Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 215, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

OSCILLATION CRITERIA FOR THIRD-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. This paper is a continuation of the recent study by Bohner et al [9] on oscillation properties of nonlinear third order functional differential equation under the assumption that the second order differential equation is nonoscillatory. We consider both the delayed and advanced case of the studied equation. The presented results correct and extend earlier ones. Several illustrative examples are included.

1. Introduction

In this article, we consider nonlinear third-order functional differential equations of the form

$$\Big(r_2\big(r_1(y')^\alpha\big)'\Big)'(t)+p(t)\big(y'(t)\big)^\alpha+q(t)f\big(y(g(t))\big)=0,\quad t\ge t_0, \tag{1.1}$$

where t_0 is fixed and $\alpha\ge 1$ is a quotient of odd positive integers. Throughout the whole paper, we assume that the following hypotheses hold:

(i) $r_1,r_2,q\in C(\mathcal{I},\mathbb{R}^+)$, where $\mathcal{I}=[t_0,\infty)$ and $\mathbb{R}^+=(0,\infty)$;
(ii) $p\in C(\mathcal{I},[0,\infty))$;
(iii) $g\in C^1(\mathcal{I},\mathbb{R})$, $g'(t)\ge 0$, $g(t)\to\infty$ as $t\to\infty$;
(iv) $f\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$ and $f(x)/x^\beta\ge k>0$ for $x\ne 0$, where k is a constant and $\beta\le\alpha$ is the ratio of odd positive integers.

By a solution of equation (1.1) we mean a function $y\in C([T_y,\infty))$, $T_y\in\mathcal{I}$, which has the property $r_1y', r_2(r_1(y')^\alpha)'\in C^1([T_y,\infty))$ and satisfies (1.1) on $[T_y,\infty)$. Our attention is restricted to those solutions y of (1.1) which exist on $\mathcal{I}$ and satisfy the condition

$$\sup\{|y(t)|:t_1\le t<\infty\}>0\quad\text{for all } t_1\ge t_0.$$

We make the standing hypothesis that (1.1) admits such a solution. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called *nonoscillatory*. Equation (1.1) is said to be *oscillatory* if all its solutions are *oscillatory*.

The study on asymptotic behavior of third-order differential equations was initiated in a pioneering paper of Birkhoff [7] which appeared in the early twentieth century. Since then, many authors contributed to the subject studying different

2010 *Mathematics Subject Classification.* 34C10, 34K11.
Key words and phrases. Oscillation; delay; advance; third order; damping; functional differential equation.

Submitted June 23, 2016. Published August 12, 2016.

classes of equations and applying various techniques. A summary of the most significant efforts on oscillation theory of third-order differential equations as well as an extensive bibliography can be found in the survey paper by Barrett [6] and monographs by Greguš [10], Swanson [13] and the recent one of Padhi and Pati [12].

The aim of this note is to complement the very recent study [9] on asymptotic and oscillatory properties of (1.1). The method and arguments used in the present paper are different than those used in [9]. We rely on the assumption that the related second-order ordinary differential equation

$$(r_2 v')'(t) + \frac{p(t)}{r_1(t)} v(t) = 0 \tag{1.2}$$

is nonoscillatory. We consider both the delay and advanced case of (1.1). While oscillation of all solutions is attained in the delay case, we state in the advanced case some new sufficient conditions for all solutions to either oscillate or converge to zero.

It is interesting to note how the asymptotic behavior of (1.1) changes when the middle term is inserted. As is customary, we choose a third-order Euler-type differential equation for demonstration.

Example 1.1. The equation

$$y'''(t) + \frac{1}{4t^2} y'(t) + \frac{1}{4t^3} y(t) = 0$$

admits oscillatory solutions and the nonoscillatory solution, where the roots of the characteristic equation are $\lambda_{1,2} = 1.5490 \pm 0.3925\mathrm{i}$ and $\lambda_3 = -0.097912$. But the corresponding equation without damping

$$y'''(t) + \frac{1}{4t^3} y(t) = 0$$

has only nonoscillatory solutions where the characteristic roots are $\lambda_1 = 1.2696$, $\lambda_2 = 1.8376$, $\lambda_3 = -0.10716$. Clearly, the middle term generates oscillation.

Because of the middle term $p(y')^\alpha$, the problem of convergence to zero as $t \to \infty$ and/or nonexistence of a nonoscillatory solution y with $yy' < 0$ seems to be especially crucial and challenging. We recall the related existing results.

Lemma 1.2 (See [4, Lemma 2.4])**.** *Assume that $\alpha = 1$. Let ρ_2 be a sufficiently smooth positive function and define*

$$\phi := (r_2 \rho_2')' r_1 + \rho_2 p.$$

Suppose that there exists $t_1 \in \mathcal{I}$ such that

$$\rho_2' \geq 0, \quad \phi \geq 0, \quad \phi' \leq 0 \quad \text{on} \quad [t_1, \infty),$$
$$\int_{t_1}^{\infty} (k\rho_2(s) q(s) - \phi'(s))\, \mathrm{d}s = \infty,$$

where $k\rho_2 q - \phi' \geq 0$ on $[t_1, \infty)$ and not identically zero on any subinterval of $[t_1, \infty)$. If (1.2) is nonoscillatory and y is a solution of (1.1) with $yL_1 y < 0$, then $\lim_{t\to\infty} y(t) = 0$.

However, since the proof of Lemma 1.2 is based on integration by parts, it cannot be generalized for $\alpha \neq 1$. The proposed method will take this problem into account. On the other hand, in [9], the authors offered a partial result for (1.1) in the sense that either (1.1) is oscillatory or $r_2(r_1(y')^\alpha)'$ is oscillatory (see [9, Theorem 3.1]). Oscillation of (1.1) has been left as an interesting open problem. So far, very little is known when $g(t) > t$. Some attempts in unifying results for both delay and advanced case have been made in [3]. We also extend these results by employing Riccati type transformation and comparison with oscillatory first-order advanced differential equations.

