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EXISTENCE OF SOLUTIONS TO NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS WITH SINGULAR POTENTIALS

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Abstract. We study the eigenvalue problem

$$(-\Delta)^s u(x) + V(x)u(x) - K(x)|u|^{p-2}u(x) = \lambda u(x) \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$, $2 < p < 2^* = \frac{2N}{N-2s}$, $V(x)$ is indefinite and allowed to be unbounded from below, and $K(x)$ is nonnegative and allowed to be unbounded from above. When $\lambda < \lambda_0 = \inf \sigma((-\Delta)^s + V(x))$ (the lowest spectrum of the operator $(-\Delta)^s + V(x)$), we obtain a positive ground state solution by using the constrained minimization method. Also we discuss the regularity of solutions.

1. Introduction and statement of main results

In this article, we consider standing waves of the nonlinear fractional Schrödinger equation

$$i\psi_t = (-\Delta)^s \psi + V(x)\psi - K(x)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $(x,t) \in \mathbb{R}^N \times (0,\infty)$, $0 < s < 1$, $V(x)$ and $K(x)$ are some real functions. The operator $(-\Delta)^s$ is the fractional Laplacian of order s.

This equation was introduced by Laskin [8, 9], and comes from fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy process. The Lévy process is widely used to model a variety of processes, such as turbulence, financial dynamics, biology and physiology, see [7, 11, 19]. When $s = 1$, the Lévy process becomes the Brownian motion, and the equation (1.1) reduces to the classical Schrödinger equation

$$i\psi_t = -\Delta\psi + V(x)\psi - K(x)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Standing wave solutions to this equation are solutions of the form $\psi(x,t) = e^{-i\lambda t}u(x)$ where $u(x)$ satisfies the equation

$$-\Delta u + (V(x) - \lambda)u - K(x)|\psi|^{p-2}u = 0 \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

which has been extensively studied in the past 20 years. We mention some earlier work here. Oh [12] studied positive multi-lump bound states, and it was assumed that $K(x) \equiv \gamma$ for some $\gamma > 0$, and $V(x)$ belongs to a class of potentials $(V)_a$ for some a and $\lambda < a$ ($V \in (V)_a$ if either $V(x) \equiv a$ or $V(x) > a$ for all $x \in \mathbb{R}^N$ and $(V(x)-a)^{-1/2} \in Lip(\mathbb{R}^N)$). Rabinowitz [15] investigated the ground state solutions

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of the problem (1.3) under the condition $\inf_{\mathbb{R}^N} V(x) > \lambda$ and after this Byeon and Wang [1] considered the case $\inf_{\mathbb{R}^N} V(x) = \lambda$ which they call it critical frequency case.

Our goal is to look for standing wave solutions of the form $\psi(x,t) = e^{-i\lambda t}u(x)$ to equation (1.1) for fractional order $s \in (0,1)$. Precisely, we will investigate the problem.

$$\begin{gathered}(-\Delta)^s u(x) + (V(x)-\lambda)u(x) - K(x)|u|^{p-2}u(x) = 0 \quad \text{in } \mathbb{R}^N,\\ u(x) \in H^s(\mathbb{R}^N).\end{gathered} \tag{1.4}$$

Where $s \in (0,1)$, $2 < p < 2^* = \frac{2N}{N-2s}$, $N > 2s$, $\lambda \in \mathbb{R}$, $V(x)$ and $K(x)$ are real functions satisfying the following conditions:

- (A1) $V(x) : \mathbb{R}^N \to \mathbb{R}$, $V(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
- (A2) for any $\epsilon > 0$, the Lebesgue measure $|\{x : |V(x)| > \epsilon\}| < \infty$.
- (A3) $K(x) \geq 0$, $K(x) \not\equiv 0$, $K(x) \in L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
- (A4) for any $\epsilon > 0$, the Lebesgue measure $|\{x : |K(x)| > \epsilon\}| < \infty$.
- (A5) $V(x) \in L^{\tilde{q}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $K(x) \in L^{\tilde{r}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for $\tilde{q} > \frac{N}{2s}$ and $\tilde{r} > \frac{2^*}{2^*-p}$.

We remark that in [9], Laskin investigated the fractional Hydrogen-like atom where $V(x) = -\frac{Ze^2}{|x|}$ (for $N = 3$ and $1/2 < s < 1$), and evaluated the corresponding energy spectrum. It is easy to check that such potential satisfies condition (A1).

In recent years, there have been a few results for nonlinear fractional Schrödinger equations like (1.4). Teng [18] investigated multiple solutions of the equation

$$(-\Delta)^s u + V(x)u = f(x,u), \tag{1.5}$$

for $V(x) \in C(\mathbb{R}^N)$, $\operatorname{ess\,inf} V(x) > 0$, and $f \in C(\mathbb{R}^N \times \mathbb{R})$. Secchi [16] studied the ground state solutions of (1.5) for the case that $V \in C^1(\mathbb{R}^N)$, $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$, and $f \in C^1(\mathbb{R}^N \times \mathbb{R})$ satisfying Ambrosetti-Rabinowitz condition. In [3], ground states and bound states of (1.5) are obtained by assuming that $V(x) > 1$ and $\lim_{|x|\to+\infty} V(x) = +\infty$, and the nonlinearity is $f(t) = |t|^{p-1}t$. Chang [2] investigated the ground state solutions for asymptotically linear fractional Schrödinger equations. In particular, Felmer [6] studied the existence of positive solutions of (1.5) for $V(x) \equiv 1$ and $f(x,u)$ is superlinear and has subcritical growth with respect to u such that there exist $1 < p < (N+2s)/(N-2s)$, so that

$$f(x,\xi) \leq C(1+|\xi|)^p \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N. \tag{1.6}$$

Furthermore, they discuss the regularity, decay and symmetry properties of solutions.

The nonlinearity $K(x)|u|^{p-2}u(x)$ in this paper is quite different from (1.6), since $K(x)$ may not be bounded by a constant C. For example, $K(x) = \frac{1}{|x-x_0|^\alpha}$ for $0 < \alpha < \frac{(2^*-p)N}{2^*}$, satisfies (A3), (A4), and has singular point $x_0 \in \mathbb{R}^N$. On the other hand, since $V(x)$ is indefinite, it is hard to use usual mountain pass arguments to obtain ground state solutions([6, 16, 2]), here we will use the constrained minimization method to obtain the ground state solutions.

We say that $u \in H^s(\mathbb{R}^N)$ is a weak solution of (1.4), if for any $\phi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}\overline{u} \cdot (-\Delta)^{s/2}\phi \, dx + \int_{\mathbb{R}^N} (V(x)-\lambda)\overline{u} \cdot \phi \, dx = \int_{\mathbb{R}^N} K(x)|u|^{p-2}\overline{u} \cdot \phi \, dx,$$

where $\overline{u}(x)$ is conjugation of $u(x)$ in the complex space $H^s(\mathbb{R}^N)$.

Solutions of (1.4) correspond to the critical points of the energy functional

$$\mathcal{I}(u) = \frac{1}{2}[\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx + \int_{\mathbb{R}^N} (V(x)-\lambda)|u|^2\,dx] - \frac{1}{p}\int_{\mathbb{R}^N} K(x)|u|^p\,dx. \quad (1.7)$$

And *a ground state* of (1.4) is a solution that minimizes the energy functional on the Nehari manifold

$$\mathcal{N} = \big\{u \in H^s(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + (V(x)-\lambda)|u|^2\,dx = \int_{\mathbb{R}^N} K(x)|u|^p\,dx\big\}. \quad (1.8)$$

Now we state our main result.

Theorem 1.1. *Let $s \in (0,1)$, $2 < p < 2^*$ ($2^* = \frac{2N}{N-2s}$), $N > 2s$. Assume that* (A1)–(A4) *are satisfied. Let*

$$\begin{aligned}\lambda_0 &= \inf \sigma((-\Delta)^s + V(x)) \\ &= \inf\{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\psi|^2 + V(x)|\psi|^2\,dx : \psi \in H^s(\mathbb{R}^N), \|\psi\|_{L^2} = 1\},\end{aligned}$$

and assume that $\lambda \le 0$ and $\lambda < \lambda_0$. Then (1.4) *admits at least one nonnegative weak solution such that this solution is a ground state.*

To prove the positive property of nonnegative weak solutions, we need to take advantage of the representation formula

$$u = \mathcal{K}^\mu * f = \int_{\mathbb{R}^N} K(x-\xi)f(\xi)\,d\xi,$$

for some $\mu > 0$, and that u satisfies the equation

$$(-\Delta)^s u + \mu u = f \quad \text{in } \mathbb{R}^N,$$

where $\mathcal{K}^\mu$ is the Bessel kernel

$$\mathcal{K}^\mu = \mathcal{F}^{-1}\big(\frac{1}{\mu + |\xi|^{2s}}\big).$$

We have the following positive property.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let $w(x)$ be a nonnegative ground state solution obtain in Theorem 1.1. If we further assume that $V(x)$ is bound from above, then $w(x)$ can be chosen positive in $\mathbb{R}^N$.*

The next step is to prove regularity of the weak solutions. Inspired by ideas in [6], We also use the representation formula above to discuss the regularity. We have the following result.

