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GOOD RADON MEASURE FOR ANISOTROPIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. We study nonlinear anisotropic problems with bounded Radon diffuse measure and variable exponent. We prove the existence and uniqueness of entropy solution.

1. Introduction

We consider the anisotropic elliptic Dirichlet boundary-value problem

$$\begin{gathered} b(u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) = \mu \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ ($N \geq 3$), with smooth boundary, $b : \mathbb{R} \to \mathbb{R}$ is a continuous, surjective and non-decreasing function, with $b(0) = 0$ and μ is a bounded Radon diffuse measure (this is μ does not charge the sets of zero $p_m(\cdot)$-capacity) such that $|\mu|(\Omega) > 0$. All papers concerned by problems like (1.1) considered particular cases of data b and measure μ. Indeed, in [13], the authors considered $b(\cdot) \equiv 0$, which permit them to exploit minimization technics to prove the existence of weak solutions and mini-max theory to prove that weak solutions are multiple. Using the same methods, Koné et *al.* (see [10]) studied the problem

$$\begin{gathered} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) = \mu \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where μ is a bounded Radon measure. Note that (1.2) is a particular case of (1.1), where $b(\cdot) \equiv 0$. Koné et *al.* also studied problem (1.2) when $\mu \in L^\infty(\Omega)$ (see [9]) and Ouaro, when $\mu \in L^1(\Omega)$ (see [15]). Ibrango and Ouaro studied the following problem (see [8])

$$\begin{gathered} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) + b(u) = \mu \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

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where $\mu \in L^1(\Omega)$. In [8], the authors used the technic of monotone operators in Banach spaces and approximation methods to prove the existence and uniqueness of entropy solution of problem (1.3).

In this paper, by using the same technics as in [8], we extend their result, taking into account a measure μ which is zero on the subsets of zero $p(\cdot)$-capacity (i.e., the capacity defined starting from $W_0^{1,p(\cdot)}(\Omega)$). In order to do that, we use a decomposition theorem proved by Nyanquini et *al.* in [14]: every bounded Radon measure that is zero on the sets of zero $p(\cdot)$-capacity can be split in the sum of an element in $W^{-1,p'(\cdot)}(\Omega)$ (the dual space of $W_0^{1,p(\cdot)}(\Omega)$), and a function in $L^1(\Omega)$, and conversely, every bounded measure in $L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$ is zero on the sets of zero $p(\cdot)$-capacity. Using the decomposition of measures result of Nyanquini et *al.* (see [14]), we prove that there exists a unique entropy solution of (1.1). The proof of our result will strongly rely on the structure of the measure μ, that is, μ belongs to $L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$.

Note that, since b is not necessarily invertible, then, the uniqueness of the entropy solution is proved in terms of $b(u)$ which is clearly equivalent to the uniqueness of u if and only if b is invertible. Note that a good Radon measure for the problem (1.1) is a Radon measure for which, the entropy solution of problem (1.1) is unique. Many papers are related to problems involving variable exponents due to their applications to elastic mechanics, electrorheological fluids or image restoration.

We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures in Ω, equiped with its standard norm $\|\cdot\|_{\mathcal{M}_b(\Omega)}$. Note that, if μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$ (the total variation of μ) is a bounded positive measure on Ω. Given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(\cdot)}(\Omega)$ ($p(\cdot)$-capacity for short) if $\mu(A)=0$, for every set A such that $Cap_{p(\cdot)}(A,\Omega)=0$. For $A\subset\Omega$, we denote

$$S_{p(\cdot)}(A) := \{u\in W_0^{1,p(\cdot)}(\Omega)\cap C_0(\Omega) : u=1 \text{ on } A, u\geq 0 \text{ on } \Omega\}.$$

The $p(\cdot)$-capacity of every subset A with respect to Ω is defined by

$$Cap_{p(\cdot)}(A,\Omega) := \inf_{u\in S_{p(\cdot)}(A)}\Big\{\int_\Omega |\nabla u|^{p(x)}dx\Big\}.$$

In the case $S_{p(\cdot)}(A)=\emptyset$, we set $Cap_{p(\cdot)}(A,\Omega)=+\infty$. The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(\cdot)}(\Omega)$. We recall the decomposition result of bounded Radon diffuse measure proved by Nyanquini et *al* (see [14]).

Theorem 1.1. *Let $p:\overline{\Omega}\to(1,+\infty)$ be a continuous function and $\mu\in\mathcal{M}_b(\Omega)$. Then, $\mu\in\mathcal{M}_b^{p(\cdot)}(\Omega)$ if and only if $\mu\in L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$.*

Recall that, in this paper, we assume that $\mu\in\mathcal{M}_b^{p_m(\cdot)}(\Omega)$, where $p_m(\cdot)$ is to be defined later.

Remark 1.2. We do not have uniqueness of entropy solution if the measure μ does not belong to the space $\mathcal{M}_b^{p_m(\cdot)}$ (see Proof of uniqueness). Therefore, $\mathcal{M}_b^{p_m(\cdot)}$ is the set of good Radon measure for problem (1.1).

The remaining part of this article is organized as follows: In Section 2, we introduce some preliminary results. In Section 3, we study the existence and uniqueness of entropy solution. We refer to [3, 4, 7] as papers dealing with measures (including the case of variable exponents).

2. Preliminaries

We study problem (1.1) under the following assumptions on the data.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary domain $\partial\Omega$ and $\overrightarrow{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot))$ such that for any $i = 1, \dots, N$, $p_i(\cdot) : \overline{\Omega} \to \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \operatorname{ess\,inf}_{x\in\Omega} p_i(x) \leq p_i^+ := \operatorname{ess\,sup}_{x\in\Omega} p_i(x) < \infty. \tag{2.1}$$

For $i = 1, \dots, N$, let $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying:

• there exists a positive constant C_1 such that

$$|a_i(x,\xi)| \leq C_1\Big(j_i(x) + |\xi|^{p_i(x)-1}\Big), \tag{2.2}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(\cdot)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$;

• for $\xi, \eta \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$, there exists a positive constant C_2 such that

$$(a_i(x,\xi) - a_i(x,\eta))(\xi - \eta) \geq \begin{cases} C_2|\xi-\eta|^{p_i(x)} & \text{if } |\xi-\eta| \geq 1 \\ C_2|\xi-\eta|^{p_i^-} & \text{if } |\xi-\eta| < 1 \end{cases} \tag{2.3}$$

• there exists a positive constant C_3 such that

$$a_i(x,\xi).\xi \geq C_3|\xi|^{p_i(x)}, \tag{2.4}$$

for $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.