2. Preliminary lemmas and definitions

As in [9], we define

$$L_0 y = y, \quad L_1 y = r_1 (y')^\alpha, \quad L_2 y = r_2 (L_1 y)', \quad L_3 y = (L_2 y)'$$

on $\mathcal{I}$. With this notation, (1.1) can be rewritten as

$$L_3 y(t) + \frac{p(t)}{r_1(t)} L_1 y(t) + q(t) f(y(g(t))) = 0. \tag{2.1}$$

Following [9], we define the functions:

$$R_1(t,t_1) = \int_{t_1}^t \frac{\mathrm{d}s}{r_1^{1/\alpha}(s)}, \quad R_2(t,t_1) = \int_{t_1}^t \frac{\mathrm{d}s}{r_2(s)},$$
$$R^*(t,t_1) = \int_{t_1}^t \frac{R_2^{1/\alpha}(s,t_1)}{r_1^{1/\alpha}(s)}\,\mathrm{d}s,$$
$$R(g(t),t_1) := \begin{cases} \frac{R^*(g(t),t_1)}{R^*(t,t_1)} & \text{if } g(t) < t, \\ \frac{R_1(g(t),t_1)}{R_1(t,t_1)} & \text{if } g(t) \geq t, \end{cases}$$

for $t_0 \leq t_1 \leq t < \infty$. Note that the above definition of $R(g(t),t_1)$ will allow us to consider delayed and advanced type equations simultaneously in the proof of our main results.

Throughout and without further mentioning, it will be assumed that

$$R_i(t,t_0) \to \infty \quad \text{as } t \to \infty \text{ for } i = 1,2.$$

All the functional inequalities considered in the paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

Now, we provide several auxiliary results that are of importance in establishing our main results.

Lemma 2.1. *Let v be a solution of (1.2) which is positive on $[t_1,\infty)$. Then*

$$v' > 0 \tag{2.2}$$

and

$$\Big(\frac{v}{R_2(\cdot,t_1)}\Big)' \leq 0 \tag{2.3}$$

on $[t_1,\infty)$.

Proof. Let v be a solution of (1.2) with $v > 0$ on $[t_1,\infty)$. Then $(r_2 v')' < 0$ on $[t_1,\infty)$ so that $r_2 v'$ is decreasing on $[t_1,\infty)$. First assume $v'(t_2) < 0$ for some $t_2 \geq t_1$. Then $r_2(t)v'(t) \leq r_2(t_2)v'(t_2) =: c < 0$ for all $t \geq t_2$ and thus

$$v(t) = v(t_2) + \int_{t_2}^t v'(s)\,\mathrm{d}s \leq v(t_2) + c\int_{t_2}^t \frac{\mathrm{d}s}{r_2(s)}$$

$$= v(t_2) - c\int_{t_1}^{t_2} \frac{\mathrm{d}s}{r_2(s)} + cR_2(t,t_1) \to -\infty \quad \text{as } t \to \infty,$$

a contradiction. Thus (2.2) holds. Now let $t \geq t_1$. Then

$$v(t) \geq v(t) - v(t_1) = \int_{t_1}^{t} \frac{1}{r_2(s)} r_2(s)v'(s)\,\mathrm{d}s \geq r_2(t)v'(t)R_2(t,t_1)$$

and we see that

$$\Big(\frac{v}{R_2(\cdot,t_1)}\Big)'(t) = \frac{r_2(t)v'(t)R_2(t,t_1) - v(t)}{r_2(t)R_2^2(t,t_1)} \leq 0.$$

Hence $v/R_2(\cdot,t_1)$ is nonincreasing on $[t_1,\infty)$. □

Lemma 2.2 (See [5, Theorem 1.1]). *Assume that v is a positive solution of* (1.2) *on $\mathcal{I}$. Then*

$$\Big(r_2(r_1(y')^\alpha)'\Big)'(t) + p(t)(y'(t))^\alpha = \frac{1}{v(t)}\Big(r_2v^2(\frac{r_1}{v}(y')^\alpha)'\Big)'(t), \tag{2.4}$$

for $t \in \mathcal{I}$.

If (1.2) is nonoscillatory, the classical work of Hartmann [11] has termed a nontrivial solution v of (1.2) *a principal solution* (unique up to a constant multiple) such that

$$\int^\infty \frac{\mathrm{d}s}{r_2(s)v^2(s)} = \infty.$$

Since every eventually positive solution of (1.2) is increasing, the principal solution of (1.2) satisfies

$$\int_{t_0}^\infty \frac{\mathrm{d}s}{r_2(s)v^2(s)} = \infty, \quad \int_{t_0}^\infty \Big(\frac{v(s)}{r_1(s)}\Big)^{1/\alpha}\mathrm{d}s = \infty. \tag{2.5}$$

In the proofs of our theorems, an equivalent binomial form of (1.1) will be used repeatedly. This will also allow us to take correctly into account the possible case of L_2y being oscillatory that was missing in the previous results.

Lemma 2.3 (See [9, Lemma 2.2]). *Suppose that* (1.2) *is nonoscillatory. If y is a nonoscillatory solution of* (1.1) *on $[t_1,\infty)$, $t_1 \geq t_0$, then there exists $t_2 \geq t_1$ such that*

$$yL_1y > 0 \tag{2.6}$$

or

$$yL_1y < 0 \tag{2.7}$$

on $[t_2,\infty)$.

Lemma 2.4. *If y is a nonoscillatory solution of* (1.1) *with $y(t)L_1y(t) > 0$ for $t \geq t_1$, $t_1 \in \mathcal{I}$. Then*

$$yL_2y \geq 0, \quad yL_3y < 0$$

on $[t_1,\infty)$.

