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NONLINEAR FREDHOLM ALTERNATIVE FOR THE p-LAPLACIAN UNDER NONHOMOGENEOUS NEUMANN BOUNDARY CONDITION

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Abstract. The nonlinear Fredholm alternative for the p-Laplacian in higher dimensions is established when nonhomogeneous terms appear in the equation and in the Neumann boundary condition. Further, the geometry of the associated energy functional is described and compared with the Dirichlet counterpart. The proofs require only variational methods.

1. Introduction

The nonlinear Fredholm alternative for the p-Laplacian under Dirichlet boundary condition has been of interest to several authors, see for instance [2, 5, 6, 7, 8, 9, 10, 12, 13, 15]. Given a bounded domain with smooth boundary $\Omega \subset \mathbb{R}^N$, $N \geq 1$, it consists of finding sufficient (and possibly necessary) conditions on $f(x)$ for the following problem to have a solution:

$$\begin{gathered} -\Delta_p u = \lambda_1 |u|^{p-2} u + f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of the p-Laplacian in $W_0^{1,p}(\Omega)$. In the case $p = 2$ it is known from the theory of linear equations that the condition

$$\int_\Omega f \varphi_1 \, dx = 0, \tag{1.2}$$

where $\varphi_1 > 0$ is the normalized principal eigenfunction corresponding to λ_1, is necessary and sufficient for the solvalility of (1.1). For $p \neq 2$, the previous condition is not necessary for the solvability of problem (1.1) as showed in [2] through an example in the case $N = 1$. Still in the one dimensional case a characterization of how should be $f(x)$ for (1.1) to have a solution is given in [5]. Characterizations on $f(x)$ in higher dimensional domains were given in [12, 13] and [10] using variational and topological methods, bifurcation theory or combinations of them.

In this article we are interested in the Neumann boundary condition counterpart. Actually, we establish the nonlinear Fredholm alternative for the p-Laplacian with

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nonhomogeneous terms appearing in the equation and in the Neumann boundary condition. More precisely, we consider the problem

$$\begin{gathered} -\Delta_p u = \mu_1 |u|^{p-2}u + f(x) \quad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where ν is the outward normal vector to the boundary $\partial\Omega$ of a smooth domain $\Omega \subset \mathbb{R}^N$, with $N \geq 2$. The number $\mu_1 = 0$ is the first eigenvalue of the p-Laplacian operator under zero Neumann boundary condition. We obtain a necessary and sufficient condition on $f(x)$ and $g(x)$ so that (1.3) can be solved, characterizing the solution set. Further, we describe the geometry of the energy functional associated with (1.3) and compare with the geometry of the functional in the Dirichlet case.

In fact, contrary to the Dirichlet boundary condition case the analogous condition of (1.1) for problem (1.3), namely,

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\mathcal{H}^{N-1} = 0 \tag{1.4}$$

(where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Haursdorff measure) besides being necessary suffices for the solvability of (1.3). To state our first result let $p > 1$ and $p^\star$, $p_\star$ be the critical Sobolev exponents for the embeddings $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, respectively. Let also $p^{\star'}$, $p'_\star$ be the corresponding conjugate exponents; that is, $1/p^\star + 1/p^{\star'} = 1$ and $1/p_\star + 1/p'_\star = 1$. We prove the following result.

Theorem 1.1. *For $(f,g) \in L^{p^{\star'}}(\Omega) \times L^{p'_\star}(\partial\Omega)$ problem (1.3) has a solution if and only if condition (1.4) holds. In this case the solution set of (1.3) is*

$$\big\{u \in W^{1,p}(\Omega) : u = \mathfrak{u} + c,\ c \in \mathbb{R}\big\} \tag{1.5}$$

where $\mathfrak{u} \in W^{1,p}(\Omega)$ is a uniquely determined function.

Theorem 1.1 establishes the nonlinear Fredholm alternative for the Neumann problem (1.3) in higher dimensions, providing a characterization of the solution set. In dimension $N = 1$ it was considered in [6, 14, 15], see also the references therein.