The hypotheses on a_i are classical in the study of nonlinear problems (see [12]). Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p}-N}{\overline{p}(N-1)}, \tag{2.5}$$

and

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1, \tag{2.6}$$

where $\frac{N}{\overline{p}} = \sum_{i=1}^N \frac{1}{p_i^-}$.

A prototype example that is covered by our assumption is the following anisotropic $\overrightarrow{p}$-harmonic problem: set

$$a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi, \text{where } p_i(x) \geq 2 \text{ for } i = 1, \dots, N.$$

Then, we obtain the problem

$$\begin{gathered} b(u) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big) = \mu \text{ in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.7}$$

which, in the particular case where $p_i = p$ for any $i = 1, \dots, N$, is the p-Laplace equation.

We also recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces. We refer to [17, 18] for details and related properties. Set

$$C_+(\overline{\Omega}) = \big\{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \quad a.e.\ x \in \Omega\big\}$$

and denote by

$$p_M(x) := \max(p_1(x), \dots, p_N(x)) \quad \text{and} \quad p_m(x) := \min(p_1(x), \dots, p_N(x)).$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) := \Big\{u : u \text{ is a measurable real valued function such that} \int_\Omega |u|^{p(x)}dx < \infty\Big\},$$

endowed with the so-called Luxembourg norm

$$|u|_{p(\cdot)} := \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\big\}.$$

The $p(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}dx.$$

For any $u \in L^{p(\cdot)}(\Omega)$, the following inequality (see [5, 6]) will be used later

$$\min\{|u|^{p^-}_{p(\cdot)}; |u|^{p^+}_{p(\cdot)}\} \le \rho_{p(\cdot)}(u) \le \max\{\ |u|^{p^-}_{p(\cdot)};\ |u|^{p^+}_{p(\cdot)}\}. \tag{2.8}$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ in Ω, we have the Hölder type inequality (see [11])

$$|\int_\Omega uv dx| \le \big(\frac{1}{p^-} + \frac{1}{q^-}\big)|u|_{p(\cdot)}|v|_{q(\cdot)}. \tag{2.9}$$

If Ω is bounded and $p, q \in C_+(\overline{\Omega})$ such that $p(x) \le q(x)$ for any $x \in \Omega$, then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous (see [11, Theorem 2.8]).

Herein, we need the anisotropic Sobolev space

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) := \big\{u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega),\, i = 1, \dots, N\big\},$$

which is a separable and reflexive Banach space (see [11]) under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|_{p_i(\cdot)}.$$

We introduce the numbers

$$q = \frac{N(\overline{p} - 1)}{N - 1}, \quad q^* = \frac{N(\overline{p} - 1)}{N - \overline{p}} = \frac{Nq}{N - q},$$

and define

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}, \quad P_-^+ = \max\{p_1^-, \dots, p_N^-\}, \quad P_{-,\infty} = \max\{P_-^+,\ P_-^*\}.$$

Remark 2.1. Since $\mu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega)$, the Theorem 1.1 implies that there exist $f \in L^1(\Omega)$ and $F \in (L^{p'_m(\cdot)}(\Omega))^N$ such that

$$\mu = f - \operatorname{div} F, \tag{2.10}$$

where $\frac{1}{p_m(x)} + \frac{1}{p'_m(x)} = 1$ for all $x \in \Omega$.

We have the following embedding result (see [13, Theorem 1]).

Theorem 2.2. *Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. Assume also that the relation* (2.6) *is fillfuled. For any $q \in C(\overline{\Omega})$ verifying*
$$1 < q(x) < P_{-,\infty} \quad \text{for any } x \in \overline{\Omega},$$
the embedding $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

The following result is due to Troisi (see [20]).

Theorem 2.3. *Let $p_1, \dots, p_N \in [1, +\infty)$; $g \in W^{1,(p_1,\dots,p_N)}(\Omega)$ and*
$$q = \begin{cases} (\overline{p})^* & \text{if } (\overline{p})^* < N \\ \in [1, +\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$
Then, there exists a constant $C_4 > 0$ depending on $N, p_1, \dots, p_N$ if $\overline{p} < N$ and also on q and $\operatorname{meas}(\Omega)$ *if $\overline{p} \geq N$ such that*
$$\|g\|_{L^q(\Omega)} \leq C_4 \prod_{i=1}^N \|\frac{\partial g}{\partial x_i}\|_{L^{p_i}(\Omega)}^{1/N}. \tag{2.11}$$

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ $(1 < q < +\infty)$ as the set of measurable function $g : \Omega \to \mathbb{R}$ for which the distribution
$$\lambda_g(k) = \operatorname{meas}(\{x \in \Omega : |g(x)| > k\}), \quad k \geq 0 \tag{2.12}$$
satisfies an estimate of the form
$$\lambda_g(k) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{2.13}$$
We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$
$$\|g\|_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \quad \forall k > 0\}. \tag{2.14}$$
Finally, we use throughout the paper, the truncation function T_k, $(k > 0)$ by
$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.15}$$
It is clear that $\lim_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$. We define $\mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. In the sequel, we denote $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = E$ to simplify notation.

3. Existence and uniqueness result

Definition 3.1. A measurable function $u \in \mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is an entropy solution of (1.1) if $b(u) \in L^1(\Omega)$ and
$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - v)dx + \int_\Omega b(u)T_k(u - v)dx \leq \int_\Omega T_k(u - v)d\mu, \tag{3.1}$$
for all $v \in E \cap L^\infty(\Omega)$ and for every $k > 0$.

The existence result is as follows.

Theorem 3.2. *Assume* (2.1)-(2.6) *and* (2.10) *hold. Then, there exists at least one entropy solution of problem* (1.1).

Proof. The proof is done in three steps.
Step 1: Approximate problem. We consider the problem

$$\begin{gathered}
-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)+T_n(b(u_n))=\mu_n \quad \text{in } \Omega \\
u_n=0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.2}$$

where $f_n = T_n(f) \in L^\infty(\Omega)$ and $\mu_n = f_n - \operatorname{div} F$. Note that $f_n \to f$ in $L^1(\Omega)$ as $n \to +\infty$, and

$$\|f_n\|_1 = \int_\Omega |f_n|dx \le \int_\Omega |f|dx = \|f\|_1. \tag{3.3}$$

Definition 3.3. A measurable function $u_n \in E$ is a weak solution for (3.2) if

$$\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx+\int_\Omega T_n(b(u_n))v\,dx=\int_\Omega f_n v dx+\int_\Omega F.\nabla v\,dx, \tag{3.4}$$

for every $v \in E$.