Proof. Let y be a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$, and $L_1y(t) > 0$ for all $t \geq t_1$. By (2.1), we see that $L_3y(t) < 0$ for all $t \geq t_1$ so L_2y is strictly decreasing on $[t_1,\infty)$. Now assume there exists $t_2 \geq t_1$ with $L_2y(t_2) < 0$. Then for $t \geq t_2$,

$$L_1y(t) = L_1y(t_2) + \int_{t_2}^{t}(L_1y)'(s)\,\mathrm{d}s = L_1y(t_2) + \int_{t_2}^{t}\frac{L_2y(s)}{r_2(s)}\,\mathrm{d}s$$

$$\le L_1y(t_2) + L_2y(t_2)R_2(t,t_2) \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction. $\square$

Lemma 2.5 (See [9, Lemma 2.3])**.** *Let y be a nonoscillatory solution of* (1.1) *with $y(t)L_1y(t) > 0$ for $t \ge t_1$, $t_1 \in \mathcal{I}$. Then*

$$L_1y(t) \ge R_2(t,t_1)L_2y(t), \quad t \ge t_1, \tag{2.8}$$

$$y(t) \ge R^*(t,t_1)L_2^{1/\alpha}y(t), \quad t \ge t_1. \tag{2.9}$$

Lemma 2.6. *Let y be a solution of* (1.1) *with $y(t)L_1y(t) > 0$ for $t \ge t_1$, $t_1 \in \mathcal{I}$. If*

$$\int_{t_1}^{\infty} \frac{1}{r_2(u)} \int_u^{\infty} (\frac{p(s)}{r_1(s)} + kq(s)R_1^{\beta}(g(s),t_1))\,\mathrm{d}s\,\mathrm{d}u = \infty, \tag{2.10}$$

then $\lim_{t\to\infty} L_1y(t) = \infty$.

Proof. Let y be a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$, and $L_1y(t) > 0$ for $t \ge t_1$. Then by Lemma 2.4, $L_2y \ge 0$ and L_1y is increasing, so $L_1y(t) \ge L_1y(t_1) =: \ell > 0$. Obviously,

$$y(g(t)) \ge \ell^{1/\alpha}R_1(g(t),t_1) \quad \text{for } t \ge t_1.$$

Setting both estimates into (1.1) and integrating from t to ∞, one gets

$$L_2y(t) \ge \ell \int_t^{\infty} \frac{p(s)}{r_1(s)}\,\mathrm{d}s + k\ell^{\beta/\alpha} \int_t^{\infty} q(s)R_1^{\beta}(g(s),t_1)\,\mathrm{d}s.$$

By integrating the last inequality from t_1 to ∞, we obtain (2.10). $\square$

Lemma 2.7. *Assume* (2.10) *holds. Let y be a solution of* (1.1) *with $y(t)L_1y(t) > 0$ for $t \ge t_1$, $t_1 \in \mathcal{I}$. Then there exists $t_2 > t_1$ such that*

$$y(g(t)) \ge R(g(t),t_1)y(t), \quad \text{for all } t \ge t_2. \tag{2.11}$$

Proof. Let y be a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$, and $L_1y(t) > 0$ for $t \ge t_1$.

We first prove (2.11) if $g(t) \le t$ holds for all $t \in \mathcal{I}$. From (2.8), we have

$$\Big(\frac{L_1y}{R_2(\cdot,t_1)}\Big)'(t) = \frac{L_2y(t)R_2(t,t_1) - L_1y(t)}{r_2(t)R_2^2(t,t_1)} \le 0.$$

Thus $\frac{L_1y}{R_2(\cdot,t_1)}$ is nonincreasing on $[t_1,\infty)$ and moreover, this fact yields

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^{t} \frac{R_2^{1/\alpha}(u,t_1)L_1^{1/\alpha}y(u)}{r_1^{1/\alpha}(u)R_2^{1/\alpha}(u,t_1)}\,\mathrm{d}u \\ &\ge \frac{L_1^{1/\alpha}y(t)}{R_2^{1/\alpha}(t,t_1)} \int_{t_1}^{t} \frac{R_2^{1/\alpha}(u,t_1)}{r_1^{1/\alpha}(u)}\,\mathrm{d}u = \frac{L_1^{1/\alpha}y(t)R^*(t,t_1)}{R_2^{1/\alpha}(t,t_1)} \end{aligned} \tag{2.12}$$

for $t \ge t_1$. Consequently,

$$\Big(\frac{y}{R^*(\cdot,t_1)}\Big)'(t) = \frac{L_1^{1/\alpha}y(t)R^*(t,t_1) - y(t)R_2^{1/\alpha}(t,t_1)}{r_1^{1/\alpha}(t)(R^*(t,t_1))^2} \le 0 \quad \text{for all } t \ge t_1,$$

which implies that $\frac{y}{R^*(\cdot,t_1)}$ is nonincreasing on $[t_1,\infty)$. Thus, if $g(t) \ge t_1$, then

$$y(g(t)) \ge \frac{R^*(g(t),t_1)}{R^*(t,t_1)}y(t) = R(g(t),t_1)y(t).$$

Now, we show that (2.11) holds in case of $g(t) \geq t$ for all $t \in \mathcal{I}$. Since $L_1^{1/\alpha} y$ is increasing on $[t_1, \infty)$, it is easy to see that, where $t_3 > t_2$,

$$\begin{aligned} y(t) &= y(t_3) + \int_{t_3}^{t} \frac{L_1^{1/\alpha} y(s)}{r_1^{1/\alpha}(s)} \, \mathrm{d}s \\ &\leq y(t_3) + L_1^{1/\alpha} y(t) R_1(t, t_3) \\ &= y(t_3) - L_1^{1/\alpha} y(t) R_1(t_3, t_1) + L_1^{1/\alpha} y(t) R_1(t, t_1), \end{aligned}$$

for all $t \geq t_3$. On the other hand, it follows from (2.10) that

$$\lim_{t \to \infty} L_1^{1/\alpha} y(t) = \infty.$$

Therefore, there exists $t_2 > t_3$ such that

$$y(t) \leq L_1^{1/\alpha} y(t) R_1(t, t_1) \tag{2.13}$$

on $[t_2, \infty)$. Now, one can see that

$$\Big(\frac{y}{R_1(\cdot, t_1)}\Big)'(t) = \frac{L_1^{1/\alpha} y(t) R_1(t, t_1) - y(t)}{r_1^{1/\alpha}(t) R_1^2(t, t_1)} \geq 0 \quad \text{for all } t \geq t_2,$$

so we conclude that $\frac{y}{R_1(\cdot, t_1)}$ is nondecreasing on $[t_2, \infty)$. Hence, if $g(t) \geq t_2$, then

$$y(g(t)) \geq \frac{R_1(g(t), t_1)}{R_1(t, t_1)} y(t) = R(g(t), t_1) y(t).$$

The proof is complete. □

Lemma 2.8. *Let y be a solution of* (1.1) *with $y(t) L_1 y(t) > 0$ for $t \geq t_1$, $t_1 \in \mathcal{I}$. Assume that*

$$\int_{t_1}^{\infty} \big(\frac{p(s)}{r_1(s)} R_2(s, t_1) + k q(s) (R^*(g(s), t_1))^{\beta}\big) \, \mathrm{d}s = \infty. \tag{2.14}$$

Then $\lim_{t \to \infty} y(t)/R^*(t, t_1) = 0$.

