u_\infty(0;\lambda)|x|^{-\frac{2}{p-1}} \quad \text{for } x \in B_{1/4}(0)\backslash\{0\}.$$

Let $a_1 = m_0^{-\frac{q}{p-1}} u_\infty(0;\lambda)$ and $\alpha_0 = \frac{2}{p-1}$. Then we find that

$$\hat{u}(x) \le a_1|x|^{-\alpha_0} \quad \text{for } x \in B_{1/4}(0)\backslash\{0\}.$$

By the second equation in (1.1), it follows that

$$-\Delta\hat{v} \ge \mu\frac{\hat{v}}{|x|^2} - a_1^s|x|^{-\frac{2s}{p-1}}\hat{v}^t, \quad x \in B_{1/4}(0)\backslash\{0\}.$$

Since $s < p-1$, we have $-\frac{2s}{p-1} > -2$. Take $a_2 = a_1^{-\frac{s}{t-1}}$. In view of Lemma 4.2 again, we obtain

$$\hat{v}(x) \ge a_2|x|^{-\frac{2-s\alpha_0}{t-1}} \quad \text{for } x \in B_{2^{-3}}(0)\backslash\{0\}.$$

Choose $\beta_1 = \frac{2-s\alpha_0}{t-1}$, and hence we see that

$$\hat{v}(x) \ge a_2|x|^{-\beta_1} \quad \text{for } x \in B_{2^{-3}}(0)\backslash\{0\}.$$

Let $a_3 = a_2^{-\frac{q}{p-1}}$ and $\alpha_2 = \frac{2-q\beta_1}{p-1}$. Using the first equation in (1.1) and Lemma 4.2 again, we have

$$\hat{u} \le a_3|x|^{-\frac{2-q\beta_1}{p-1}} \quad \text{in } B_{2^{-4}}(0)\backslash\{0\}.$$

Proceeding in this way inductively, we obtain

$$\begin{gathered}\hat{u}(x) \le a_{2n-1}|x|^{-\frac{2-q\beta_{n-1}}{p-1}}, \quad x \in B_{2^{-2n}}(0)\backslash\{0\},\\ \hat{v}(x) \ge a_{2n}|x|^{-\frac{2-q\alpha_{n-1}}{t-1}}, \quad x \in B_{2^{-(2n+1)}}(0)\backslash\{0\},\end{gathered}$$

where $n \ge 1$, and

$$\begin{gathered}\alpha_0 = \frac{2}{p-1}, \quad \beta_0 = 0, \quad \alpha_n = \frac{2-q\beta_n}{p-1}, \quad \beta_n = \frac{2-s\alpha_{n-1}}{t-1},\\ a_1 = m_0^{-\frac{q}{p-1}} u_\infty(0;\lambda), \quad a_{2n} = a_{2n-1}^{-\frac{s}{t-1}}, \quad a_{2n+1} = a_{2n}^{-\frac{q}{p-1}}.\end{gathered}$$

From the relation between $\alpha_n$ and $\beta_n$, we can deduce that $\{\alpha_n\}$ is a decreasing sequence and $\{\beta_n\}$ is an increasing sequence. By letting $n \to \infty$, it follows that

$$\lim_{n\to\infty}\alpha_n = \frac{2(t-1-q)}{(p-1)(t-1)-qs} \quad \text{and} \quad \lim_{n\to\infty}\beta_n = \frac{2(p-1-s)}{(p-1)(t-1)-qs}.$$

From the relation between $a_n$ and $a_{n+1}$, we can deduce that

$$a_{2n+1} = a_1^{\frac{q^n s^n}{(t-1)^n(p-1)^n}}.$$

In view of $(p-1)(t-1) > qs$ and $\min\{a_1, 1\} \le a_{2n+1} \le \max\{a_1, 1\}$ for all $n$, we have

$$\min\{a_1, a_1^{-\frac{s}{t-1}}\} \le a_n \le \max\{a_1, a_1^{-\frac{s}{t-1}}\} \quad \text{for all } n.$$

Thus, for any $\tau > 0$ there are $\delta := \delta(\tau)$ and $C_1$, $C_2 > 0$ such that

$$\begin{gathered}C_1|x|^{-\frac{2(p-1-s)}{(p-1)(t-1)-qs}+\tau} \le \hat{v}(x) \quad \text{for } x \in B_\delta(0)\backslash\{0\},\\ \hat{u}(x) \le C_2|x|^{-\frac{2(t-1-q)}{(p-1)(t-1)-qs}-\tau} \quad \text{for } x \in B_\delta(0)\backslash\{0\}.\end{gathered}$$

Consequently, the second inequality of (1.10) and the first inequality of (1.11) hold. By a similar argument, we can prove the remaining cases of (1.10) and (1.11). $\square$