Lemma 3.4. *There exists at least one weak solution u_n for problem* (3.2) *and*

$$|b(u_n)| \le \|\mu_n\|_\infty.$$

Proof. We define the operators A_n and B_n as follows.

$$\langle A_n u, v\rangle = \langle Au, v\rangle + \int_\Omega T_n(b(u))v dx \quad \forall u, v \in E, \tag{3.5}$$

where

$$\langle Au, v\rangle = \int_\Omega \sum_{i=1}^{N} a_i\big(x,\frac{\partial u}{\partial x_i}\big)\frac{\partial v}{\partial x_i}dx, \tag{3.6}$$

$$\langle B_n, v\rangle = \int_\Omega v d\mu_n. \tag{3.7}$$

The operator A_n is onto (see [8, Lemma 3.1] and [19, Corollary 2.2]). Therefore, for $B_n \in E^*$, we can deduce the existence of a function $u_n \in E$ such that $\langle A_n u_n, v\rangle = \langle B_n, v\rangle$.

Now, we show that $|b(u_n)| \le \|\mu_n\|_\infty$. Indeed, let us denote by

$$H_\epsilon = \min\big(\frac{s^+}{\epsilon};1\big), \quad \operatorname{sign}_0^+(s)=\begin{cases}1 & \text{if } s>0\\ 0 & \text{if } s\le 0.\end{cases}$$

If γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s)=\begin{cases}\text{minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \ne \emptyset\\ +\infty & \text{if } [s,+\infty)\cap D(\gamma)=\emptyset\\ -\infty & \text{if } (-\infty,s]\cap D(\gamma)=\emptyset.\end{cases}$$

We remark that, as ϵ approaches 0, $H_\epsilon(s)$ apaprocahes $\operatorname{sign}_0^+(s)$. We take $v = H_\epsilon(u_n - M)$ as test function in (3.4), for the weak solution u_n, where $M > 0$ (a constant to be chosen later), to obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i\big(x,\frac{\partial u_n}{\partial x_i}\big)\frac{\partial}{\partial x_i}H_\epsilon(u_n-M)dx+\int_{\Omega}T_n(b(u_n))H_\epsilon(u_n-M)dx\\&=\int_{\Omega}H_\epsilon(u_n-M)d\mu_n\,.\end{aligned}\tag{3.8}$$

By (2.4) we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i\big(x,\frac{\partial u_n}{\partial x_i}\big)\frac{\partial}{\partial x_i}H_\epsilon(u_n-M)dx\\&=\frac{1}{\epsilon}\sum_{i=1}^{N}\int_{\{\frac{(u_n-M)^+}{\epsilon}<1\}}a_i\big(x,\frac{\partial u_n}{\partial x_i}\big)\frac{\partial}{\partial x_i}(u_n-M)^+dx\\&=\frac{1}{\epsilon}\sum_{i=1}^{N}\int_{\{0<u_n-M<\epsilon\}}a_i\big(x,\frac{\partial u_n}{\partial x_i}\big)\frac{\partial u_n}{\partial x_i}dx\geq 0\,.\end{aligned}$$

Then, (3.8) gives

$$\int_{\Omega}T_n(b(u_n))H_\epsilon(u_n-M)dx\leq\int_{\Omega}H_\epsilon(u_n-M)d\mu_n,$$

which is equivalent to

$$\int_{\Omega}\Big(T_n(b(u_n))-T_n(b(M))\Big)H_\epsilon(u_n-M)dx\leq\int_{\Omega}\Big(\mu_n-T_n(b(M))\Big)H_\epsilon(u_n-M)dx.$$

We now let ϵ approach 0 in the inequality above to obtain

$$\int_{\Omega}\Big(T_n(b(u_n))-T_n(b(M))\Big)^+dx\leq\int_{\Omega}\Big(\mu_n-T_n(b(M))\Big)\operatorname{sign}_0^+(u_n-M)dx.\tag{3.9}$$

Choosing $M=b_0^{-1}(\|\mu_n\|_\infty)$ in the above inequality (since b is surjective), we obtain

$$\begin{aligned}&\int_{\Omega}\Big(T_n(b(u_n))-T_n(\|\mu_n\|_\infty)\Big)^+dx\\&\leq\int_{\Omega}\Big(\mu_n-T_n(\|\mu_n\|_\infty)\Big)\operatorname{sign}_0^+(u_n-b_0^{-1}(\|\mu_n\|_\infty))dx.\end{aligned}\tag{3.10}$$

For any $n\geq\|\mu_n\|_\infty$, we have

$$\begin{aligned}&\int_{\Omega}\Big(\mu_n-T_n(\|\mu_n\|_\infty)\Big)\operatorname{sign}_0^+(u_n-b_0^{-1}(\|\mu_n\|_\infty))dx\\&=\int_{\Omega}\Big(\mu_n-\|\mu_n\|_\infty\Big)\operatorname{sign}_0^+(u_n-b_0^{-1}(\|\mu_n\|_\infty))dx\leq 0.\end{aligned}$$

Then, (3.10) gives

$$\int_{\Omega}\Big(T_n(b(u_n))-\|\mu_n\|_\infty\Big)^+dx\leq 0.$$

Hence, for all $n>\|\mu_n\|_\infty$, we have $\Big(T_n(b(u_n))-\|\mu_n\|_\infty\Big)^+=0$ a.e. in Ω, which is equivalent to

$$T_n(b(u_n))\leq\|\mu_n\|_\infty,\quad\text{for all } n>\|\mu_n\|_\infty.\tag{3.11}$$

Let us remark that as u_n is a weak solution of (3.4), then $(-u_n)$ is a weak solution of the problem

$$\begin{gathered}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\tilde{a}_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)+T_n(\tilde{b}(u_n))=\tilde{\mu}_n \quad \text{in } \Omega\\ u_n=0 \quad \text{on } \partial\Omega,\end{gathered}\tag{3.12}$$

where $\tilde{a}_i(x,\xi)=-a_i(x,-\xi)$, $\tilde{b}(s)=-b(-s)$ and $\tilde{\mu}_n=-\mu_n$. From (3.11) we deduce that

$$T_n(-b(u_n))\le\|\mu_n\|_\infty,\quad \text{for all } n>\|\mu_n\|_\infty.$$

Therefore,

$$T_n(b(u_n))\ge-\|\mu_n\|_\infty,\quad \text{for all } n>\|\mu_n\|_\infty.\tag{3.13}$$

It follows from (3.11) and (3.13) that for all $n>\|\mu_n\|_\infty$, $|T_n(b(u_n))|\le\|\mu_n\|_\infty$ which implies $|b(u_n)|\le\|\mu_n\|_\infty$ a.e. in Ω. $\square$

We now consider the problem

$$\begin{gathered}-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)+b(u_n)=\mu_n \quad \text{in } \Omega\\ u_n=0 \quad \text{on } \partial\Omega,\end{gathered}\tag{3.14}$$

where $f_n=T_n(f)\in L^\infty(\Omega)$ and $\mu_n\in L^\infty(\Omega)$. It follows from Lemma 3.4 that there exists $u_n\in E$ with $b(u_n)\in L^\infty(\Omega)$ such that

$$\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx+\int_\Omega b(u_n)v\,dx=\int_\Omega f_n v\,dx+\int_\Omega F.\nabla v\,dx,\tag{3.15}$$

for every $v\in E$.