Proof. Let y be a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$, and $L_1 y(t) > 0$ for $t \geq t_1$. By l'Hospital's rule, it is easy to see that

$$\lim_{t \to \infty} \frac{y(t)}{R^*(t, t_1)} = \lim_{t \to \infty} L_2 y(t).$$

Assume to the contrary that $L_2 y(t) \geq \ell > 0$ for all $t \geq t_1$. Integrating (1.1) from t_1 to t and using (2.8) and (2.9), we find

$$\begin{aligned} L_2 y(t_1) &\geq \int_{t_1}^{t} \frac{p(s)}{r_1(s)} L_1 y(s) \, \mathrm{d}s + \int_{t_1}^{t} q(s) f(y(g(s))) \, \mathrm{d}s \\ &\geq \ell \int_{t_1}^{t} \frac{p(s)}{r_1(s)} R_2(s, t_1) \, \mathrm{d}s + k \ell^{\beta/\alpha} \int_{t_1}^{t} q(s) (R^*(g(s), t_1))^{\beta} \, \mathrm{d}s. \end{aligned}$$

Letting $t \to \infty$, one gets a contradiction with (2.14) and so $\ell = 0$. □

3. Main results

Now, we are prepared to present the main results of this paper.

Lemma 3.1. *Let* (1.2) *be nonoscillatory. If*

$$\int_{t_1}^{\infty} \frac{R_2^{1/\alpha}(x,t_1)}{r_1^{1/\alpha}(x)} \Big(\int_x^{\infty} \frac{\int_u^{\infty} q(s)\, \mathrm{d}s}{r_2(u) R_2(u,t_1)}\, \mathrm{d}u \Big)^{1/\alpha} \mathrm{d}x = \infty, \tag{3.1}$$

then any solution y of (1.1) *with $yL_1y < 0$ converges to zero as $t \to \infty$.*

Proof. Assume to the contrary that y is a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$, and $L_1y(t) < 0$ for $t \geq t_1$, $t_1 \in \mathcal{I}$ such that

$$\lim_{t\to\infty} y(t) = \ell > 0.$$

Using assumption (iv) on f and (2.4) in (1.1), we have

$$\Big(r_2 v^2 (\frac{r_1}{v} (y')^{\alpha})' \Big)'(t) + kq(t)v(t)y^{\beta}(g(t)) \leq 0. \tag{3.2}$$

Then by [5, Lemma 1.6], y satisfies

$$y' < 0, \quad \big(\frac{r_1}{v}(y')^{\alpha}\big)' > 0, \quad \Big(r_2 v^2 \big(\frac{r_1}{v}(y')^{\alpha}\big)' \Big)' < 0 \tag{3.3}$$

on $[t_1, \infty)$. Integrating (3.2) from t to ∞ and using $y(g(t)) \geq \ell$, we obtain

$$(\frac{r_1}{v}(y')^{\alpha})'(t) \geq \frac{k\ell^{\beta}}{r_2(t)v^2(t)} \int_t^{\infty} q(s)v(s)\, \mathrm{d}s. \tag{3.4}$$

Taking (2.2) into account, (3.4) becomes

$$\big(\frac{r_1}{v}(y')^{\alpha}\big)'(t) \geq \frac{\ell_1}{r_2(t)v(t)} \int_t^{\infty} q(s)\, \mathrm{d}s,$$

where $\ell_1 = k\ell^{\beta} > 0$. Integrating the last inequality from t to ∞ and using (2.3) from Lemma 2.1, we arrive at

$$\begin{aligned} -(y'(t))^{\alpha} &\geq \ell_1 \frac{v(t)}{r_1(t)} \int_t^{\infty} \frac{\int_u^{\infty} q(s)\, \mathrm{d}s}{r_2(u)v(u)}\, \mathrm{d}u \\ &\geq \ell_1 \frac{R_2(t,t_1)}{r_1(t)} \int_t^{\infty} \frac{\int_u^{\infty} q(s)\, \mathrm{d}s}{r_2(u)R_2(u,t_1)}\, \mathrm{d}u. \end{aligned}$$

Finally, by integrating the above inequality from t_1 to t, we have

$$y(t_1) \geq \ell_1^{1/\alpha} \int_{t_1}^{t} \frac{R_2^{1/\alpha}(x,t_1)}{r_1^{1/\alpha}(x)} \Big(\int_x^{\infty} \frac{\int_u^{\infty} q(s)\, \mathrm{d}s}{r_2(u)R_2(u,t_1)}\, \mathrm{d}u \Big)^{1/\alpha} \mathrm{d}x.$$

Letting $t \to \infty$, we obtain a contradiction with (3.1). Hence $\ell = 0$. The proof is complete. □

Theorem 3.2. *Suppose that* (1.2) *is nonoscillatory and that* (2.10) *and* (2.14) *hold. If there exists a constant $c > 0$ and a function $\rho \in C^1(\mathcal{I}, \mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \Big(k\rho(s)q(s)R^{\beta}(g(s),t_1) - \frac{A^2(s)}{4B(s)} \Big)\, \mathrm{d}s = \infty, \tag{3.5}$$

where, for $t \geq t_1$,

$$\begin{aligned} A(t) &= \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r_1(t)} R_2(t, t_1), \\ B(t) &= \beta c^{\beta-\alpha} \rho^{-1}(t) (R^*(t, t_1))^{\beta-1} \Big(\frac{R_2(t, t_1)}{r_1(t)}\Big)^{1/\alpha}, \end{aligned} \tag{3.6}$$

then any solution y of (1.1) *is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Let y be a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$, $t_1 \geq t_0$. From Lemma 2.3, it follows that $L_1 y < 0$ or $L_1 y > 0$ on $[t_1, \infty)$.