**Remark 4.3.** From the proof of (i) in Theorem 1.2, it follows that (1.1) has a positive solution $(u,v)$ satisfying

$$C_1|x|^{-\frac{2+\gamma}{p-1}} \le u(x) \le C_2|x|^{-\frac{2+\gamma}{p-1}} \quad \text{for } x \in \Omega\backslash\{0\},$$
$$C_1|x|^{-\frac{2+\sigma}{t-1}} \le v(x) \le C_2|x|^{-\frac{2+\sigma}{t-1}} \quad \text{for } x \in \Omega\backslash\{0\},$$

where $\gamma$ and $\sigma$ satisfies (4.8). In fact, a simple calculation shows that

$$\frac{2+\gamma}{p-1} = \frac{2(t-1-q)}{(p-1)(t-1)-qs} \quad \text{and} \quad \frac{2+\sigma}{t-1} = \frac{2(p-1-s)}{(p-1)(t-1)-qs}.$$

## 5. Proof of Theorem 1.3

*Proof.* Suppose that $(u(x),v(x))$ is a positive radial solution of (1.1) with condition (1.2). For convenience, we denote $u(r)=u(x)$ and $v(r)=v(x)$ as $|x|=r$. Define

$$t_0 = \inf\{r_0 \in (0,1) : u'(r) < 0 \text{ and } v'(r) < 0 \text{ for } r \in (r_0,1)\}.$$

By Hopf's lemma, when $r<1$ is close to 1, we have $u'(r)<0$ and $v'(r)<0$. So, this leads to $t_0<1$. We claim $t_0=0$. Otherwise, we will see that

Case (1) $u'(t_0)=0$, $u'(r)<0$ and $v'(r)<0$ for $r\in(t_0,1)$, and
Case (2) $v'(t_0)=0$, $v'(r)<0$ and $u'(r)<0$ for $r\in(t_0,1)$.

Without loss of generality, we assume that Case (1) occurs. Denote

$$\tilde{u} = u(t_0) - u(x) \quad \text{for } t_0 < |x| < 1.$$

Then, $\tilde{u}(x)>0$ when $t_0<|x|<1$ and $\tilde{u}'(t_0)=\tilde{u}(t_0)=0$. A simple calculation gives

$$-\Delta\tilde{u} - \lambda\frac{\tilde{u}}{|x|^2} = -\lambda\frac{u(t_0)}{|x|^2} + u^pv^q > 0 \quad \text{when } t_0 < |x| < 1.$$

By Hopf's lemma, there holds $\tilde{u}'(t_0)>0$, which contradicts $\tilde{u}'(t_0)=0$. The conclusion $t_0=0$ implies that

$$u'(r) < 0 \text{ and } v'(r) < 0 \text{ for } r \in (0,1).$$

Since $u'(r)<0$ in $(0,1)$ and $\lambda\le 0$, we have $u''\ge u^pv^q$ in $(0,1)$, which implies that $u(r)$ is a convex function in $(0,1)$. Analogously, we may show that $v$ is a convex function. So, the conclusion (i) is proved.

Now we prove the conclusion (ii). From (1.1), we see that

$$-(r^{N-1}u')' = \lambda r^{N-3}u - r^{N-1}u^pv^q \text{ for } r \in (0,r).$$

Integrating on $[r,1]$ gives

$$r^{N-1}u'(r) - u'(1) = \lambda\int_r^1 \tau^{N-3}u(\tau)d\tau - \int_r^1 \tau^{N-1}u(\tau)^pv(\tau)^q d\tau \quad \text{for } r \in (0,1).$$

That is,

$$\begin{aligned} u'(r) &= r^{1-N}u'(1) + \lambda r^{1-N}\int_r^1 \tau^{N-3}u(\tau)d\tau \\ &\quad - r^{1-N}\int_r^1 \tau^{N-1}u(\tau)^pv(\tau)^q d\tau \quad \text{for } r \in (0,1). \end{aligned}$$

Integrating on $[r,1]$ yields

$$u(r) = -\int_r^1 s^{1-N}u'(1)ds - \lambda\int_r^1 s^{1-N}\int_s^1 \tau^{N-3}u(\tau)d\tau ds$$

$$
\begin{aligned}
&+\int_r^1 s^{1-N}\int_s^1 \tau^{N-1}u(\tau)^p v(\tau)^q d\tau ds\\
&\geq -\int_r^1 s^{1-N}u'(1)ds.
\end{aligned}
$$

This inequality implies that $u(r)\to\infty$ as $r\to 0$. In a similar manner, one can see that $v(r)\to\infty$ as $r\to 0$. □

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