Our aim is to prove that these approximated solutions u_n tend, as n approaches infinity, to a measurable function u which is an entropy solution of the problem (1.1). To start with, we establish some a priori estimates.

Step 2: A priori estimates.

Lemma 3.5. *There exists a positive constant C_5 which does not depend on n, such that*

$$\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i^-}dx\le C_5(k+1),\tag{3.16}$$

for every $k>0$.

Proof. We take $v=T_k(u_n)$ as test function in (3.15) to obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial u_n}{\partial x_i}dx+\int_\Omega b(u_n)T_k(u_n)dx\\&=\int_\Omega f_nT_k(u_n)dx+\int_\Omega F\nabla T_k(u_n)dx.\end{aligned}$$

Using relation (2.4) and the fact that $\int_\Omega b(u_n)T_k(u_n)dx\ge 0$, we obtain

$$c_3\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx\le\Big|\int_\Omega f_nT_k(u_n)dx+\int_\Omega F\nabla T_k(u_n)dx\Big|.\tag{3.17}$$

Since

$$|\int_\Omega f_n T_k(u_n)dx + \int_\Omega F\nabla T_k(u_n)dx| = |\int_\Omega T_k(u_n)d\mu_n| \le k|\mu|(\Omega) \le Ck,$$

we deduce that

$$c_3 \sum_{i=1}^N \int_{\{|u_n|\le k\}} |\frac{\partial u_n}{\partial x_i}|^{p_i(x)}dx \le Ck. \tag{3.18}$$

We have

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n|\le k\}} |\frac{\partial u_n}{\partial x_i}|^{p_i^-}\,dx \\
&= \sum_{i=1}^N \int_{\{|u_n|\le k; |\frac{\partial u_n}{\partial x_i}|>1\}} |\frac{\partial u_n}{\partial x_i}|^{p_i^-}\,dx + \sum_{i=1}^N \int_{\{|u_n|\le k; |\frac{\partial u_n}{\partial x_i}|\le 1\}} |\frac{\partial u_n}{\partial x_i}|^{p_i^-}\,dx \\
&\le \sum_{i=1}^N \int_{\{|u_n|\le k\}} |\frac{\partial u_n}{\partial x_i}|^{p_i(x)}dx + N.\operatorname{meas}(\Omega) \\
&\le \frac{C}{C_3}k + N.\operatorname{meas}(\Omega) \text{ due to relation (3.18)} \\
&\le C_5(1+k)
\end{aligned}$$

with $C_5 = \max\big\{\frac{C}{C_3}; N.\operatorname{meas}(\Omega)\big\}$. □

We also have the following lemma (see [9], [10]).

Lemma 3.6. *For any $k>0$, there exists some positive constants C_6 and C_7 such that:*

- (i) $\|u_n\|_{\mathcal{M}^{q^*}(\Omega)} \le C_6$;
- (ii) $\|\frac{\partial u_n}{\partial x_i}\|_{\mathcal{M}^{p_i^- \frac{q}{p}}(\Omega)} \le C_7, \quad \forall i = 1,\dots,N.$

Step 3: Convergence. According to [2] (see also [10]), we have the following lemma.

Lemma 3.7. *For $i=1,\dots,N$, as $n\to+\infty$, we have*

$$a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) \to a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \quad \text{in } L^1(\Omega), \text{ a.e. } x\in\Omega. \tag{3.19}$$

Proposition 3.8. *Assume (2.1)-(2.6) hold. If $u_n \in E$ is a weak solution of (3.2) then, the sequence $(u_n)_{n\in\mathbb{N}^*}$ is Cauchy in measure. In particular, there exists a measurable function u and a sub-sequence still denoted by u_n such that $u_n\to u$ in measure.*

Proposition 3.9. *Assume (2.1)-(2.6) hold. If $u_n\in E$ is a weak solution of (3.2), then*

- (i) *for $i=1,\dots,N$, $\frac{\partial u_n}{\partial x_i}$ converges in measure to the weak partial gradient of u;*
- (ii) *for $i=1,\dots,N$ and $k>0$, $a_i\big(x, \frac{\partial}{\partial x_i}T_k(u_n)\big)$ converges to $a_i\big(x, \frac{\partial}{\partial x_i}T_k(u)\big)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(\cdot)}(\Omega)$ weakly.*

We can now pass to the limit in (3.4). Let $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$; we choose $T_k(u_n - v)$ as test function in (3.15) to obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u_n - v)dx + \int_\Omega b(u_n)T_k(u_n - v)dx \\ &= \int_\Omega f_n T_k(u_n - v)dx + \int_\Omega F\nabla T_k(u_n - v)dx. \end{aligned} \tag{3.20}$$

For the first term of the right-hand side of (3.20), we have

$$\int_\Omega f_n(x)T_k(u_n - v)dx \to \int_\Omega f(x)T_k(u - v)dx, \tag{3.21}$$

since f_n converges strongly to f in $L^1(\Omega)$ and $T_k(u_n - v)$ converges weakly-* to $T_k(u - v)$ in $L^\infty(\Omega)$ and a.e. in Ω.

For the second term of the right-hand side of (3.20), we have

$$\int_\Omega F\nabla T_k(u_n - v)dx \to \int_\Omega F\nabla T_k(u - v)dx, \tag{3.22}$$

since $\nabla T_k(u_n - v) \rightharpoonup \nabla T_k(u - v)$ in $(L^{p_m(\cdot)}(\Omega))^N$ and $F \in (L^{p'_m(\cdot)}(\Omega))^N$. For the first term of (3.20), we have (see [2]):

$$\begin{aligned} &\lim_{n\to+\infty} \inf \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u_n - v)dx \\ &\geq \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - v)dx. \end{aligned} \tag{3.23}$$

For the second term of (3.20), we have

$$\int_\Omega b(u_n)T_k(u_n - v)dx = \int_\Omega (b(u_n) - b(v))T_k(u_n - v)dx + \int_\Omega b(v)T_k(u_n - v)dx.$$