First, we assume $L_1 y > 0$. By Lemma 2.4, $L_2 y(t) \geq 0$ for $t \geq t_1$. Setting the estimate (2.11) into (2.1) and using the assumption (iv) on f, we obtain

$$L_3 y(t) + \frac{p(t)}{r_1(t)} L_1 y(t) + k R^\beta(g(t), t_1) q(t) y^\beta(t) \leq 0 \tag{3.7}$$

on $[t_2, \infty)$ for some $t_2 > t_1$. We define

$$\omega = \rho \frac{L_2 y}{y^\beta} > 0 \quad \text{on } [t_2, \infty). \tag{3.8}$$

Differentiating the function ω and using (3.7) and (2.8) in the resulting equation, we have

$$\omega'(t) \leq -k\rho(t) q(t) R^\beta(g(t), t_1) + A(t)\omega(t) - \beta \frac{y'(t)}{y(t)} \omega. \tag{3.9}$$

From the definition of $L_1 y$ and (2.8), we obtain

$$y'(t) = \Big(\frac{L_1 y(t)}{r_1(t)}\Big)^{1/\alpha} \geq \Big(\frac{R_2(t, t_1)}{r_1(t)}\Big)^{1/\alpha} L_2^{1/\alpha} y(t).$$

Thus

$$\begin{aligned} \frac{y'(t)}{y(t)} &\geq \Big(\frac{R_2(t, t_1)}{\rho(t) r_1(t)}\Big)^{1/\alpha} \frac{\rho^{1/\alpha}(t) L_2^{1/\alpha} y(t)}{y^{\beta/\alpha}(t)} y^{\beta/\alpha - 1}(t) \\ &= \Big(\frac{R_2(t, t_1)}{\rho(t) r_1(t)}\Big)^{1/\alpha} w^{1/\alpha}(t) y^{\beta/\alpha - 1}(t), \end{aligned}$$

and the inequality (3.9) becomes

$$\begin{aligned} \omega'(t) &\leq -k\rho(t) q(t) R^\beta(g(t), t_1) + A(t)\omega(t) \\ &\quad - \beta \omega^{1+1/\alpha}(t) y^{\beta/\alpha - 1}(t) \Big(\frac{R_2(t, t_1)}{\rho(t) r_1(t)}\Big)^{1/\alpha}. \end{aligned} \tag{3.10}$$

By Lemma 2.8, it follows from (2.14) that

$$0 < \frac{y(t)}{R^*(t, t_1)} \leq L_2 y(t_1) =: c \quad \text{for all } t \geq t_1.$$

Hence

$$y^{\beta/\alpha - 1}(t) \geq c^{\beta/\alpha - 1} (R^*(t, t_1))^{\beta/\alpha - 1}. \tag{3.11}$$

From the definition of ω and (2.9), we obtain

$$\omega(t) = \rho(t) \frac{L_2 y(t)}{y^\beta(t)} \leq \rho(t) (R^*(t, t_1))^{-\alpha} y^{\alpha - \beta}(t), \quad t \geq t_2.$$

Using (3.11) in the above inequality, we have

$$\omega(t) \leq c^{\alpha-\beta}\rho(t)(R^*(t,t_1))^{-\beta},$$

and since $\alpha \geq 1$,

$$w^{1/\alpha-1}(t) \geq c^{(\alpha-\beta)(1/\alpha-1)}\rho^{1/\alpha-1}(t)(R^*(t,t_1))^{-\beta(1/\alpha-1)}. \tag{3.12}$$

Using (3.11) and (3.12) in (3.10), we have

$$\begin{aligned}
\omega'(t) &\leq -k\rho(t)q(t)R^\beta(g(t),t_1) + A(t)\omega(t) \\
&\quad - \beta c^{\beta-\alpha}\rho^{-1}(t)(R^*(t,t_2))^{\beta-1}\Big(\frac{R_2(t,t_1)}{r_1(t)}\Big)^{1/\alpha} w^2(t) \\
&= -k\rho(t)q(t)R^\beta(g(t),t_1) + A(t)\omega(t) - B(t)\omega^2(t) \\
&= -k\rho(t)q(t)R^\beta(g(t),t_1) - \Big(\sqrt{B(t)}\omega(t) - \frac{A(t)}{2\sqrt{B(t)}}\Big)^2 + \frac{A^2(t)}{4B(t)} \\
&\leq -k\rho(t)q(t)R^\beta(g(t),t_1) + \frac{A^2(t)}{4B(t)}
\end{aligned} \tag{3.13}$$

for all $t \geq t_2$, where A and B are as in (3.6). Integrating the inequality (3.13) from t_2 to t, we find

$$\int_{t_2}^t \Big(k\rho(s)q(s)R^\beta(g(s),t_1) - \frac{A^2(s)}{4B(s)}\Big)\,\mathrm{d}s \leq \omega(t_2) - \omega(t) \leq \omega(t_2),$$

which contradicts condition (3.5).

Assume $L_1 y < 0$. By Lemma 3.1, condition (4.1) ensures that any solution of (1.1) tends to zero as $t \to \infty$. The proof is complete. ☐

For $t \geq t_1 \geq t_0$, we let

$$P(t) = \frac{1}{r_2(t)}\int_t^\infty \frac{p(s)}{r_1(s)}\,\mathrm{d}s, \quad Q_1(t) = \frac{(R^*(g(t),t_1))^\beta}{r_2(t)R_2^{\beta/\alpha}(g(t),t_1)}\int_t^\infty kq(s)\,\mathrm{d}s,$$
$$\mu(t) = \exp\Big(-\int_{t_1}^t P(s)\,\mathrm{d}s\Big).$$

Now, we present the following comparison result for the advanced case, which complements [9, Theorem 3.5].

Theorem 3.3. *Assume that $g(t) \geq t$ holds for all $t \in \mathcal{I}$. Let all the hypotheses of Theorem 3.2 hold, except* (3.5)*. If every solution of the first-order advanced equation*

$$z'(t) - (\mu(g(t)))^{1-\beta/\alpha}Q_1(t)z^{\beta/\alpha}(g(t)) = 0 \tag{3.14}$$

is oscillatory, then any solution y of (1.1) *is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Let y be a nonoscillatory solution of (1.1) on $[t_1,\infty)$, $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From Lemma 2.3, it follows that $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$.