Theorem 1.3. *Let $u(x) \in H^s(\mathbb{R}^N)$ be a solution of* (1.4)*, assume that $\lambda < 0$ and* (A5) *holds, i.e., $V(x) \in L^{\tilde q}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $K(x) \in L^{\tilde r}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for $\tilde q > \frac{N}{2s}$ and $\tilde r > \frac{2^*}{2^*-p}$. Then $u \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Moreover, $u(x) \to 0$ as $|x| \to \infty$.*

We remark that having the regularities above, by the same arguments as in [6, Theorem 1.5], it is easy to show that the positive ground state solutions $u(x)$ behave at infinity like $\frac{1}{|x|^{N+2s}}$.

The rest of the article is originated as follows. In section 2 we give some preliminary and show some properties of the operator $(-\Delta)^s + V(x)$. In section 3 we will show that weak convergence in $H^s(\mathbb{R}^N)$ implies strong convergence on finite measure sets, which is important to prove our result. In section 4 we prove that weak continuity of the potential energies. In section 5 we prove the Theorem 1.1. In section 6 we give the proof of Theorem 1.2 and 1.3.

Notation. To coincide with the book [10], the Banach spaces $L^p(\mathbb{R}^N)$, $H^s(\mathbb{R}^N)$ used here are complex Banach spaces. And the inner product is defined by

$$(f(x), g(x)) = \int_{\mathbb{R}^N} \overline{f(x)} g(x)\, dx, \quad \text{for any } f(x), g(x) \in L^2(\mathbb{R}^N), \tag{1.9}$$

where $\overline{f}$ denotes conjugation of $f(x)$.

$\widehat{u}$ denotes the Fourier transform of $u \in L^2(\mathbb{R}^N)$.

$$L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) := \{u = u_0 + u_1 : u_0 \in L^q(\mathbb{R}^N), u_1 \in L^\infty(\mathbb{R}^N)\}.$$

$\rightharpoonup$ denotes weakly converge. $C^{0,\alpha}(\mathbb{R}^N)$ denotes Hölder continuous with exponent $\alpha \in (0, 1)$.

2. Preliminaries

The fractional Laplacian $(-\Delta)^s$ of a rapidly decaying test function u is defined as

$$(-\Delta)^s u(x) = C_{N,s}\, \text{P.V.} \int \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dx\, dy, \tag{2.1}$$

where P.V. denotes the principal value of the singular integral, and $C_{N,s}$ is a constant.

We recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ (e.g., see [16]) is defined for any $p \in [1, \infty)$ and $s \in (0, 1)$ as

$$W^{s,p}(\mathbb{R}^N) = \Big\{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{s\,p+N}}\, dxdy < \infty\Big\},$$

endowed with the norm

$$\|u\|_W^{s,p} = \Big(\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{s\,p+N}}\, dxdy\Big)^{1/p}.$$

When $p = 2$, these spaces are also denoted by $H^s(\mathbb{R}^N)$.

When $p = 2$, there is an equivalent definition of fractional Sobolev spaces based on Fourier analysis that

$$H^s(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N) : \int (1 + |\xi|^{2s})|\widehat{u}(\xi)|^2 d\xi < \infty\Big\},$$
$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\widehat{u}, \quad \text{for } u \in H^s(\mathbb{R}^N),$$

and the norm can be equivalently written

$$\|u\|_{H^s} = \Big(\|u\|_{L^2}^2 + \int |\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi\Big)^{1/2} = \Big(\|u\|_{L^2}^2 + \|(-\Delta)^{s/2}u\|_{L^2}^2\Big)^{1/2}$$

Therefore, we see that $H^s(\mathbb{R}^N)$ is just $L^2(\mathbb{R}^N, d\mu)$, where μ is a measure defined by

$$\mu(dx) = (1 + |x|^{2s})dx.$$

A sequence $f^j(x)$ *converges weakly* to $f(x)$ (we write $f^j \rightharpoonup f$) in $H^s(\mathbb{R}^N)$ in the following sense (see [10, §7.18] or [4]): for any $g(x) \in H^s(\mathbb{R}^N)$, when $j \to \infty$, one has

$$\frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} [\widehat{f^j}(\xi) - \widehat{f}(\xi)]\widehat{g}(\xi)(1 + |\xi|^{2s})\, d\xi \to 0. \tag{2.2}$$

The following lemma is obvious. There are more details about $H^{1/2}(\mathbb{R}^N)$ in [10], so the case for general $s \in (0,1)$ is just the same arguments to $H^{1/2}(\mathbb{R}^N)$.

Lemma 2.1. *If a sequence f^j converges weakly to f in $H^s(\mathbb{R}^N)$. Then, there exists a constant M independent of number j, such that*

$$\|f^j\|_{H^s} \le M, \quad \|f\|_{H^s} \le M\,. \tag{2.3}$$

Proof. Since $H^s(\mathbb{R}^N)$ is just $L^2(\mathbb{R}^N, d\mu)$, thus by uniform boundedness principle and Lower semicontinuity of L^p-norm respectively, we obtain (2.3). $\square$

Now we review the Sobolev inequality and Sobolev-Gagliardo-Nirenberg inequality for fractional Sobolev spaces, we only show the case for $H^s(\mathbb{R}^N)$.

Lemma 2.2 (Sobolev inequality [17])**.** *Let $s \in (0,1)$ be such that $N > 2s$. Then*

$$\|u\|_{L^{2^*}} \le S_{N,s}\|(-\Delta)^{s/2}u\|_{L^2}$$

for every $u \in H^s(\mathbb{R}^N)$, where $S_{N,s}$ is sharp constants depending only on N,s, and

$$2^* = \frac{2N}{N-2s}$$

is the factional critical exponent.

Lemma 2.3 (Sobolev-Gagliardo-Nirenberg inequality [16])**.** *Let $q \in (2, 2^*)$. Then there exists a constant $C > 0$ such that*

$$\|u\|_{L^q}^q \le C\|u\|_{H^s}^{\frac{(q-2)N}{2s}} \|u\|_{L^2}^{q-\frac{(q-2)N}{2s}}$$

for every $u \in H^s(\mathbb{R}^N)$.

Next we show some properties of fractional Schrödinger operator $(-\Delta)^s + V(x)$. For any $\psi(x) \in H^s(\mathbb{R}^N)$, let λ_0 be defined in Theorem 1.1, define

$$\mathcal{E}(\psi) := \|(-\Delta)^{s/2}\psi\|_{L^2}^2 + \int_{\mathbb{R}^N} V(x)|\psi|^2\, dx. \tag{2.4}$$

then $\lambda_0 = \inf\{\mathcal{E}(\psi) : \psi \in H^s(\mathbb{R}^N), \|\psi\|_{L^2} = 1\}$. We have the following theorem.

Theorem 2.4. *For $s \in (0,1)$, $N > 2s$, if $V(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, then*

(i) *λ_0 is finite.*
(ii) *$\|(-\Delta)^{s/2}\psi\|_{L^2}^2 \le C\mathcal{E}(\psi) + D\|\psi\|_{L^2}^2$ for $\psi \in H^s(\mathbb{R}^N)$ and suitable constants C and D.*

Proof. Since $V(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, we can write $V(x) = v(x) + w(x)$ with $v(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ and $w(x) \in L^\infty(\mathbb{R}^N)$.

First we claim that we can choose $v(x)$ satisfying $\|v(x)\|_{L^{\frac{N}{2s}}} \le \frac{1}{2}(S_{N,s})^{-2}$. In fact, for $M > 0$, define $S_v(M)$ by

$$S_v(M) = \{x \in \mathbb{R}^N : |v(x)| > M\},$$

then by Chebyshev inequality (see [5])

$$|S_v(M)| \le \Big(\frac{C\|v\|_{L^{\frac{N}{2s}}}}{M}\Big)^{N/(2s)}. \tag{2.5}$$

Let χ_A be the characteristic function on subset $A \subset \mathbb{R}^N$. Decompose $v(x)$ into

$$v(x) = \chi_{S_v(M)} v(x) + (1 - \chi_{S_v(M)}) v(x).$$

Let $v_1 = \chi_{S_v(M)} v(x)$, $v_2 = (1 - \chi_{S_v(M)}) v(x)$, then $v_1 \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, $v_2 \in L^\infty(\mathbb{R}^N)$, and by (2.5) we have $\|v_1\|_{L^{\frac{N}{2s}}} < \frac{1}{2}(S_{N,s})^{-2}$ for large enough M. Replace $v(x)$ by v_1, then the claim holds.