The proof of Theorem 1.1 requires only variational methods and it is performed as follows. For $p > 1$ the energy functional associated with (1.3) is $\mathcal{J}_p : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$, given by

$$\mathcal{J}_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega fu\,dx - \int_{\partial\Omega} gu\,d\mathcal{H}^{N-1}. \tag{1.6}$$

If (1.4) holds then it is clear that $\mathcal{J}_p$ is not coercive on $W^{1,p}(\Omega)$. Restricting $\mathcal{J}_p$ to a subspace of $W^{1,p}(\Omega)$ of codimension one induced by (1.4) then $\mathcal{J}_p$ turns out to be coercive and strictly convex. Thus $\mathcal{J}_p$ has a global minimizer in that subspace, which is proved to be a critical point over $W^{1,p}(\Omega)$ using the Lagrange multiplier theorem and will help us to precisely describe the solution set of (1.3).

Another question of interest is understanding the geometries of the energy functionals corresponding to (1.1) and (1.3). The associated energy functional for the Dirichlet problem (1.1) is $\mathcal{E}_p : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, $p > 1$, defined by

$$\mathcal{E}_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda_1}{p}\int_\Omega |u|^p\,dx - \int_\Omega fu\,dx. \tag{1.7}$$

When $p \neq 2$, so that (1.1) is driven by a nonlinear operator, the geometry of $\mathcal{E}_p$ changes strongly according to $p \in (1,2)$ or $p \in (2,\infty)$. Actually, it was showed in [8] that for $p \in (1,2)$, $\mathcal{E}_p$ is unbounded above and below and has a saddle point geometry. For $p \in (1,2)$ it follows that $\mathcal{E}_p$ is bounded below and has the global minimizer geometry. Further, for $p \neq 2$ the set of critical points of $\mathcal{E}_p$ is a priori bounded, see [8, 9].

Concerning the Neumann problem (1.3) and the Dirichlet problem (1.1) from the viewpoint of the geometry of its energy functionals, the conclusion one can draw is $\mathcal{J}_p$ behaves like $\mathcal{E}_2$, for all $p > 1$. Indeed, the strategy used for the proof of Theorem 1.1 helps to infer that $\mathcal{J}_p$ and $\mathcal{E}_2$ have the global minimizer geometry for all $p > 1$ and also have unbounded sets of critical points. Thus from such a perspective nonlinear problem (1.3) behaves like the linear one (1.1) (for $p = 2$). That is the content of the following theorem.

Theorem 1.2. *The energy functional $\mathcal{J}_p$ for the nonlinear Neumann problem* (1.3) *and the energy functional $\mathcal{E}_2$ for the linear Dirichlet problem* (1.1) *have the global minimizer geometry for all $p > 1$. Further, their sets of critical points are unbounded.*

The rest of this article is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3, after proving a necessary lemma to apply the ideas used in the proof of Theorem 1.1, we prove Theorem 1.2.

2. Proof of Theorem 1.1

For $p > 1$ the critical Sobolev exponents $p^\star$, $p_\star$ for the embeddings $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, respectively, are defined by (see [1])

$$p^\star := \begin{cases} \frac{pN}{N-p}, & \text{for } 1 < p < N \\ \infty, & \text{for } p > N \\ \text{arbitrary } q \in (1,\infty), & \text{for } p = N \end{cases}$$

and

$$p_\star := \begin{cases} \frac{p(N-1)}{N-p}, & \text{for } 1 < p < N \\ \infty, & \text{for } p > N \\ \text{arbitrary } q \in (1,\infty) & \text{for } p = N. \end{cases}$$

Given $(f,g) \in L^{p^{\star\prime}}(\Omega) \times L^{p_\star'}(\partial\Omega)$, a function $u \in W^{1,p}(\Omega)$ is a (weak) solution of (1.3) when

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = \int_\Omega f\phi \, dx + \int_{\partial\Omega} g\phi \, d\mathcal{H}^{N-1} \tag{2.1}$$

for all $\phi \in W^{1,p}(\Omega)$, that is, if and only if $u \in W^{1,p}(\Omega)$ is a critical point of $\mathcal{J}_p$. We want to prove that problem (1.3) has a solution if and only if (1.4) holds and, in this case, the solution set of (1.3) is given by (1.5)

Proof of Theorem 1.1. As a matter of fact, taking $\phi = 1$ in (2.1) it is easy to see that condition (1.4) is necessary for the solvability of (1.3).