First, we assume $L_1y > 0$. Then by Lemma 2.4, $L_2y > 0$ on $[t_1, \infty)$. Integrating (1.1) from t to ∞ and using the assumption (iv), we obtain

$$\begin{aligned} L_2y(t) &\ge \int_t^\infty \frac{p(s)}{r_1(s)} L_1y(s)\,\mathrm{d}s + \int_t^\infty kq(s)y^\beta(g(s))\,\mathrm{d}s \\ &\ge L_1y(t)\int_t^\infty \frac{p(s)}{r_1(s)}\,\mathrm{d}s + y^\beta(g(t))\int_t^\infty kq(s)\,\mathrm{d}s \end{aligned} \tag{3.15}$$

for $t \ge t_1$. If $g(t) \ge t_1$, we have from (2.12) that

$$y(g(t)) \ge \frac{R^*(g(t), t_1)}{R_2^{1/\alpha}(g(t), t_1)} L_1^{1/\alpha} y(g(t)). \tag{3.16}$$

Setting (3.16) into (3.15), we obtain

$$L_2y(t) \ge L_1y(t)\int_t^\infty \frac{p(s)}{r_1(s)}\,\mathrm{d}s + L_1^{\beta/\alpha}y(g(t))\frac{(R^*(g(t), t_1))^\beta}{R_2^{\beta/\alpha}(g(t), t_1)}\int_t^\infty kq(s)\,\mathrm{d}s,$$

which can be written as

$$w'(t) - P(t)w(t) - Q_1(t)w(g(t)) \ge 0,$$

where $w(t) = r_2(t)L_1y(t)$. Setting $z(t) = \mu(t)w(t) > 0$ in the above inequality and noting that $\mu(t) \ge \mu(g(t))$, we obtain

$$z'(t) - (\mu(g(t)))^{1-\beta/\alpha}Q_1(t)z^{\beta/\alpha}(g(t)) \ge 0.$$

By [2, Lemma 2.2.10], the corresponding differential equation (3.14) also possesses an eventually positive solution, which is a contradiction.

Assume $L_1y < 0$. By Lemma 3.1, condition (4.1) ensures that any solution tends to zero as $t \to \infty$. The proof is complete. □

The following corollary is immediate.

Corollary 3.4. *Assume that $g(t) \ge t$ and $\alpha = \beta$. Let all the hypotheses of Theorem 3.2 hold, except* (3.5)*. If*

$$\liminf_{t\to\infty} \int_t^{g(t)} Q_1(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{3.17}$$

then any solution y of (1.1) *is either oscillatory or converges to zero as $t \to \infty$.*

4. Oscillation of (1.1)

For delay equations, we are able to ensure nonexistence of possible nonoscillatory solutions y with $yL_1y < 0$.

Theorem 4.1. *Assume that $g(t) < t$ for all $t \in \mathcal{I}$. Let the hypotheses of Theorem 3.2 hold. If, moreover, there exists $c_* > 0$ such that*

$$\limsup_{t\to\infty} \int_{g(t)}^t \frac{R_2^{1/\alpha}(s, t_1)}{r_1^{1/\alpha}(s)} \Big(\int_s^t \frac{\int_u^t q(x)\,\mathrm{d}x}{r_2(u)R_2(u, t_1)}\,\mathrm{d}u \Big)^{1/\alpha}\,\mathrm{d}s = c_*, \tag{4.1}$$

then (1.1) *is oscillatory.*

Proof. Assume to the contrary that y is a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$ and $L_1y(t) < 0$ for $t \ge t_1$, $t_1 \in \mathcal{I}$ with $\lim_{t\to\infty} y(t) = 0$. As in the proof of Lemma 3.1, we obtain that y is a solution of the inequality (3.2) satisfying (3.3) on $[t_1, \infty)$. Since $\alpha \ge \beta$, there exists $t_2 \ge t_1$ such that

$$y^{\beta-\alpha}(g(t)) \ge c^{\beta-\alpha} \tag{4.2}$$

for all $t \ge t_2$ and every $c > 0$. Using (4.2) in (3.2), we obtain

$$\Big(r_2 v^2 (\frac{r_1}{v}(y')^\alpha)'\Big)'(t) + kc^{\beta-\alpha} q(t)v(t)y^\alpha(g(t)) \le 0. \tag{4.3}$$

Integrating (4.3) twice from s to t, $t > s$, one obtains

$$-y'(s) \ge kc^{\beta-\alpha}\Big(\frac{v(s)}{r_1(s)}\Big)^{1/\alpha}\Big(\int_s^t \frac{\int_u^t q(x)v(x)y^\beta(g(x))\,\mathrm{d}x}{r_2(u)v^2(u)}\,\mathrm{d}u\Big)^{1/\alpha}. \tag{4.4}$$

Using the property (2.3) of v, (4.4) becomes

$$-y'(s) \ge kc^{\beta-\alpha}\Big(\frac{R_2(s,t_1)}{r_1(s)}\Big)^{1/\alpha}\Big(\int_s^t \frac{\int_u^t q(x)y^\alpha(g(x))\,\mathrm{d}x}{r_2(u)R_2(u,t_1)}\,\mathrm{d}u\Big)^{1/\alpha}.$$

Integrating the above inequality from $g(t)$ to t, we obtain

$$y(g(t)) \ge kc^{\beta-\alpha}y(g(t))\int_{g(t)}^t \frac{R_2^{1/\alpha}(s,t_1)}{r_1^{1/\alpha}(s)}\Big(\int_s^t \frac{\int_u^t q(x)\,\mathrm{d}x}{r_2(u)R_2(u,t_1)}\,\mathrm{d}u\Big)^{1/\alpha}\,\mathrm{d}s,$$

which is a contradiction with (4.1). The proof is complete. $\square$

We propose one condition in which the function $p(t)$ is directly included.