For any function $\psi \in H^s(\mathbb{R}^N)$, combing with $\|v(x)\|_{L^{\frac{N}{2s}}} \le \frac{1}{2}(S_{N,s})^{-2}$ and using Hölder inequality and Sobolev inequality (Lemma 2.2), we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} v(x)|\psi|^2\,dx\Big| &\le \|v(x)\|_{L^{\frac{N}{2s}}} \|\psi\|_{L^{2^*}}^2 \\ &\le S_{N,s}^2 \|v(x)\|_{L^{\frac{N}{2s}}} \|(-\Delta)^{s/2}\psi\|_{L^2}^2 \\ &\le \frac{1}{2}\|(-\Delta)^{s/2}\psi\|_{L^2}^2, \end{aligned}$$

it follows that

$$\begin{aligned} \mathcal{E}(\psi) &= \|(-\Delta)^{s/2}\psi\|_{L^2}^2 + \int_{\mathbb{R}^N} v(x)|\psi|^2\,dx + \int_{\mathbb{R}^N} w(x)|\psi|^2\,dx \\ &\ge \frac{1}{2}\|(-\Delta)^{s/2}\psi\|_{L^2}^2 - \|w(x)\|_{L^\infty}\|\psi\|_{L^2}^2 \ge -\|w(x)\|_{L^\infty}\|\psi\|_{L^2}^2\,, \end{aligned} \tag{2.6}$$

and we see that $-\|w(x)\|_{L^\infty}$ is a lower bound to λ_0, i.e., (i) holds. Furthermore, the first inequality of (2.6) implies

$$\|(-\Delta)^{s/2}\psi\|_{L^2}^2 \le 2(\mathcal{E}(\psi) + \|w(x)\|_{L^\infty}\|\psi\|_{L^2}^2),$$

i.e., (ii) holds. □

3. Weak convergence implies strong convergence on small sets

Consider the semigroup $\{e^{-(-\Delta)^s t}\}_{t>0}$. We know that, for any function $f(x) \in H^s(\mathbb{R}^N)$,

$$\widehat{e^{-(-\Delta)^s t} f}(\xi) = e^{-|\xi|^{2s} t}\hat{f}(\xi).$$

Now we define the heat kernel for $s \in (0,1)$, $t > 0$, and $x \in \mathbb{R}^N$ as

$$\mathcal{H}(x,t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i\,x\cdot\xi - t|\xi|^{2s}}\,d\xi, \tag{3.1}$$

and we know that

$$e^{-(-\Delta)^s t} f(x) = \int_{\mathbb{R}^N} \mathcal{H}(x-y,t) f(y)\,dy. \tag{3.2}$$

It is well known that $\mathcal{H}(x,t)$ has the following properties, see [6, Appendix A] and references therein.

Lemma 3.1. *$\mathcal{H}(x,t)$ is radially symmetric in x, and there exists two constants c_1 and c_2 such that*

$$c_1 \min\big\{t^{-\frac{N}{2s}}, t|x|^{-N-2s}\big\} \le \mathcal{H}(x,t) \le c_2 \min\big\{t^{-\frac{N}{2s}}, t|x|^{-N-2s}\big\}. \tag{3.3}$$

Now we use the properties of semigroup with respect to $(-\Delta)^s$ to prove that weak convergence in $H^s(\mathbb{R}^N)$ implies strong convergence on any finite measure set (not just on a bounded domain $\Omega \in \mathbb{R}^N$, see compact embeddings in [13, 14]). This result can also be found in [4], here we give a different proof along the ideas in [10, Theorem8.6].

Theorem 3.2. *Let $\{f^j\} \subset H^s(\mathbb{R}^N)$ such that f^j converges weakly to f in $H^s(\mathbb{R}^N)$. Let $A \subset \mathbb{R}^N$ be any set of finite Lebesgue measure, i.e., $|A| < \infty$, and let χ_A be its characteristic function. Then*

$$\chi_A f^j \to \chi_A f \quad \text{strongly in } L^q(\mathbb{R}^N)$$

for $1 \le q < 2^* = \frac{2N}{N-2s}$, when $N > 2s$.

Proof. We take three steps to prove the theorem.

Step 1. We claim that, for any $f \in H^s(R^N)$,

$$\|f - e^{-(-\Delta)^s t} f\|_{L^2} \le \|(-\Delta)^{s/2} f\|_{L^2} \sqrt{t}. \tag{3.4}$$

In fact, we know that

$$1 - \exp[-(|\xi|)^{2s} t] \le \min\left\{1, (|\xi|)^{2s} t\right\} \le |\xi|^s \sqrt{t},$$

and it follows that

$$\begin{aligned} \|f - e^{-(-\Delta)^s t} f\|_{L^2}^2 &= \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (1 - \exp[-(|\xi|)^{2s} t])^2 \, d\xi \\ &\le \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (|\xi|^s \sqrt{t})^2 \, d\xi = \|(-\Delta)^{s/2} f\|_{L^2}^2 \, t, \end{aligned}$$

this proves (3.4).

Step 2. We first prove that $\chi_A f^j \to \chi_A f$ strongly in $L^2(\mathbb{R}^N)$. Let $g^j := e^{-(-\Delta)^s t} f^j$, by Lemma 2.1, we note that

$$\|(-\Delta)^{s/2} f^j\|_{L^2} \le \|f^j\|_{H^s} \le M, \tag{3.5}$$

$$\|(-\Delta)^{s/2} f\|_{L^2} \le \|f\|_{H^s} \le M, \tag{3.6}$$

where M is a constant independent of j. Then by (3.4) we have

$$\begin{gathered} \|f^j - g^j\|_{L^2} = \|f^j - e^{-(-\Delta)^s t} f^j\|_{L^2} \le M\sqrt{t}, \\ \|f - g\|_{L^2} = \|f - e^{-(-\Delta)^s t} f\|_{L^2} \le M\sqrt{t}. \end{gathered}$$

Simply note that

$$\begin{aligned} \|\chi_A(f^j - f)\|_{L^2} &\le \|\chi_A(f^j - g^j)\|_{L^2} + \|\chi_A(g^j - g)\|_{L^2} + \|\chi_A(g - f)\|_{L^2} \\ &\le 2M\sqrt{t} + \|\chi_A(g^j - g)\|_{L^2} . \end{aligned}$$

For $\epsilon > 0$ given, first choose $t > 0$ (depending on ϵ) such that $2M\sqrt{t} < \epsilon/2$ and if for j (depending on ϵ) we have $\|\chi_A(g^j - g)\|_{L^2} < \epsilon/2$, then we have $\|\chi_A(f^j - f)\|_{L^2} < \epsilon$. Therefore, it remains to prove that $\chi_A g^j \to \chi_A g$ strongly in $L^2(\mathbb{R}^N)$.

To prove $\chi_A g^j \to \chi_A g$ strongly in $L^2(\mathbb{R}^N)$, first we note that, if $|y - x| \ge t^{\frac{1}{2s}}$, we have $t|x - y|^{-N-2s} \le t^{-\frac{N}{2s}}$, and then by Lemma 3.1, we have

$$\begin{aligned} \mathcal{H}(x - y, t) &\le c_2 \, t|x - y|^{-N-2s} = 2c_2 \frac{t}{2|x - y|^{N+2s}} \\ &\le 2c_2 \frac{t}{t^{\frac{N+2s}{2s}} + |x - y|^{N+2s}} . \end{aligned} \tag{3.7}$$

Then, for every fix x, we have $\mathcal{H}(x-y,t) \in L^{(2^*)'}(\mathbb{R}^N)$, where $(2^*)' = 2N/(N+2s)$, is dual index to 2^*. In fact, let $B(x, t^{\frac{1}{2s}})$ denote a ball center at x and has radius $t^{\frac{1}{2s}}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\mathcal{H}(x-y,t))^{(2^*)'}\,dy \\
&= \int_{B(x,t^{\frac{1}{2s}})} (\mathcal{H}(x-y,t))^{(2^*)'}\,dy + \int_{\mathbb{R}^N \setminus B(x,t^{\frac{1}{2s}})} (\mathcal{H}(x-y,t))^{(2^*)'}\,dy \\
&\le \int_{B(x,t^{\frac{1}{2s}})} (c_2\, t^{-\frac{N}{2s}})^{(2^*)'} dy + \int_{\mathbb{R}^N \setminus B(x,t^{\frac{1}{2s}})} (c_2 t|x-y|^{-N-2s})^{(2^*)'}\,dy \\
&\le M_1 + \int_{\mathbb{R}^N} (2c_2 \frac{t}{t^{\frac{N+2s}{2s}} + |x-y|^{N+2s}})^{(2^*)'}\,dy \\
&\le M_1 + \int_{\mathbb{R}^N} (2c_2 \frac{t}{t^{\frac{N+2s}{2s}} + |y|^{N+2s}})^{(2^*)'}\,dy \le M_2,
\end{aligned}$$

where M_2 is a constant independent of x.