The quantity $(b(u_n)-b(v))T_k(u_n-v)$ is nonnegative and since for all $s \in \mathbb{R}$, $s \mapsto b(s)$ is continuous, we obtain

$$(b(u_n) - b(v))T_k(u_n - v) \to (b(u) - b(v))T_k(u - v) \quad \text{a.e. in } \Omega.$$

Then, it follows by Fatou's Lemma that

$$\lim_{n\to+\infty} \inf \int_\Omega (b(u_n) - b(v))T_k(u_n - v)dx \geq \int_\Omega (b(u) - b(v))T_k(u - v)dx. \tag{3.24}$$

We have $b(v) \in L^1(\Omega)$. Since $T_k(u_n - v)$ converges weakly -* to $T_k(u - v) \in L^\infty(\Omega)$ and $b(v) \in L^1(\Omega)$, it follows that

$$\lim_{n\to+\infty} \int_\Omega b(v)T_k(u_n - v)dx = \int_\Omega b(v)T_k(u_n - v)dx. \tag{3.25}$$

From (3.21), (3.22), (3.23), (3.24) and (3.25), we pass to the limit in (3.20) to obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - v)dx + \int_\Omega b(u)T_k(u - v)dx \\ &\leq \int_\Omega fT_k(u - v)dx + \int_\Omega F\nabla T_k(u - v)dx. \end{aligned}$$

Then u is an entropy solution of (1.1). □

Theorem 3.10. *Assume* (2.1)-(2.6) *hold and let* u *be an entropy solution of* (1.1). *Then,* u *is unique.*

The proof of the above theorem is done in two steps.

Step 1: A priori estimates.

Lemma 3.11. *Assume* (2.1)-(2.6) *hold. Let* u *be an entropy solution of* (1.1). *Then*

$$\sum_{i=1}^{N} \int_{\{|u|\le k\}} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i^-} dx \le \frac{k}{C_3}|\mu|(\Omega), \quad \forall k > 0 \tag{3.26}$$

and there exists a positive constant C_8 *such that*

$$\|b(u)\|_1 \le C_8 \operatorname{meas}(\Omega) + |\mu|(\Omega). \tag{3.27}$$

Proof. Let us take $v = 0$ in the entropy inequality (3.1).
• By the fact that $\int_\Omega b(u)T_k(u)dx \ge 0$ and using relations (2.3) and (2.4), we deduce (3.26).
• As

$$\sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u)dx = \sum_{i=1}^{N} \int_{\{|u|\le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx \ge 0,$$

relation (3.1) gives

$$\int_\Omega b(u)T_k(u)dx \le \int_\Omega fT_k(u)dx + \int_\Omega F\nabla T_k(u)dx. \tag{3.28}$$

By (3.28), we deduce

$$\int_{\{|u|\le k\}} b(u)T_k(u)dx + \int_{\{|u|>k\}} b(u)T_k(u)dx \le k|\mu|(\Omega)$$

or

$$\int_{\{|u|>k\}} b(u)T_k(u)dx = -k\int_{\{u<-k\}} b(u)dx + k\int_{\{u>k\}} b(u)dx \le k|\mu|(\Omega).$$

Therefore,

$$\int_{\{|u|>k\}} |b(u)|dx \le |\mu|(\Omega).$$

So, we obtain

$$\begin{aligned}\int_\Omega |b(u)|dx &= \int_{\{|u|>k\}} |b(u)|dx + \int_{\{|u|\le k\}} |b(u)|dx \\ &\le \int_{\{|u|\le k\}} |b(u)|dx + |\mu|(\Omega).\end{aligned}$$

Since b is non-decreasing, we have

$$|u| \le k \Leftrightarrow b(-k) \le b(u) \le b(k) \Rightarrow |b(u)| \le \max\{b(k), |b(-k)|\}.$$

Then, we have

$$\int_{\{|u|\le k\}} |b(u)|dx \le \int_\Omega \max\{b(k), |b(-k)|\}dx = \max\{b(k), |b(-k)|\} \operatorname{meas}(\Omega).$$

Consequently, there exists a constant $C_8 = \max\{b(k), |b(-k)|\}$ such that

$$\|b(u)\|_1 \le C_8 \operatorname{meas}(\Omega) + |\mu|(\Omega).$$

□

Lemma 3.12. *Assume* (2.1)-(2.6) *hold true and* $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$. *If* u *is an entropy solution of* (1.1), *then there exists a constant* D *which depends on* μ *and* Ω *such that*

$$\operatorname{meas}\{|u| > k\} \le \frac{D}{\min(b(k), |b(-k)|)}, \forall k > 0 \tag{3.29}$$

and a constant D' *which depends on* μ *and* Ω *such that*

$$\operatorname{meas}\big\{|\frac{\partial u}{\partial x_i}| > k\big\} \le \frac{D'}{k^{\frac{1}{(p_M)'}}}, \quad \forall k \ge 1. \tag{3.30}$$

Proof. Since b is non-decreasing, we have

$$\forall k > 0, |u| > k \Rightarrow |b(u)| \ge \min(b(k), |b(-k)|).$$

For any $k > 0$, the relation (3.27) and the fact that $|b(u)| \ge \min(b(k), |b(-k)|)$ give

$$\int_{\{|u|>k\}} \min(b(k), |b(-k)|) dx \le \int_{\{|u|>k\}} |b(u)| dx \le C_8 \operatorname{meas}(\Omega) + |\mu|(\Omega).$$

Therefore,

$$\min(b(k), |b(-k)|) \operatorname{meas}(|u| > k) \le C_8 \operatorname{meas}(\Omega) + |\mu|(\Omega) = D;$$

that is

$$\operatorname{meas}(|u| > k) \le \frac{D}{\min(b(k), |b(-k)|)}.$$

For the proof of (3.30), we refer to [1]. □

We have the following two lemmas whose proofs can be found in [8].

Lemma 3.13. *Assume* (2.1)-(2.6) *hold, and let* $f \in L^1(\Omega)$. *If* u *is an entropy solution of* (1.1), *then*

$$\lim_{h \to +\infty} \int_\Omega |f| \chi_{\{|u|>h-t\}} dx = 0,$$

where $h > 0$ *and* $t > 0$.

Lemma 3.14. *Assume* (2.1)-(2.6) *hold, and let* $f \in L^1(\Omega)$. *If* u *is an entropy solution of* (1.1), *then there exists a positive constant* K *such that*

$$\rho_{p_i'(\cdot)}\Big(|\frac{\partial u}{\partial x_i}|^{p_i(x)-1} \chi_{F_{h,k}}\Big) \le K, \quad \textit{for } i = 1, \dots, N, \tag{3.31}$$

where $F_{h,k} = \{h < |u| \le h + k\}$, $h > 0$, $k > 0$.