Now assume (1.4) holds and consider the closed subspace of $W^{1,p}(\Omega)$,

$$\mathscr{M} \doteq \big\{u \in W^{1,p}(\Omega) : \int_\Omega u \, dx = 0\big\}.$$

From Poincaré-Wirtinger inequality, see [3], the norm $\|u\| := (\int_\Omega |\nabla u|^p\,dx)^{\frac{1}{p}}$ is equivalent to the usual norm $\|\cdot\|_{W^{1,p}}$ of $W^{1,p}(\Omega)$ in $\mathscr{M}$.

Let $\varphi \in \mathscr{M}^\star$ given by

$$\langle \varphi, u\rangle := \int_\Omega fu\,dx + \int_{\partial\Omega} gu\,d\mathcal{H}^{N-1}.$$

Using Hölder inequality and the embeddings $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{p_\star}(\partial\Omega)$ one can show the linear functional φ is really continuous on $\mathscr{M}$.

Define the functional $\mathcal{J}_p : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ by

$$\mathcal{J}_p(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \langle\varphi, u\rangle, \tag{2.2}$$

which is of class C^1 in $W^{1,p}(\Omega)$. We split the rest of the proof into 5 steps.

Step 1: $\mathcal{J}_p\big|_{\mathscr{M}}$ is coercive and strictly convex on $\mathscr{M}$. Indeed, since $\|\cdot\|$ is equivalent to $\|\cdot\|_{W^{1,p}}$ in $\mathscr{M}$, from the embeddings above and Hölder inequality we have

$$\begin{aligned}|\mathcal{J}_p\big|_{\mathscr{M}}(u)| &\geq \frac{1}{p}\|u\|^p - \|f\|_{p^{\star\prime}}\|u\|_{p^\star} - \|g\|_{p_\star'}\|u\|_{p_\star}\\ &\geq \|u\|\Big[\frac{1}{p}\|u\|^{p-1} - const.(\|f\|_{p^{\star\prime}} + \|g\|_{p_\star'})\Big] \to \infty \quad \text{as } \|u\| \to \infty\,,\end{aligned}$$

proving that $\mathcal{J}_p\big|_{\mathscr{M}}$ is coercive on $\mathscr{M}$. The strict convexity of $\mathcal{J}_p\big|_{\mathscr{M}}$ can be deduced since the function $x \mapsto |x|^p$, $x \in \mathbb{R}^N$, is strictly convex and φ is linear.

Step 2: $\mathcal{J}_p\big|_{\mathscr{M}}$ has a global minimizer $\bar u \in \mathscr{M}$. Note that from the expression in (2.2), which is the difference between a norm and a bounded linear functional, we obtain $\mathcal{J}_p\big|_{\mathscr{M}}$ is weakly lower semicontinuous. Last information and coercivity imply $\mathcal{J}_p\big|_{\mathscr{M}}$ has a global minimizer $\bar u \in \mathscr{M}$, i.e.,

$$\mathcal{J}_p\big|_{\mathscr{M}}(\bar u) = \inf_{u\in\mathscr{M}} \mathcal{J}_p\big|_{\mathscr{M}}(u). \tag{2.3}$$