Since for every x, we have $\mathcal{H}(x-y,t) \in L^{(2^*)'}(\mathbb{R}^N)$, by Hölder inequality

$$\chi_A |g^j(x)| \le \|\mathcal{H}(x-y,t)\|_{(2^*)'} \|f^j\|_{2^*} \chi_A(x).$$

Using Lemma 2.2 and (3.5), $\|f^j\|_{2^*} \le S_{N,s} \|(-\Delta)^{s/2} f^j\|_{L^2} \le S_{N,s} M$. Hence $\chi_A g^j$ is dominated by a constant multiple of the square integrable function $\chi_A(x)$. On the other hand, if $g^j(x)$ converges pointwise to $g(x)$ for every $x \in \mathbb{R}^N$, Then by general dominated convergence theorem, we have $\chi_A g^j \to \chi_A g$ strongly in $L^2(\mathbb{R}^N)$. Next we shall prove $g^j(x)$ converges pointwise for every $x \in \mathbb{R}^N$. We note that, for fixed x,

$$H(\widehat{x-y}, t)(\xi) = (e^{-ix\cdot\xi}) e^{-t|\xi|^{2s}},$$

and

$$\begin{aligned}
g^j(x) = e^{-(-\Delta)^s t} f^j(x) &= \int_{\mathbb{R}^N} \mathcal{H}(x-y,t) f^j(y)\,dy \\
&= \int_{\mathbb{R}^N} H(\widehat{x-y}, t)(\xi) \hat{f}^j(\xi)\,d\xi = \int_{\mathbb{R}^N} (e^{-ix\cdot\xi}) e^{-t|\xi|^{2s}} \hat{f}^j(\xi)\,d\xi \\
&= \int_{\mathbb{R}^N} \frac{(e^{-ix\cdot\xi}) e^{-t|\xi|^{2s}}}{1+|\xi|^{2s}} \hat{f}^j(\xi)(1+|\xi|^{2s})\,d\xi.
\end{aligned}$$

Let $h(y)$ be a function satisfying $\hat{h}(\xi) = \frac{(e^{-ix\cdot\xi}) e^{-t|\xi|^{2s}}}{1+|\xi|^{2s}}$, it is easy to see that $h(y) \in H^s(\mathbb{R}^N)$. Since f^j converges weakly to f in $H^s(\mathbb{R}^N)$, by (2.2), then we have $g^j(x)$ converges pointwise to $g(x)$ for every $x \in \mathbb{R}^N$. Hence we complete the proof of Step 2.

Step 3. The inequality

$$\|\chi_A(f^j - f)\|_{L^q} \le \|\chi_A\|_{L^r} \|\chi_A(f - f^j)\|_{L^2}$$

for $1/q = 1/r + 1/2$ proves the theorem for $1 \le q \le 2$. Again by Hölder inequality, Lemma 2.2,

$$\begin{aligned}
\|\chi_A(f^j - f)\|_{L^q} &\le \|\chi_A(f^j - f)\|_{L^2}^{\alpha} \|\chi_A(f - f^j)\|_{L^{2^*}}^{1-\alpha} \\
&\le \|\chi_A(f^j - f)\|_{L^2}^{\alpha} \|f - f^j\|_{L^{2^*}}^{1-\alpha} \\
&\le \|\chi_A(f^j - f)\|_{L^2}^{\alpha} (S_{N,s})^{1-\alpha} \|(-\Delta)^{s/2}(f - f^j)\|_{L^2}^{1-\alpha}
\end{aligned}$$

$$\leq \|\chi_A(f^j - f)\|_{L^2}^{\alpha}(2M\,S_{N,s})^{1-\alpha},$$

where $\alpha = (1/q - 1/2^*)(1/2 - 1/2^*)$, and this proves the theorem for $2 \leq q < 2^*$. The proof is complete. □

4. Weak continuity of the potential energies

Lemma 4.1. *Let $2 \leq q < 2^*$, $F_\psi := \int_{\mathbb{R}^N} F(x)|\psi|^q\,dx$, $F(x)$ be a real function on $\mathbb{R}^N$ such that $F(x) \in L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ and $|\{x : |F(x)| > \epsilon\}| < \infty$ for any $\epsilon > 0$. Then F_ψ is weakly continuous in $H^s(\mathbb{R}^N)$, i.e., if $\psi^j \rightharpoonup \psi$ as $j \to \infty$ in $H^s(\mathbb{R}^N)$, then $F_{\psi^j} \to F_\psi$ as $j \to \infty$.*

Proof. Note that by assumption, $\|\psi^j\|_{H^s}$ is uniformly bounded, i.e., there is a constant $M > 0$ independent of j such that $\|\psi^j\|_{H^s} \leq M$ for all j. For any $\delta > 0$, define

$$F^\delta(x) = \begin{cases} F(x) & \text{if } |F(x)| \leq \frac{1}{\delta}, \\ 0 & \text{if } |F(x)| > \frac{1}{\delta}. \end{cases}$$

First we claim that $F - F^\delta \in L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$. Indeed, let $\Omega = \{x : |F(x)| > \frac{1}{\delta}\}$, by assumption above we know $|\Omega| < \infty$. Writing $F(x) = f_1(x) + f_2(x)$ with $f_1 \in L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$ and $f_2(x) \in L^\infty(\mathbb{R}^N)$, then we have

$$F - F^\delta = \chi_\Omega F(x) = \chi_\Omega f_1(x) + \chi_\Omega f_2(x)\,,$$

where χ_Ω be the characteristic function on Ω. Since $|\Omega| < \infty$, by Hölder inequality, we have $\chi_\Omega f_2(x) \in L^{\frac{2^*}{2^*-q}}(\Omega)$. It follows that $\chi_\Omega f_2(x) \in L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$, thus the claim holds.

Moreover, $F - F^\delta \to 0$ in $L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$ as $\delta \to 0$(by dominated convergence). Since $\|\psi^j\|_{H^s} \leq M$, by Sobolev inequality (Lemma 2.2)

$$\|\psi^j\|_{L^{2^*}} \leq S_{N,s}\|(-\Delta)^{s/2}\psi^j\|_{L^2} \leq S_{N,s}\|\psi^j\|_{H^s} \leq C_1. \tag{4.1}$$

By Hölder inequality, we have

$$\int (F - F^\delta)|\psi^j|^q \leq \|F - F^\delta\|_{\frac{2^*}{2^*-q}}\|\psi^j\|_{2^*}^q \leq C_1\|F - F^\delta\|_{\frac{2^*}{2^*-q}} = C_\delta,$$

with C_δ independent of j, moreover, $C_\delta \to 0$ as $\delta \to 0$. Thus, our goal of showing that $F_{\psi^j} \to F_\psi$ as $j \to \infty$ would be achieved if we can prove that $F^\delta_{\psi^j} \to F^\delta_\psi$ as $j \to \infty$ for each $\delta > 0$.

To prove that $F^\delta_{\psi^j} \to F^\delta_\psi$ as $j \to \infty$, now fix δ and define the set

$$A_\epsilon = \{x : |F^\delta(x)| > \epsilon\}$$

for $\epsilon > 0$. By assumption, $|A_\epsilon| < \infty$. Then

$$F^\delta_{\psi^j} = \int_{A_\epsilon} F^\delta|\psi^j|^q + \int_{A_\epsilon^c} F^\delta|\psi^j|^q. \tag{4.2}$$

Since $2 \leq q < 2^*$, $\|\psi^j\|_2 \leq \|\psi^j\|_{H^s} \leq M$, by Lemma 2.3, we have

$$\|\psi^j\|_{L^q} \leq C\|\psi^j\|_{H^s}^{\frac{(k-2)N}{2sq}}\|\psi^j\|_{L^2}^{1-\frac{(q-2)N}{2sq}} \leq CM\,, \tag{4.3}$$

by weak lower semicontinuity of the norm, we also have

$$\|\psi\|_{L^q} \leq \liminf_{j\to\infty}\|\psi^j\|_{L^q} \leq CM. \tag{4.4}$$

Then

$$\int_{A_\epsilon^c} F^\delta |\psi^j|^q \le \epsilon \int_{\mathbb{R}^N} |\psi^j|^q \le \epsilon (C\,M)^q\,,$$

i.e., the last term of (4.2) tend to zero as $\epsilon \to 0$, and hence it suffices to show that the first term of (4.2) converges to $\int_{A_\epsilon} F^\delta |\psi|^q$.