Step 2: Uniqueness of the entropy solution. Let $h > 0$ and u, v be two entropy solutions of (1.1). We write the entropy inequality corresponding to the solution u, with $T_h(v)$ as test function, and to the solution v, with $T_h(u)$ as test function. We obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) dx + \int_\Omega b(u) T_k(u - T_h(v)) dx \\ &\le \int_\Omega f T_k(u - T_h(v)) dx + \int_\Omega F \nabla T_k(u - T_h(v)) dx \end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial v}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx+\int_\Omega b(v)T_k(v-T_h(u))dx\\&\le\int_\Omega fT_k(v-T_h(u))dx+\int_\Omega F\nabla T_k(v-T_h(u))dx.\end{aligned}\tag{3.33}$$

Adding (3.32) and (3.33), we obtain

$$\begin{aligned}&\Big[\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx-\int_\Omega F.\nabla T_k(u-T_h(v))dx\Big]\\&+\Big[\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial v}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx-\int_\Omega F.\nabla T_k(v-T_h(u))dx\Big]\\&+\int_\Omega b(u)T_k(u-T_h(v))dx+\int_\Omega b(v)T_k(v-T_h(u))dx\\&\le\int_\Omega f(x)[T_k(u-T_h(v))+T_k(v-T_h(u))]dx.\end{aligned}\tag{3.34}$$

Let us define

$$E_1=\{|u-v|\le k;|v|\le h\},\quad E_2=E_1\cap\{|u|\le h\},\quad E_3=E_1\cap\{|u|>h\},$$
$$E_1'=\{|v-u|\le k;|u|\le h\},\quad E_3'=E_1'\cap\{|v|>h\}.$$

Assertion 1. $E_3\subset F_{h,k}$ and

$$B=\{|u-h\,\mathrm{sign}_0(v)|\le k,|v|>h\}\subset F_{h-k,\ 2k}.$$

Indeed, We decompose E_3 as $E_3=E_3^+\cup E_3^-$ where $E_3^+=\{|u-v|\le k,|v|\le h,u>h\}$ and $E_3^-=\{|u-v|\le k,|v|\le h,u<-h\}$. In E_3^+, we have $-h\le v\le h$ and $-k\le u-v\le k$ so that $v-k\le u\le v+k\le h+k$. Since $u>h$ and $v+k\le h+k$, we obtain $h\le u\le h+k$. Hence, $E_3^+\subset F_{h,k}$.

In E_3^-, we have $-h\le v\le h$ and $-k\le u-v\le k$ so that $v-k\le u\le v+k\le h+k$; since $u<-h$ and $-k-h\le v-k\le h-k$, we obtain $-h-k\le u\le -h$ so that $h\le|u|\le h+k$.
Hence, $E_3^-\subset F_{h,k}$.

We split B as $B=B^+\cup B^-=\{|u-h|\le k,v>h\}\cup\{|u+h|\le k,v<-h\}$. In B^+ (B^- can be treated in the same way), we have $-k\le u-h\le k$ and $v>h$ so that $h-k\le u\le h+k$. Hence $B^+\subset F_{h-k,2k}$.

Assertion 2. On E_3 (and on B) we have according to Hölder inequality

$$\int_{E_3}F.\nabla u dx\le\Big(\int_{E_3}|F|^{(p_m')^-}dx\Big)^{\frac{1}{(p_m')^-}}\Big(\int_{E_3}|\nabla u|^{p_m^-}dx\Big)^{\frac{1}{(p_m)^-}},\tag{3.35}$$

with

$$\lim_{h\to+\infty}\Big(\int_{E_3}|F|^{(p_m')^-}dx\Big)^{\frac{1}{(p_m')^-}}\Big(\int_{E_3}|\nabla u|^{(p_m^-)}dx\Big)^{\frac{1}{(p_m^-)}}=0,$$

where $\frac{1}{(p_m^-)}+\frac{1}{(p_m')^-}=1$. Indeed,

$$\lim_{n\to+\infty}\Big(\int_{E_3}|F|^{(p_m')^-}dx\Big)^{\frac{1}{(p_m')^-}}=0,$$

as $|F|^{(p'_m)^-}\chi_{E_3}$ belongs to $L^1(\Omega)$ and as $E_3 \subset F_{h,k}$, then $|F|^{(p'_m)^-}\chi_{E_3}$ converges to zero as $h \to +\infty$. Then, by Lebesgue dominated convergence theorem,

$$\lim_{h\to+\infty}\int_{E_3}|F|^{(p'_m)^-}dx = 0.$$

Now, it remains to prove that $\int_{E_3}|\nabla u|^{(p_m^-)}dx$ is bounded with respect to h. We use the notation.

$$\mathcal{I}_1 = \{i \in \{1,\dots,N\} : |\frac{\partial u}{\partial x_i}| \le 1\}, \quad \mathcal{I}_2 = \{i \in \{1,\dots,N\} : |\frac{\partial u}{\partial x_i}| > 1\}.$$

Then we have

$$\begin{aligned}
\sum_{i=1}^N\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_i(x)}dx &= \sum_{i\in\mathcal{I}_1}\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_i(x)}dx + \sum_{i\in\mathcal{I}_2}\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_i(x)}dx\\
&\ge \sum_{i\in\mathcal{I}_2}\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_i(x)}dx\\
&\ge \sum_{i\in\mathcal{I}_2}\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_m^-}dx\\
&\ge \sum_{i=1}^N\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_m^-}dx - \sum_{i\in\mathcal{I}_1}\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_m^-}dx\\
&\ge \sum_{i=1}^N\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_m^-}dx - N\,\mathrm{meas}(\Omega)\\
&\ge \sum_{i=1}^N\|\frac{\partial u}{\partial x_i}\|^{p_m^-}_{L^{p_m^-}(F_{h,k})} - N\,\mathrm{meas}(\Omega)\\
&\ge C_9\|\nabla u\|^{p_m^-}_{(L^{p_m^-}(F_{h,k}))^N} - N\,\mathrm{meas}(\Omega),
\end{aligned}$$

where we used Poincaré inequality. We deduce that

$$\sum_{i=1}^N\int_{F_{h,k}}|\frac{\partial u}{\partial x_i}|^{p_i(x)}dx \ge C_9\int_{F_{h,k}}|\nabla u|^{p_m^-} - N\,\mathrm{meas}(\Omega). \tag{3.36}$$

Choosing $T_h(u)$ as test fonction in (3.1), we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u-T_h(u))dx + \int_\Omega b(u)T_k(u-T_h(u))dx\\
&\le \int_\Omega fT_k(u-T_h(u))dx + \int_\Omega F.\nabla T_k(u-T_h(u))dx.
\end{aligned}\tag{3.37}$$