Theorem 4.2. *Assume that $g(t) < t$ for all $t \in \mathcal{I}$. Let the hypotheses of Theorem 3.2 hold. If, moreover, there exists a constant $c_* > 0$ such that*

$$\limsup_{t\to\infty}\Big\{\int_{g(t)}^t \frac{1}{r_1^{1/\alpha}(s)}\Big(\int_s^t \frac{1}{r_2(v)}\int_v^t Q(u)\,\mathrm{d}u\,\mathrm{d}v\Big)^{1/\alpha}\,\mathrm{d}s\Big\} > 1, \tag{4.5}$$

where

$$Q(t) = kc_*^{\beta-\alpha}q(t) - \frac{p(t)R_2(t,t_1)}{r_1(t)(R^*(t,g(t)))^\alpha} > 0 \quad \text{for all } t \ge t_1,$$

then (1.1) *is oscillatory.*

Proof. Assume to the contrary that y is a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(g(t)) > 0$ and $L_1y(t) < 0$ for $t \ge t_1$, $t_1 \in \mathcal{I}$ with $\lim_{t\to\infty} y(t) = 0$. We consider $L_2y(t)$. The case $L_2y(t) \le 0$ cannot holds for all large t, say $t \ge t_2 \ge t_1$, since by integrating this inequality, we see

$$y'(t) = \Big(\frac{L_1y(t_2)}{r_1(t)}\Big)^{1/\alpha} \le \Big(\frac{L_1y(t_2)}{r_1(t)}\Big)^{1/\alpha} \quad \text{for all } t \ge t_2, \tag{4.6}$$

which contradicts the positivity of $y(t)$. Therefore, either $L_2y(t) > 0$ or $L_2y(t)$ changes sign on $[t_2, \infty)$. We claim that $Q(t) > 0$ implies $L_2y(t) > 0$ on $[t_2, \infty)$.

Similarly to the proof of Lemma 3.1, we obtain that y is a positive solution of (3.2) satisfying (3.3) on $[t_1,\infty)$. Now, for $x\ge u\ge t_1$, we obtain

$$\begin{aligned} y(u)-y(x) &= -\int_u^x \Big(\frac{v(s)}{r_1(s)}\Big)^{1/\alpha}\Big(\frac{r_1(s)}{v(s)}(y'(s))^\alpha\Big)^{1/\alpha}\,\mathrm{d}s \\ &\ge -y'(x)\Big(\frac{r_1(x)}{v(x)}\Big)^{1/\alpha}\int_u^x\Big(\frac{v(s)}{r_1(s)}\Big)^{1/\alpha}\,\mathrm{d}s \\ &\ge -\frac{L_1^{1/\alpha}y(x)}{R_2^{1/\alpha}(x,t_1)}\int_u^x\Big(\frac{R_2(s,t_1)}{r_1(s)}\Big)^{1/\alpha}\,\mathrm{d}s \\ &= -\frac{L_1^{1/\alpha}y(x)R^*(x,u)}{R_2^{1/\alpha}(x,t_1)}. \end{aligned} \tag{4.7}$$

Using (4.7) with $u=g(t)$, $x=t$ and $-L_1y(t)>0$, we obtain

$$y(g(t))\ge\frac{R^*(t,g(t))}{R_2^{1/\alpha}(t,t_1)}(-L_1^{1/\alpha}y(t)),\quad \text{for } t\ge t_1,$$

e.g.,

$$L_1y(t)\ge-\frac{R_2(t,t_1)}{(R^*(t,g(t)))^\alpha}y^\alpha(g(t)).$$

Using this inequality in (2.1), we obtain

$$-L_3y(t)\ge\Big(kq(t)y^{\beta-\alpha}(g(t))-\frac{p(t)R_2(t,t_1)}{r_1(t)(R^*(t,g(t)))^\alpha}\Big)y^\alpha(g(t)). \tag{4.8}$$

In view of (3.1) and the fact that $\alpha\ge\beta$, there exists $t_2\ge t_1$ such that

$$y^{\beta-\alpha}(g(t))\ge c^{\beta-\alpha} \tag{4.9}$$

for every $c>0$ and for all $t\ge t_2$. Thus we have

$$\begin{aligned} -L_3y(t) &\ge \Big(kc^{\beta-\alpha}q(t)-\frac{p(t)R_2(t,t_1)}{r_1(t)(R^*(t,g(t)))^\alpha}\Big)y^\alpha(g(t)) \\ &= Q(t)y^\alpha(g(t))>0. \end{aligned} \tag{4.10}$$

Hence $L_3y<0$ and similarly as in the proof of Lemma 2.4, we see that $L_2y\ge0$ on $[t_2,\infty)$. Integrating (4.10) from s to t, $t>s$, we obtain

$$L_2y(s)\ge\int_s^t Q(u)y^\alpha(g(u))\,\mathrm{d}u.$$

Integrating again from s to t, we obtain

$$-L_1^{1/\alpha}y(s)\ge\Big(\int_s^t\frac{1}{r_2(v)}\int_v^t Q(u)y^\alpha(g(u))\,\mathrm{d}u\,\mathrm{d}v\Big)^{1/\alpha}.$$

Finally, integrating the above inequality from $g(t)$ to t, we arrive at

$$y(g(t))\ge y(g(t))\int_{g(t)}^t\frac{1}{r_1^{1/\alpha}(s)}\Big(\int_s^t\frac{1}{r_2(v)}\int_v^t Q(u)\,\mathrm{d}u\,\mathrm{d}v\Big)^{1/\alpha}\,\mathrm{d}s,$$

which in view of (4.5) results in contradiction. The proof is complete. □

The following corollary is immediate.

Corollary 4.3. *Assume that $g(t)<t$ for all $t\in\mathcal{I}$. Let the hypotheses of Theorem 3.2 hold. If, moreover, there exists a constant $c_*>0$ such that* (4.1) *or* (4.5) *holds, then* (1.1) *is oscillatory.*

Remark 4.4. Estimate (4.5) slightly differs from the one used in [9] but it correctly takes into account a class of nonoscillatory solutions with yL_2y oscillatory.