This is accomplished as follows. By Theorem 3.2 (in the Appendix below), on any finite measure set (that we take to be A_ϵ) $\psi^j \to \psi$ strongly in $L^r(A_\epsilon)$, for $r \in [1, 2^*)$. Here we can choose $r = q$. Since $q \ge 2$, using the inequality

$$\big||\psi^j|^q - |\psi|^q\big| \le C_q(|\psi^j|^{q-1} + |\psi|^{q-1})|\psi^j - \psi|,$$

where C_q is a constant only dependent of q, and by (4.3), (4.4) and Hölder inequality, we have

$$\begin{aligned}
\int_{A_\epsilon} \big||\psi^j|^q - |\psi|^q\big| &\le \int_{A_\epsilon} C_q(|\psi^j|^{q-1} + |\psi|^{q-1})|\psi^j - \psi| \\
&\le C_q \||\psi^j|^{q-1} + |\psi|^{q-1}\|_{L^{\frac{q}{q-1}}(A_\epsilon)} \|\psi^j - \psi\|_{L^q(A_\epsilon)} \\
&\le C_3 \|\psi^j - \psi\|_{L^q(A_\epsilon)},
\end{aligned}$$

so $|\psi^j|^q \to |\psi|^q$ strongly in $L^1(A_\epsilon)$. Since $F^\delta \in L^\infty(\mathbb{R}^N)$ (see the definition above), we conclude that

$$\int_{A_\epsilon} F^\delta |\psi^j|^q \to \int_{A_\epsilon} F^\delta |\psi|^q, \quad \text{as } j \to \infty.$$

This completes the proof. □

5. Proof of Theorem 1.1

We will give the proof by a series of lemmas. Firstly, for any $\beta > 0$, we set

$$\Sigma_\beta := \big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} K(x)|u|^p\,dx = \beta\big\}.$$

Lemma 5.1. *Assume that $K(x)$ satisfies* (A3)*, then Σ_β is not empty.*

Proof. Since $K(x) \ge 0$ and $K(x) \not\equiv 0$, for any fixed $u \in H^s(\mathbb{R}^N) \setminus \{0\}$, we have

$$\int_{\mathbb{R}^N} K(x)|u|^p\,dx > 0.$$

Write $K(x) = K_1 + K_2$ with $K_1 \in L^{\frac{2^*}{2^*-p}}(\mathbb{R}^N)$ and $K_2 \in L^\infty(\mathbb{R}^N)$. For any fixed $u \in H^s(\mathbb{R}^N)\backslash\{0\}$, Since $2 < p < 2^*$, by Hölder inequality, Lemma 2.2, Lemma 2.3 we have

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)|u|^p\,dx &\le \|K_1\|_{L^{\frac{2^*}{2^*-p}}} \|u\|_{L^{2^*}}^p + \|K_2\|_{L^\infty} \|u\|_{L^p}^p \\
&\le C_1 \|(-\Delta)^{s/2} u\|_{L^2}^p + C_2 \|u\|_{H^s}^{\frac{(p-2)N}{2s}} \|u\|_{L^2}^{p-\frac{(p-2)N}{2s}} < \infty,
\end{aligned}$$

where C_1 and C_2 are some constants. Then we can choose $t > 0$ such that $tu(x) \in \Sigma_\beta$, where

$$t = \Big(\frac{\beta}{\int_{\mathbb{R}^N} K(x)|u|^p\,dx}\Big)^{1/p}.$$

□

Let $\mathcal{I}(u)$ be the energy functional defined by (1.7), we want to consider the minimizing problem

$$\inf_{\Sigma_\beta} \mathcal{I}(u) = \frac{1}{2} \inf_{\Sigma_\beta} \Big\{ \int_{\mathbb{R}^N} (|(-\Delta)^{s/2}u|^2 + (V(x)-\lambda)|u|^2)\,dx \Big\} - \frac{1}{p}\beta.$$

Let

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} (|(-\Delta)^{s/2}u|^2 + (V(x)-\lambda)|u|^2)\,dx, \tag{5.1}$$

with $m_\beta = \inf_{u\in\Sigma_\beta} \mathcal{J}(u)$, so we have

$$\inf_{u\in\Sigma_\beta} \mathcal{I}(u) = \frac{1}{2}m_\beta - \frac{1}{p}\beta.$$

Thus minimizing $\mathcal{I}(u)$ on Σ_β is equivalent to considering just m_β.

Lemma 5.2. *With the assumptions of Theorem 1.1, let $\{u_k\}_k \subset \Sigma_\beta$ be a minimizing sequence for m_β. Then $\{u_k\}$ is bounded in $H^s(\mathbb{R}^N)$.*

Proof. Since $\{u_k\}_k$ is a minimizer sequence for m_β, it follows that

$$\lim_{k\to\infty} \mathcal{J}(u_k) = m_\beta.$$

Then, $\mathcal{J}(u_k)$ is bounded by a constant independent of k, i.e., $\mathcal{J}(u_k) \le M$. Since $V(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ by Theorem 2.4, we know that λ_0 is finite. By the assumption $\lambda < \lambda_0$ in Theorem 1.1, we have

$$\mathcal{J}(u_k) \ge \lambda_0 \int_{\mathbb{R}^N} |u_k|^2\,dx - \lambda \int_{\mathbb{R}^N} |u_k|^2\,dx = (\lambda_0 - \lambda)\|u_k\|_2^2\,,$$

it follows that $\|u_k\|_2 \le M/(\lambda_0 - \lambda)$. i.e., $\{u_k\}_k$ is bounded in $L^2(\mathbb{R}^N)$.

Since $\lambda \le 0$, by (ii) of Theorem 2.4, we have

$$\begin{aligned}
\|(-\Delta)^{s/2}u_k\|_{L^2}^2 &\le C\mathcal{E}(u_k) + D\|u_k\|_{L^2}^2 \\
&\le C(\mathcal{E}(u_k) - \lambda\|u_k\|_{L^2}^2) + D\|u_k\|_{L^2}^2 \\
&= C\mathcal{J}(u_k) + D\|u_k\|_{L^2}^2 \\
&\le C\,M + D\,M/(\lambda_0 - \lambda).
\end{aligned}$$

Therefore, $\{u_k\}_k$ is bounded in $H^s(\mathbb{R}^N)$. □

Lemma 5.3. *With the assumptions of Theorem 1.1, for every $\beta > 0$, m_β is attained by a nonnegative function, namely there exists $u_0 \in \Sigma_\beta$, $u_0(x) \ge 0$ a.e. in $\mathbb{R}^N$, such that*

$$m_\beta = \mathcal{J}(u_0).$$

Moreover, $m_\beta > 0$.

Proof. Let $\{u_k\}_k \subset \Sigma_\beta$ be a minimizing sequence for m_β. In Section 2, we know that $\|(-\Delta)^{s/2}u_k\|_{L^2}^2$ is equivalent to

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{2s+N}}\,dx\,dy,$$

it follows that $\|(-\Delta)^{s/2}|u_k|\|_{L^2}^2 \le \|(-\Delta)^{s/2}u_k\|_{L^2}^2$, hence the sequence $\{|u_k|\}_k$ is still a minimizing sequence and we can assume from the beginning that $u_k \ge 0$ a.e. in $\mathbb{R}^N$ for all k. By Lemma 5.2, this minimizing sequence is bounded in $H^s(\mathbb{R}^N)$, so up to subsequences,

$$u_k \rightharpoonup u_0 \quad \text{in } H^s(\mathbb{R}^N),$$

by Lemma 4.1, we have

$$\int_{\mathbb{R}^N} K(x)|u_k|^p\,dx \to \int_{\mathbb{R}^N} K(x)|u_0|^p\,dx,$$

and then

$$\int_{\mathbb{R}^N} K(x)|u_0|^p\,dx = \int_{\mathbb{R}^N} K(x)|u_k|^p\,dx = \beta,$$

thus $u_0 \in \Sigma_\beta$.

Applying Lemma 4.1 for $p = 2$, we have

$$\int_{\mathbb{R}^N} V(x)|u_k|^2\,dx \to \int_{\mathbb{R}^N} V(x)|u_0|^2\,dx.$$

Since $\lambda \le 0$, by weak lower semicontinuity of the norm, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_0|^2\,dx + \int_{\mathbb{R}^N} V(x)|u_0|^2\,dx - \lambda\int_{\mathbb{R}^N} |u_0|^2\,dx\\
&= \|(-\Delta)^{s/2}u_0\|_{L^2}^2 + (-\lambda)\|u_0\|_{L^2}^2 + \int_{\mathbb{R}^N} V(x)|u_0|^2\,dx\\
&\le \liminf_{k\to\infty}\, [\, \|(-\Delta)^{s/2}u_k\|_{L^2}^2 + (-\lambda)\|u_k\|_{L^2}^2 + \int_{\mathbb{R}^N} V(x)|u_k|^2\,dx] = m_\beta,
\end{aligned}$$

together with $u_0 \in \Sigma_\beta$, this shows that

$$m_\beta = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_0|^2\,dx + \int_{\mathbb{R}^N} V(x)|u_0|^2\,dx - \lambda\int_{\mathbb{R}^N} |u_0|^2\,dx = \mathcal{J}(u_0).$$

Note that $u_0 \in \Sigma_\beta$ implies that $u_0 \not\equiv 0$, then from the definition of λ_0 given in Theorem 1.1 it follows that

$$m_\beta = \mathcal{J}(u_0) \ge (\lambda_0 - \lambda)\|u_0\|_{L^2}^2 > 0.$$

This completes the proof. □

Lemma 5.4. *With the assumptions of Theorem 1.1, let u_0 be a minimizer for m_β. Then u_0 satisfies*

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{s/2}\overline{u_0}\cdot(-\Delta)^{s/2}v\,dx + \int_{\mathbb{R}^N} (V(x)-\lambda)\overline{u_0}\cdot v\,dx\\
&= \frac{m_\beta}{\beta}\int_{\mathbb{R}^N} K(x)|u_0|^{p-2}\overline{u_0}\cdot v\,dx
\end{aligned} \tag{5.2}$$

for all $v \in H^s(\mathbb{R}^N)$.