According to the fact that $\nabla T_k(u-T_h(u)) = \nabla u$ on $\{h \le |u| \le h+k\}$, and zero elsewhere, and $\int_\Omega b(u)T_k(u-T_h(u))dx \ge 0$, we deduce from (3.37) that

$$\begin{aligned}
&\sum_{i=1}^N\int_{F_{h,k}}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial u}{\partial x_i}dx\\
&\le k\int_{\{|u|\ge h\}}|f|dx + \int_{F_{h,k}}|\Big(\frac{2}{C_3C_9p_m^-}\Big)^{\frac{1}{p_m^-}}F||\Big(\frac{C_3C_9p_m^-}{2}\Big)^{\frac{1}{p_m^-}}\nabla u|dx.
\end{aligned}\tag{3.38}$$

Using (2.4) (in the left hand side of (3.38)), Young inequality (in the right hand side of (3.38)) and setting

$$C_{10} = \Big(\frac{2}{C_3 C_9 p_m^-}\Big)^{\frac{(p'_m)^-}{(p_m^-)}} \frac{p_m^- - 1}{p_m^-},$$

we obtain

$$\begin{aligned} &C_3 \sum_{i=1}^N \int_{F_{h,k}} |\frac{\partial u}{\partial x_i}|^{p_i(x)} dx \\ &\le k \int_{\{|u| \ge h\}} |f| dx + C_{10} \int_{F_{h,k}} |F|^{(p'_m)^-} dx + \frac{C_3 C_9}{2} \int_{F_{h,k}} |\nabla u|^{p_m^-} dx. \end{aligned} \tag{3.39}$$

From (3.36) and (3.39), we obtain

$$\begin{aligned} &C_3 C_9 \int_{F_{h,k}} |\nabla u|^{p_m^-} dx \\ &\le k \int_{\{|u| \ge h\}} |f| dx + C_{10} \int_{F_{h,k}} |F|^{(p'_m)^-} dx + \frac{C_3 C_9}{2} \int_{F_{h,k}} |\nabla u|^{p_m^-} dx + N \operatorname{meas}(\Omega). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{C_3 C_9}{2} \int_{F_{h,k}} |\nabla u|^{p_m^-} dx \\ &\le k \int_{\{|u| \ge h\}} |f| dx + C_{10} \int_{F_{h,k}} |F|^{(p'_m)^-} dx + N \operatorname{meas}(\Omega). \end{aligned} \tag{3.40}$$

Since $E_3 \subset F_{h,k}$, we deduce from (3.40) that $\int_{E_3} |\nabla u|^{p_m^-} dx$ is bounded. Since $B \subset F_{h-k,2k}$, reasoning as before, we obtain

$$\int_B F.\nabla u dx \le \Big(\int_B |F|^{(p'_m)^-} dx\Big)^{\frac{1}{(p'_m)^-}} \Big(\int_B |\nabla u|^{p_m^-} dx\Big)^{\frac{1}{p_m^-}},$$

with

$$\lim_{h \to +\infty} \Big(\int_B |F|^{(p'_m)^-} dx\Big)^{\frac{1}{(p'_m)^-}} \Big(\int_B |\nabla u|^{(p_m^-)} dx\Big)^{\frac{1}{(p_m^-)}} = 0.$$

We have

$$\begin{aligned} &\sum_{i=1}^N \int_{\{|u - T_h(v)| \le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) dx \\ &- \int_{\{|u - T_h(v)| \le k\}} F \nabla T_k(u - T_h(v)) dx \\ &= \sum_{i=1}^N \int_{\{|u - T_h(v)| \le k\} \cap \{|v| \le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) dx \\ &\quad + \sum_{i=1}^N \int_{\{|u - T_h(v)| \le k\} \cap \{|v| > h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) dx \\ &\quad - \int_{\{|u - T_k(v)| \le k\} \cap \{|v| \le h\}} F \nabla T_k(u - T_h(v)) dx \\ &\quad - \int_{\{|u - T_k(v)| \le k\} \cap \{|v| > h\}} F \nabla T_k(u - T_h(v)) dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_{\{|u-v|\le k\}\cap\{|v|\le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u-h\operatorname{sign}_0(v)|\le k\}\cap\{|v|\le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial u}{\partial x_i}dx \\
&\quad - \int_{\{|u-v|\le k\}\cap\{|v|\le h\}} F\nabla(u-v)dx \\
&\quad - \int_{\{|u-h\operatorname{sign}_0(v)|\le k\}\cap\{|v|>h\}} F\nabla T_k(u-h\operatorname{sign}_0(v))dx \\
&\ge \sum_{i=1}^{N} \int_{E_1} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx \\
&\quad - \int_{E_1} F\nabla(u-v)dx - \int_{\{|u-h\operatorname{sign}_0(v)|\le k\}\cap\{|v|>h\}} F\nabla u\, dx \\
&= \sum_{i=1}^{N} \int_{E_2} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx + \sum_{i=1}^{N} \int_{E_3} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx \\
&\quad - \int_{E_2} F\nabla(u-v)dx - \int_{E_3} F\nabla(u-v)dx - \int_{\{|u-h\operatorname{sign}_0(v)|\le k\}\cap\{|v|>h\}} F\nabla u\, dx.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\
&- \int_{\{|u-T_k(v)|\le k\}} F.\nabla T_k(u-T_h(v))dx \\
&\ge \sum_{i=1}^{N} \int_{E_2} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx + \sum_{i=1}^{N} \int_{E_3} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx \\
&\quad - \int_{E_2} F\nabla(u-v)dx + \int_{E_3} F\nabla v dx - \int_{E_3} F\nabla u dx - \int_{B} F\nabla u dx.
\end{aligned}
$$

We deduce from (3.35) that

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}T_k(u-T_h(v))dx \\
&\quad - \int_{\{|u-T_k(v)|\le k\}} F\nabla T_k(u-T_h(v))dx \\
&\ge \sum_{i=1}^{N} \int_{E_2} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v)dx \\
&\quad - \Big[\sum_{i=1}^{N} \int_{E_3} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}v dx - \int_{E_3} F\nabla v dx\Big] \\
&\quad - \int_{E_2} F\nabla(u-v)dx - \Big(\int_{E_3} |F|^{(p'_m)^-} dx\Big)^{\frac{1}{(p'_m)^-}} \Big(\int_{E_3} |\nabla u|^{(p_m^-)} dx\Big)^{\frac{1}{(p_m^-)}}
\end{aligned}
$$

$$- \Big(\int_B |F|^{(p'_m)^-}\,dx \Big)^{\frac{1}{(p'_m)^-}} \Big(\int_B |\nabla u|^{(p_m^-)} dx \Big)^{\frac{1}{p_m^-}}. \tag{3.41}$$