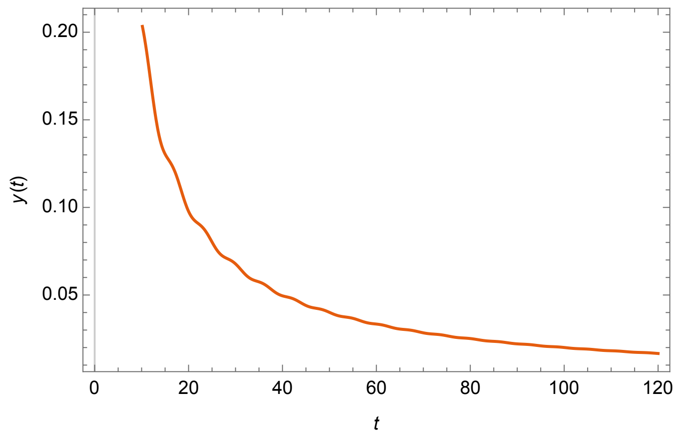

FIGURE 1. $y(t) = \frac{2}{t} - \frac{\sin(t)}{t^2}$

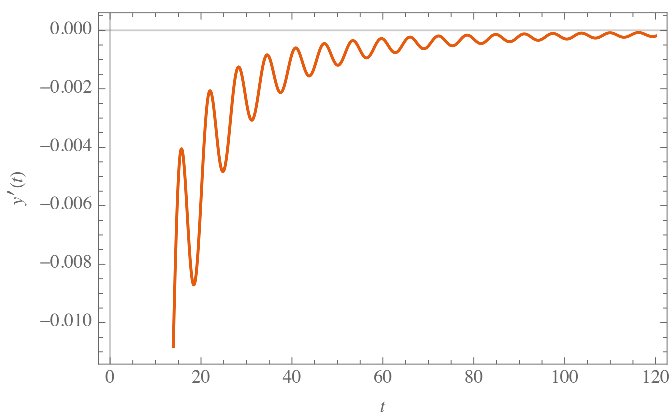

FIGURE 2. $y'(t) = \frac{2\sin(t)}{t^3} - \frac{2}{t^2} - \frac{\cos(t)}{t^2}$

5. Examples

We give a couple of examples to illustrate our main results.

Example 5.1. Consider the equation of Euler type

$$y'''(t) + \frac{a}{t^2}y'(t) + \frac{b}{t^3}y(\lambda t) = 0, \quad t \geq 1,\ \lambda > 0,\ a \leq 1/4, \tag{5.1}$$

where a, b are some positive constants. Setting $k = 1$ and $\rho(t) = t^2$, we can conclude from Theorem 3.2 that any solution y of (5.1) is oscillatory or converges to zero as $t \to \infty$ for

$$b > \frac{(2-a)^2}{4\lambda^2} \text{ for } \lambda \in (0,1); \quad b > \frac{(2-a)^2}{4\lambda} \text{ for } \lambda \geq 1.$$

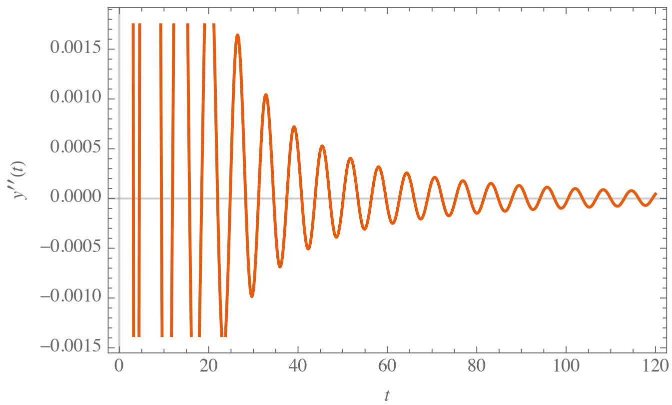

FIGURE 3. $y''(t) = -\frac{6\sin(t)}{t^4} + \frac{4}{t^3} + \frac{4\cos(t)}{t^3} + \frac{\sin(t)}{t^2}$

If we take $\lambda \in (0,1)$ and, moreover,

$$b(\lambda^2(1-\ln\lambda) - \ln\lambda - 1) > 4$$

or

$$\frac{b(1-\lambda^2)-a}{(1-\lambda^2)}\Big(\lambda - \frac{\ln\lambda}{2} - \frac{\lambda^2}{4} - \frac{3}{4}\Big) > 1,$$

then it follows from Corollary 4.3 that (5.1) is oscillatory. We note that none of the results in [1, 3, 4, 8, 9, 14] can guarantee oscillation of (1.1).

Example 5.2. We consider the equation

$$\Big(t^{1/4}(y'(t))^{1/3}\Big)'' + \frac{3}{16t^{7/4}}(y'(t))^{1/3} + \frac{a}{t^{25/12}}y^{1/3}(\lambda t) = 0, \tag{5.2}$$

for $t \ge 1$, $\lambda > 0$. In [5], the authors deduced that (5.2) is oscillatory for $\lambda = 0.4$ provided that $a > 16.1197$. The same conclusion follows from Corollary 4.3 for $a > 8.1263$, which is a significantly better result. We also stress that in contrast to [5], we do not require any information about the auxiliary solution v of (1.2). On the other hand, if we set $\lambda > 1$ say $\lambda = 2$, then, from Theorem 3.2, any solution of (5.2) is either oscillatory or converges to zero as $t \to \infty$ for $a > 0.2589$.

6. General Remarks

The results of this note complement those obtained in a recent paper [9] and can be applied to both delayed and advanced third-order differential equations with damping. As is well known, it is only the delay in (1.1) that can generate oscillation of all solutions.

The class of positive solutions with L_2y oscillatory has been eliminated under the essential assumption that (1.2) is nonoscillatory. It appears that the case when (1.2) is oscillatory is still open. For instance, the equation

$$y'''(t) + y'(t) + \frac{2(t^3 + 2t^2\sin(t) + 6t - 12\sin(t) + 9t\cos(t))}{t^3(2t - \sin(t))}y(t) = 0 \tag{6.1}$$

admits a nonoscillatory solution y satisfying (2.7) with L_2y oscillatory, as depicted on Figures 1–3. Eliminating such a case seems to be the major challenge.

It might be also interesting to extend results of this paper to higher-order differential equations of the form

$$\Big(r_2\Big(r_1\big(y^{(n-2)}\big)^{\alpha}\Big)'\Big)'(t) + p(t)\Big(y^{(n-2)}(t)\Big)^{\alpha} + q(t)f(y(g(t))) = 0$$

for n odd. This would be left to further research.

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