Proof. Let $\mathcal{J}(u_0)$ be energy functional defined by (5.1). Fix $v(x) \in H^s(\mathbb{R}^N)$, for $\varepsilon \in \mathbb{R}$ small enough, when $r \in (-\varepsilon, \varepsilon)$, the function $u_0 + rv$ is not identically zero. Therefore there exists a function $t(r) : (-\varepsilon, \varepsilon) \to (0, \infty)$ such that

$$\int_{\mathbb{R}^N} |K(x)t(r)(u_0 + rv)|^p\,dx = \beta.$$

Precisely,

$$t(r) = \Big(\frac{\beta}{\int_{\mathbb{R}^N} |K(x)(u_0 + rv)|^p\,dx}\Big)^{1/p}.$$

Note that the map $r \mapsto t(r)(u_0 + rv)$ defines a curve on Σ_β that passes through u_0 when $r = 0$. The function $t(r)$ is differentiable on $(-\varepsilon, \varepsilon)$,

$$t'(r) = -\beta^{1/p}\Big(\int_{\mathbb{R}^N} |K(x)(u_0 + rv)|^p\,dx\Big)^{-\frac{1}{p}-1}\operatorname{Re}(K(x)|u_0 + rv|^{p-2}(u_0 + rv), v),$$

where Re denotes real part of inner product $(\cdot,\cdot)$ (defined in (1.9)), and

$$\operatorname{Re}(|u_0+rv|^{p-2}(u_0+rv),v)=\operatorname{Re}\int_{\mathbb{R}^N}K(x)|(u_0+rv)|^{p-2}\overline{(u_0+rv)}\cdot v\,dx.$$

Then we have

$$t(0)=1 \quad\text{and}\quad t'(0)=-\beta^{-1}\operatorname{Re}(K(x)|u_0|^{p-2}u_0,v). \tag{5.3}$$

We define $\gamma:(-\varepsilon,\varepsilon)\to R$ as

$$\begin{aligned}\gamma(r)&=\mathcal{J}(t(r)(u_0+rv))=t^2(r)\mathcal{J}(u_0+rv)\\&=t^2(r)((-\Delta)^{s/2}(u_0+rv),(-\Delta)^{s/2}(u_0+rv))\\&\quad+t^2(r)((V(x)-\lambda)(u_0+rv),(u_0+rv)).\end{aligned}$$

Since $t(r)(u_0+rv)\in\Sigma_\beta$ for every $r\in(-\varepsilon,\varepsilon)$, the point $r=0$ is a local minimum for γ, such that

$$\gamma(0)=\mathcal{J}(u_0)=m_\beta. \tag{5.4}$$

The function γ is differentiable and

$$\begin{aligned}\gamma'(r)=&2t(r)t'(r)\mathcal{J}(u_0+rv)\\&+2t^2(r)\operatorname{Re}[((-\Delta)^{s/2}(u_0+rv),(-\Delta)^{s/2}v)+((V(x)-\lambda)(u_0+rv),v)].\end{aligned}$$

by (5.3), (5.4), then

$$\begin{aligned}0=\gamma'(0)&=2t(0)t'(0)\mathcal{J}(u_0)+2t^2(0)\operatorname{Re}\big[((-\Delta)^{s/2}u_0,(-\Delta)^{s/2}v)\\&\quad+((V(x)-\lambda)u_0,v)\big]\\&=-2\beta^{-1}\operatorname{Re}(K(x)|u_0|^{p-2}u_0,v)m_\beta\\&\quad+2\operatorname{Re}[((-\Delta)^{s/2}u_0,(-\Delta)^{s/2}v)+((V(x)-\lambda)u_0,v)].\end{aligned} \tag{5.5}$$

Since v is an arbitrary complex function in $H^s(\mathbb{R}^N)$, it follows that

$$-\beta^{-1}(K(x)|u_0|^{p-2}u_0,v)m_\beta+[((-\Delta)^{s/2}u_0,(-\Delta)^{s/2}v)+((V(x)-\lambda)u_0,v)]=0,$$

i.e. (5.2) holds. □

Let u_0 be a minimizer for m_β. Set $u_0(x)=c\,w(x)$, where $c\in\mathbb{R}$ will be determined later. By Lemma 5.4, $w(x)$ satisfies

$$c[((-\Delta)^{s/2}w,(-\Delta)^{s/2}v)+((V(x)-\lambda)w,v)]=\frac{m_\beta}{\beta}\,c^{p-1}(K(x)|w|^{p-2}w,v)$$

for all $v\in H^s(\mathbb{R}^N)$. Choosing $c=(\frac{\beta}{m_\beta})^{\frac{1}{p-2}}$, we see that $w(x)$ is nonnegative by Lemma 5.3 and satisfies

$$\big((-\Delta)^{s/2}w,(-\Delta)^{s/2}v\big)+\big((V(x)-\lambda)w,v\big)=\big(K(x)|w|^{p-2}w,v\big)\quad\forall v\in H^s(\mathbb{R}^N),$$

namely $w(x)$ is a weak (nonzero) solution of (1.4), such that

$$u_0(x)=(\frac{\beta}{m_\beta})^{\frac{1}{p-2}}w(x). \tag{5.6}$$

Thus we obtain the existence of the solution.

Let $\mathcal{N}$ be the Nehari manifold defined by (1.8), note that $w(x)\in\mathcal{N}$. We mention in Section 1 that a ground state of (1.4) is a solution that minimizes the energy functional $\mathcal{I}(u)$ on the Nehari manifold $\mathcal{N}$, next we will prove that $w(x)$ is a ground state, that is, we need to prove that

$$\mathcal{I}(w) \leq \mathcal{I}(\phi), \quad \text{for any } \phi \in \mathcal{N}. \tag{5.7}$$

For any function $\phi \in \mathcal{N}$, then by the definition of $\mathcal{N}$ we have

$$\mathcal{I}(\phi) = (\frac{1}{2} - \frac{1}{p})\mathcal{J}(\phi), \tag{5.8}$$

where $\mathcal{J}(\phi)$ is energy functional defined in (5.1).

Fix any $\phi \in \mathcal{N}$ and let $\theta := \int_{\mathbb{R}^N} K(x)|\phi|^p\,dx$, then $\phi \in \Sigma_\theta$. Let $v_0 = \tilde{c}w(x)$ with $\tilde{c} = (\frac{\theta}{m_\theta})^{\frac{1}{p-2}}$, we claim that v_0 is a minimizer for m_θ. Indeed, for any $u \in \Sigma_\beta$, the scaling $v = (\frac{\theta}{\beta})^{1/p}u \in \Sigma_\theta$, then $\mathcal{J}(v) = (\frac{\theta}{\beta})^{2/p}\mathcal{J}(u)$, it follows that

$$\frac{m_\beta}{\beta^{2/p}} = \frac{m_\theta}{\theta^{2/p}}, \quad \text{for any } \theta > 0 \text{ such that } \theta \neq \beta. \tag{5.9}$$

Note that, by (5.6) we know $w(x) = (\frac{m_\beta}{\beta})^{\frac{1}{p-2}}u_0$, then by (5.9) we have

$$v_0 = \tilde{c}w(x) = (\frac{\theta}{m_\theta})^{\frac{1}{p-2}}(\frac{m_\beta}{\beta})^{\frac{1}{p-2}}u_0 = (\frac{\theta}{\beta})^{1/p}u_0. \tag{5.10}$$

Since u_0 is the minimizer for m_β, it follows that $u_0 \in \Sigma_\beta$, and that that $v_0 \in \Sigma_\theta$. Moreover, using (5.9) again,

$$\mathcal{J}(v_0) = (\frac{\theta}{\beta})^{2/p}\mathcal{J}(u_0) = (\frac{\theta}{\beta})^{2/p}m_\beta = m_\theta,$$

thus v_0 is the minimizer for m_θ.