According to (2.2) and the Hölder type inequality, we have

$$\begin{aligned}
&| \int_{E_3} \Big[\sum_{i=1}^N a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v) - F.\nabla v \Big] dx | \\
&\le C_1 \sum_{i=1}^N \int_{E_3} \Big(j_i(x) + |\frac{\partial u}{\partial x_i}|^{p_i(x)-1} \Big) |\frac{\partial v}{\partial x_i}| dx + \int_{E_3} |F \nabla v| dx \\
&\le C_1 \sum_{i=1}^N \int_{E_3} \Big(|j_i(x)| + \big| |\frac{\partial u}{\partial x_i}|^{p_i(x)-1} \big|_{p'_i(\cdot),\{h<|u|<h+k\}} \Big) |\frac{\partial v}{\partial x_i}|_{p'_i(\cdot),\{h-k<|u|<h\}} \\
&\quad + \Big(\int_{E_3} |F|^{(p'_m)^-}\, dx \Big)^{\frac{1}{(p'_m)^-}} \Big(\int_{E_3} |\nabla v|^{(p_m^-)} dx \Big)^{\frac{1}{(p_m^-)}},
\end{aligned}$$

where

$$\big| |\frac{\partial u}{\partial x_i}|^{p_i(x)-1} \big|_{p'_i(\cdot),\{h<|u|<h+k\}} = \big\| |\frac{\partial u}{\partial x_i}|^{p_i(x)-1} \big\|_{L^{p'_i(\cdot)}(\{h<|u|<h+k\})}.$$

For $i = 1, \dots, N$, the quantity $\Big(|j_i(x)| + \big| |\frac{\partial u}{\partial x_i}|^{p_i(x)-1} \big|_{p'_i(\cdot),\{h<|u|<h+k\}} \Big)$ is finite according to (2.8) and Lemma 3.14.

Thanks to Lemma 3.13 and Assertion 2, the quantity $|\frac{\partial v}{\partial x_i}|_{p'_i(\cdot),\{h-k<|u|<h\}}$ and

$$\Big(\int_{E_3} |F|^{(p'_m)^-}\, dx \Big)^{\frac{1}{(p'_m)^-}} \Big(\int_{E_3} |\nabla v|^{(p_m^-)} dx \Big)^{\frac{1}{(p_m^-)}}$$

converge to zero as h approaches infinity. Consequently, the second term at the right-hand side of (3.41) converges to zero as h approaches infinity. Therefore,

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u-T_h(v)|\le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) dx \\
&- \int_{\{|u-T_h(v)|\le k\}} F.\nabla T_k(u - T_h(v)) dx \\
&\ge I_h + \sum_{i=1}^N \int_{E_2} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(u-v) dx - \int_{E_2} F.\nabla(u-v) dx,
\end{aligned}$$

with $\lim_{h\to+\infty} I_h = 0$. We adopt the same process (replacing respectively E_1, E_3 by E'_1 and E'_3) to treat the second term of (3.34), which give

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|v-T_h(u)|\le k\}} a_i\Big(x, \frac{\partial v}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(v - T_h(u)) dx \\
&- \int_{\{|v-T_h(u)|\le k\}} F.\nabla T_k(v - T_h(u)) dx \\
&\ge J_h + \sum_{i=1}^N \int_{E_2} a_i\Big(x, \frac{\partial v}{\partial x_i}\Big) \frac{\partial}{\partial x_i}(v-u) dx - \int_{E_2} F.\nabla(v-u) dx,
\end{aligned}$$

with $\lim_{h\to+\infty} J_h = 0$.

The other two terms in the left-hand side of (3.34) are denoted by

$$K_h = \int_\Omega b(u)T_k(u - T_h(v))dx + \int_\Omega b(v)T_k(v - T_h(u))dx.$$

We have

$$b(u)T_k(u - T_h(v)) \to b(u)T_k(u - v) \quad \text{a.e. in } \Omega \text{ as } h \to +\infty,$$
$$|b(u)T_k(u - T_h(v))| \le k|b(u)| \in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\to+\infty} \int_\Omega b(u)T_k(u - T_h(v))dx = \int_\Omega b(u)T_k(u - v)dx.$$

In the same way, we obtain

$$\lim_{h\to+\infty} \int_\Omega b(v)T_k(v - T_h(u))dx = \int_\Omega b(u)T_k(v - u)dx.$$

Then

$$\lim_{h\to+\infty} K_h = \int_\Omega (b(u) - b(v))T_k(u - v)dx.$$

Now, let us consider the integral of the right-hand side of (3.34). We have

$$\lim_{h\to+\infty} f(x)\Big(T_k(u - T_h(v)) + T_k(v - T_h(u))\Big) = 0 \quad \text{a.e. in } \Omega,$$
$$|f(x)\Big(T_k(u - T_h(v)) + T_k(v - T_h(u))\Big)| \le 2k|f| \quad \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\to+\infty} \int_\Omega f(x)\Big(T_k(u - T_h(v)) + T_k(v - T_h(u))\Big)dx = 0.$$

After passing to the limit as h tends to infinity in (3.34), we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u-v|\le k\}} \Big(a_i(x, \frac{\partial u}{\partial x_i}) - a_i(x, \frac{\partial v}{\partial x_i})\Big)\frac{\partial}{\partial x_i}(u - v)dx \\
&+ \int_\Omega (b(u) - b(v))T_k(u - v)dx \le 0.
\end{aligned}$$

Since b and $a_i(x,\cdot)$ are monotone, we have

$$\sum_{i=1}^N \int_{\{|u-v|\le k\}} \Big(a_i(x, \frac{\partial u}{\partial x_i}) - a_i(x, \frac{\partial v}{\partial x_i})\Big)\frac{\partial}{\partial x_i}(u - v)dx = 0, \tag{3.42}$$

$$\int_\Omega (b(u) - b(v))T_k(u - v)dx = 0. \tag{3.43}$$

We deduce from (3.43) that

$$\lim_{k\to 0} \int_\Omega (b(u) - b(v))\frac{1}{k}T_k(u - v)dx = \int_\Omega |b(u) - b(v)|dx = 0. \tag{3.44}$$

We deduce from (2.3) and (3.42) that

$$\frac{\partial}{\partial x_i}(u - v) = 0 \quad \text{a.e. in } \Omega, \text{ for } i = 1, \dots, N.$$

Therefore, there exists $c \in \mathbb{R}$ such that $u - v = c$ a.e. in Ω and using (3.44) we obtain $b(u) = b(v)$.

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