Since $w \in \mathcal{N}$, v_0, $\phi \in \Sigma_\theta$ and v_0 is the minimizer for m_θ, by (5.8) we have

$$\begin{aligned}
\mathcal{I}(w) &= (\frac{1}{2} - \frac{1}{p})\mathcal{J}(w) = (\frac{1}{2} - \frac{1}{p})\mathcal{J}(\tilde{c}^{-1}v_0) \\
&= (\frac{1}{2} - \frac{1}{p})\tilde{c}^{-2}\mathcal{J}(v_0) = (\frac{1}{2} - \frac{1}{p})(\frac{m_\theta}{\theta})^{\frac{2}{p-2}}\mathcal{J}(v_0) \\
&\leq (\frac{1}{2} - \frac{1}{p})(\frac{m_\theta}{\theta})^{\frac{2}{p-2}}\mathcal{J}(\phi) = (\frac{m_\theta}{\theta})^{\frac{2}{p-2}}\mathcal{I}(\phi),
\end{aligned}$$

hence to prove $\mathcal{I}(w) \leq \mathcal{I}(\phi)$, it is sufficient to show that $\frac{m_\theta}{\theta} \leq 1$. Since $\phi \in \mathcal{N} \cap \Sigma_\theta$, we obtain

$$\mathcal{J}(\phi) = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi|^2\,dx + \int_{\mathbb{R}^N} (V(x) - \lambda)|\phi|^2\,dx = \int_{\mathbb{R}^N} K(x)|\phi|^p\,dx = \theta.$$

Thus

$$m_\theta = \inf_{u \in \Sigma_\theta} \mathcal{J}(u) \leq \mathcal{J}(\phi) = \theta,$$

i.e., $\frac{m_\theta}{\theta} \leq 1$. Thus $w(x)$ is a ground state of (1.4). This completes the proof of Theorem 1.1.

6. Proof of Theorems 1.2 and 1.3

In this section we prove that weak solutions of (1.4) are of class $C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. First we give some properties of $L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ which will be used below.

Proposition 6.1. *The space $L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ has following properties.*

(i) *$L^r(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for any $1 \leq q \leq r \leq \infty$.*

(ii) $L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ *for any* $1 \le q \le r \le \infty$.

Proof. (i) Let $f(x) \in L^r(\mathbb{R}^N)$, for a given constant $M > 0$ we have $f(x) = f_0 + f_1$, where

$$f_0 = \chi_{\{x:|f(x)|>M\}} f(x), \quad f_1 = \chi_{\{x:|f(x)|\le M\}} f(x).$$

by the Chebyshev inequality [5]

$$|\{x : |f(x)| > M\}| \le \Big(\frac{C\|f\|_{L^r}}{M}\Big)^r < \infty.$$

Since $q \le r$, then $L^r(\{x : |f(x)| > M\}) \subset L^q(\{x : |f(x)| > M\})$, then $f_0 \in L^q(\mathbb{R}^N)$. It is obvious to see that $f_1 \in L^\infty(\mathbb{R}^N)$. Then $f(x) \in L^q(\mathbb{R}^N)+L^\infty(\mathbb{R}^N)$. Therefore, the case (i) holds.

The case (ii) is easy to obtain from case (i). □

Recall that the definition of fractional Sobolev spaces (e.g. see [6]) for $p \ge 1$ and $\beta > 0$:

$$\mathcal{L}^{\beta,p} = \{u \in L^p(\mathbb{R}^N) | \mathcal{F}^{-1}[(1 + |\xi|^2)^{\beta/2}\widehat{u}] \in L^p(\mathbb{R}^N)\},$$

and associated to the fractional Laplacian, the space

$$\mathcal{W}^{\beta,p} = \{\in L^p(\mathbb{R}^N) | \mathcal{F}^{-1}[(1 + |\xi|^\beta)\widehat{u}] \in L^p(\mathbb{R}^N)\}.$$

The following two theorems are basic results for these spaces which can be found in [6].

Theorem 6.2 ([6]). *Assume that* $p \ge 1$ *and* $\beta > 0$. *The following hold:*

(i) $\mathcal{L}^{\beta,p} = \mathcal{W}^{\beta,p}$, *and* $\mathcal{L}^{n,p} = W^{n,p}(\mathbb{R}^N)$ *for all* $n \in \mathbb{N}$, *where* $W^{n,p}$ *is the usual Sobolev space.*

(ii) *For* $\alpha \in (0, 1)$ *and* $2\alpha < \beta$, *we have* $(-\Delta)^\alpha : W^{\beta,p} \to W^{\beta-2\alpha,p}$.

(iii) *For* $\alpha, \gamma \in (0, 1)$ *and* $0 < \mu \le \gamma - 2\alpha$, *we have*

$$(-\Delta)^\alpha : C^{0,\gamma}(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N) \quad \text{if } 2\alpha < \gamma,$$

and, for $0 \le \mu \le 1 + \gamma - 2\alpha$,

$$(-\Delta)^\alpha : C^{1,\gamma}(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N) \quad \text{if } 2\alpha > \gamma.$$

Theorem 6.3 ([6]). (i) *If* $0 \le \alpha$, *and either* $1 < p \le q \le Np/(N - \alpha p) < \infty$ *or* $p = 1$ *and* $1 \le q < N/(N - \alpha)$, *then* $\mathcal{L}^{\alpha,p}$ *is continuously embedded in* $L^q(\mathbb{R}^N)$.

(ii) *Assume that* $0 \le \alpha \le 2$ *and* $\alpha > N/p$. *If* $\alpha - N/p > 1$ *and* $0 < \mu \le \alpha-N/p-1$, *then* $\mathcal{L}^{\alpha,p}$ *is continuously embedded in* $C^{1,\mu}(\mathbb{R}^N)$. *If* $\alpha-N/p < 1$ *and* $0 < \mu \le \alpha - N/p$, *then* $\mathcal{L}^{\alpha,p}$ *is continuously embedded in* $C^{0,\mu}(\mathbb{R}^N)$.

Let $\mathcal{H}(x, t)$ be defined in (3.1) (in the Appendix below), then we define the kernel $\mathcal{K}$, $\mathcal{K}^\mu$ with $\mu > 0$ as

$$\mathcal{K}(x) = \int_0^\infty e^{-t}\mathcal{H}(x, t)\, dt, \quad \mathcal{K}^\mu(x) = \int_0^\infty e^{-\mu t}\mathcal{H}(x, t)\, dt. \tag{6.1}$$

By the rescaling property of $\mathcal{H}(x, t)$,

$$\mathcal{H}(x, \frac{t}{\mu}) = \mu^{\frac{N}{2s}}\mathcal{H}(\mu^{\frac{1}{2s}}\, x, t),$$

we have

$$\mathcal{K}^\mu(x) = \mu^{\frac{N}{2s}-1}\mathcal{K}(\mu^{\frac{1}{2s}} x). \tag{6.2}$$

On the other hand, In the Appendix of [6], we know that $\mathcal{K}(x) = \mathcal{F}^{-1}\big(\frac{1}{1+|\xi|^{2s}}\big)$, then in the same way, we have

$$\mathcal{K}^{\mu}(x) = \mathcal{F}^{-1}\big(\frac{1}{\mu + |\xi|^{2s}}\big). \tag{6.3}$$

The following theorem can be found in [6].

Theorem 6.4 ([6])**.** *Let $N \geq 2$ and $s \in (0,1)$. Then we have the following:*

(i) *$\mathcal{K}$ is positive, radically symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$. Moreover, it is nonincreasing as a function of $r = |x|$.*

(ii) *For appropriate constants C_1 and C_2,*

$$\begin{aligned} \mathcal{K}(x) &\leq \frac{C_1}{|x|^{N+2s}} \quad \text{if } |x| \geq 1, \\ \mathcal{K}(x) &\leq \frac{C_2}{|x|^{N-2s}} \quad \text{if } |x| \leq 1. \end{aligned} \tag{6.4}$$

Corollary 6.5. *For $N \geq 2$ and $s \in (0,1)$, we have $\mu > 0$ and $\mathcal{K}^{\mu}$ satisfies Theorem 6.4 (i)-(ii).*

Since (6.2) holds, then it is easy to verify the above corollary.

Proof of Theorem 1.2. Since $V(x)$ is bound from above, then there exists a constant $M > 0$ such that $V(x) \leq M$. Note that $u(x) \in H^s(\mathbb{R}^N)$ is a nonnegative solution of (1.4) satisfying

$$(-\Delta)^s u(x) + V(x)u(x) - K(x)|u|^{p-2}u(x) = \lambda u(x),$$

then

$$(-\Delta)^s u(x) + (M - \lambda)u(x) = (M - V(x))u(x) + K(x)|u|^{p-2}u(x).$$

Let $\mu_0 = M - \lambda$, since $\lambda \leq 0$, we have $\mu_0 > 0$. Let $h(x) = (M - V(x))u(x) + K(x)|u|^{p-2}u(x)$, then we have

$$u(x) = \mathcal{K}^{\mu_0} * h(x).$$

Note that $u(x)$ is nonnegative and nontrivial, $V(x) \leq M$, $K(x) \neq 0$, we have $h(x) \geq 0$ such that $h(x) \neq 0$. By the corollary 6.5, we know that K^{μ_0} is positive, it follows that $u(x)$ is positive in $\mathbb{R}^N$. The proof is complete. □

To discuss the regularity of the weak solution (1.4), first we discuss the following result about liner equations.

Theorem 6.6. *Let $s \in (0,1)$, assume that $u \in H^s(\mathbb{R}^N)$, $N > 2s$ such that*

$$(-\Delta)^s u(x) + \mu u(x) = V(x)u(x) \quad \text{in } \mathbb{R}^N, \tag{6.5}$$

for $\mu > 0$, $V(x) \in L^q(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N)$ with $q > \frac{N}{2s}$. Then $u \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Moreover, $u(x) \to 0$ as $|x| \to \infty$.

Proof. First we know that $u \in H^s(\mathbb{R}^N) = \mathcal{W}^{s,2}$. Let $1 = r_0 > r_1 > r_2 > \cdots$, and consider $B_i = B(0, r_i)$, the ball of radius r_i and centered at the origin. We define $h(x) = V(x)u(x)$, since $V(x) \in L^q(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N)$, we have $V(x) = V_1 + V_2$ such that $V_1 \in L^q(\mathbb{R}^N)$ and $V_2 \in L^{\infty}(\mathbb{R}^N)$, then $h(x) = h_1 + h_2$ with $h_1 = V_1 u(x)$ and $h_2 = V_2 u(x)$. Since $u \in H^s(\mathbb{R}^N)$, by Sobolev inequality we have $u \in L^{2^*}(\mathbb{R}^N)$ with $2^* = 2N/(N-2s)$. Since $V_1 \in L^q(\mathbb{R}^N)$, by Hölder inequality, then we have $h_1 \in$

$L^{k_0}(\mathbb{R}^N)$ with $k_0 = (1/q + 1/2^*)^{-1}$. Therefore, $h(x) = h_1 + h_2$ with $h_1 \in L^{k_0}(\mathbb{R}^N)$ and $h_2 \in L^{2^*}(\mathbb{R}^N)$.

Now let $\eta_1 \in C^\infty$ with $0 \le \eta_1 \le 1$, with support in B_0 and such that $\eta_1 \equiv 1$ in $B_{1/2}$, where $B_{1/2} = B(0, r_{1/2})$ with $r_1 < r_{1/2} < r_0$. Let u_1 be the solution of the equation

$$(-\Delta)^s u_1 + \mu u_1 = \eta_1 h(x) \quad \text{in } \mathbb{R}^N, \tag{6.6}$$

then

$$(-\Delta)^s (u - u_1) + \mu(u - u_1) = (1 - \eta_1) h(x) \quad \text{in } \mathbb{R}^N, \tag{6.7}$$

so that

$$u - u_1 = \mathcal{K}^\mu * \{(1 - \eta_1) h\}. \tag{6.8}$$

Using the Hölder inequality and (6.4) we have

$$\begin{aligned} &|u(x) - u_1(x)| \\ &\le C\big\{\|\mathcal{K}^\mu\|_{L^{l_0}(B^c_{1/2})} \|(1 - \eta_1) h_1\|_{L^{k_0}} + \|\mathcal{K}^\mu\|_{L^{l_1}(B^c_{1/2})} \|(1 - \eta_1) h_2\|_{L^{2^*}}\big\}, \end{aligned} \tag{6.9}$$

for all $x \in B_1$, where $l_0 = k_0/(k_0 - 1)$, k_0 is given above, and $l_1 = 2^*/(2^* - 1)$. In view of this inequality we have to concentrate our attention in $u_1(x)$.

Since B_0 is bound and $V(x) \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, we obtain that $\eta_1 V(x) \in L^q(B_0)$. With the assumption $q > \frac{N}{2s}$, we have $\eta_1 V(x) \in L^{q_0}(B_0)$ for $\frac{N}{2s} < q_0 \le \min\{q, \frac{N}{s}\}$. Since $u \in L^{2^*}(\mathbb{R}^N)$, by Hölder inequality, we have $\eta_1 V(x) u \in L^{k_1}(\mathbb{R}^N)$, for $k_1 = (1/q_0 + 1/2^*)^{-1}$ such that $k_1 > 1$. Since η_1 has support in B_0, we have $\eta_1 V(x) u \in L^{p_1}(\mathbb{R}^N)$, for any $1 < p_1 < \min\{k_1, N/(2s)\}$. Note that u_1 satisfies (6.6), thus by the definition of the space $\mathcal{W}^{2s,p_1}$, we have $u_1 \in \mathcal{W}^{2s,p_1}$. Then, using Sobolev embedding of the Theorem 6.3 (i) and (6.9), we have $u \in L^{q_1}(B_1)$ for $q_1 = p_1 N/(N - 2s\, p_1)$.

Now we repeat the procedure, but consider a smooth function η_2 such that $0 \le \eta_2 \le 1$, with support in B_1 and $\eta_2 \equiv 1$ in $B_{3/2}$, where $B_{3/2} = B(0, r_{3/2})$ with $r_2 < r_{3/2} < r_1$. We also have $\eta_2 V(x) \in L^{q_0}(B_1)$ for any $\frac{N}{2s} < q_0 \le \min\{q, \frac{N}{s}\}$, we can set $\frac{1}{q_0} = \frac{2s}{N} - \epsilon$ with $0 < \epsilon \le \frac{s}{N}$. By Hölder inequality again, we have $\eta_2 V(x) u \in L^{p_2}(B_2)$ for any

$$1 \le p_2 < p_1/(1 - \epsilon) \quad \text{where } p_2 = (1/q_0 + 1/q_1)^{-1}.$$

Proceeding as above, with the obvious changes we obtain that

$$u_2 = \mathcal{K}^\mu * (\eta_2 h(x)),$$

satisfying $u_2 \in \mathcal{W}^{2s,p_2}$. Then we have $u \in L^{q_2}(B_2)$ for $q_2 = p_2 N/(N - 2s\, p_2)$.

Repeating the argument, for sequences η_j, p_j and $q_j = p_j N/(N - 2s\, p_j)$, we have $\eta_j V(x) u \in L^{p_j}(B_j)$ for any

$$1 \le p_j < p_{j-1}/(1 - \epsilon) \quad \text{where } p_j = (1/q_0 + 1/q_j)^{-1}.$$

It follows that for some finite j, $\eta_j V(x) u \in L^{p_j}(B_j)$ such that $p_j > N/(2s)$. Then by Theorem 6.3(ii), we have $u_j \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Since u_j satisfies the inequality that similar to (6.9), we have $u \in C^{0,\alpha}(B_{j+1})$.

The ball B_j is centered at the origin, but we may arbitrarily move it around $\mathbb{R}^N$. Covering $\mathbb{R}^N$ with these balls, we obtain that $u \in C^{0,\alpha}(\mathbb{R}^N)$. Finally, the fact that $u \in L^{2^*}(\mathbb{R}^N) \cap C^{0,\alpha}(\mathbb{R}^N)$ implies that $u(x) \to 0$ as $|x| \to \infty$, completing the proof. □

Proof of Theorem 1.3. Note that $u(x)$ satisfies

$$(-\Delta)^s u(x) + V(x)u(x) - K(x)|u|^{p-2}u(x) = \lambda u(x),$$

for $2 < p < 2^*$. Let $\widetilde{V}(x) = -V(x) + K(x)|u|^{p-2}$, then the equation becomes

$$(-\Delta)^s u(x) - \lambda u(x) = \widetilde{V}(x)u(x).$$

We claim that $\widetilde{V}(x) \in L^l(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $l > \frac{N}{2s}$.

Since the condition (A5) holds, $K(x) \in L^{\tilde{r}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for $\tilde{r} > \frac{2^*}{2^*-p}$, then $K(x) = K_1 + K_2$ with $K_1 \in L^{\tilde{r}}(\mathbb{R}^N)$ and $K_2 \in L^\infty(\mathbb{R}^N)$. Then Since $u \in L^{2^*}(\mathbb{R}^N)$, we have $K_2|u|^{p-2} \in L^{r_0}(\mathbb{R}^N)$ for $r_0 = \frac{2^*}{p-2} > \frac{2^*}{2^*-2} = \frac{N}{2s}$. By Hölder inequality, we have $K_1|u|^{p-2} \in L^{r_1}(\mathbb{R}^N)$ with $r_1 = (\frac{1}{\tilde{r}} + \frac{p-2}{2^*})^{-1}$ such that $r_1 > \frac{N}{2s}$. Then by Proposition 6.1 (i), we have $K(x)|u|^{p-2} \in L^{l_1}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ with $l_1 = \min\{r_0, r_1\}$ such that $l_1 > \frac{N}{2s}$. Then by Proposition 6.1 (ii), we have $\widetilde{V}(x) \in L^l(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ with $l = \min\{\tilde{q}, l_1\}$(where $\tilde{q}$ given in (A5)), such that $l > \frac{N}{2s}$. Then by Theorem 6.6, we obtain the regular result of Theorem 1.3. $\square$

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