Indeed, by coercivity one gets $\rho > 0$ such that $\mathcal{J}_p\big|_{\mathscr{M}}(u) \geq \mathcal{J}_p\big|_{\mathscr{M}}(0)$ for all $u \in (B_\rho^{\mathscr{M}}(0))^c$, where $B_\rho^{\mathscr{M}}(0) \doteq \{u \in \mathscr{M} : \|u\| < \rho\}$. If $\mathcal{J}_p\big|_{\mathscr{M}}$ were unbounded from below in $B_\rho^{\mathscr{M}}(0)$ one could obtain $(u_k) \subset B_\rho^{\mathscr{M}}(0)$ verifying $\mathcal{J}_p\big|_{\mathscr{M}}(u_k) \to -\infty$, as $k \to \infty$. The reflexivity of $W^{1,p}(\Omega)$, $p > 1$, allows one to use Banach-Alaoglu theorem (see [3]) and pass to a subsequence (u_{k_j}) satisfying $u_{k_j} \rightharpoonup \tilde u$ (weakly) for some $\tilde u \in \mathscr{M}$, and then

$$\mathcal{J}_p\big|_{\mathscr{M}}(\tilde u) \leq \liminf_{j\to\infty} \mathcal{J}_p\big|_{\mathscr{M}}(u_{k_j}) = -\infty$$

what is impossible. Hence $\mathcal{J}_p\big|_{\mathscr{M}}$ is bounded from below in a such way that the infimum in (2.3) is finite and can be attained through a minimizing sequence by coercivity and weak lower semicontinuity.

Step 3: $\bar u$ is the unique global minimizer and is the only critical point of $\mathcal{J}_p\big|_{\mathscr{M}}$. Stricty convexity assures uniqueness of the global minimizer $\bar u \in \mathscr{M}$ in (2.3). In fact, if $\bar u_1 \neq \bar u_2$ were two global minimizers in (2.3) one would have

$$\inf_{u\in\mathscr{M}} \mathcal{J}_p\big|_{\mathscr{M}}(u) \leq \mathcal{J}_p\big|_{\mathscr{M}}\Big(\frac{1}{2}(\bar u_1 + \bar u_2)\Big) < \frac{1}{2}\mathcal{J}_p\big|_{\mathscr{M}}(\bar u_1) + \frac{1}{2}\mathcal{J}_p\big|_{\mathscr{M}}(\bar u_2) = \inf_{u\in\mathscr{M}} \mathcal{J}_p\big|_{\mathscr{M}}(u),$$

a contradiction. Now let $\zeta \in \mathscr{M}$ be a critical point of $\mathcal{J}_p\big|_{\mathscr{M}}$. Given $w \in \mathscr{M}$, define $\sigma(t) := \mathcal{J}_p\big|_{\mathscr{M}}(\zeta + tw)$ for $t \in \mathbb{R}$. It is not difficult to infer that σ is differentiable, strictly convex and satisfies $\sigma'(0) = 0$. Thus from the fact that σ' is strictly increasing one can deduce $\sigma'(t) \neq 0$ for $t \neq 0$; that is, $\langle \mathcal{J}_p\big|'_{\mathscr{M}}(\zeta + tw), w\rangle \neq 0$.

Hence $\mathcal{J}_p\big|'_{\mathscr{M}}(\zeta + tw) \neq 0$ for $t \neq 0$ and since $w \in \mathscr{M}$ is arbitrary $\mathcal{J}_p\big|_{\mathscr{M}}$ has no other critical point than ζ. It follows from step 2 that $\mathcal{J}_p\big|_{\mathscr{M}}$ has $\bar{u}$ as its unique critical point.

Step 4: $\bar{u}$ is a weak solution to (1.3). Let $F(u) := \int_\Omega u\,dx$, for $u \in W^{1,p}(\Omega)$. Thanks to (2.3) the Lagrange multiplier theorem (see [11]) yields $\mu \in \mathbb{R}$ verifying $\mathcal{J}_p'(\bar{u}) = \mu F'(\bar{u})$; that is,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\,dx - \int_\Omega f\phi\,dx - \int_{\partial\Omega} g\phi\,d\mathcal{H}^{N-1} = \mu \int_\Omega \phi\,dx$$

for all $\phi \in W^{1,p}(\Omega)$. Using $\phi \equiv 1$ as a test function in previous relation one obtains

$$\mu = -\frac{1}{|\Omega|}\Big[\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\mathcal{H}^{N-1}\Big]$$

and by (1.4) it follows that $\mu = 0$. Hence (2.1) holds, and $\bar{u} \in W^{1,p}(\Omega)$ is a weak solution to (1.3).

Step 5: The set (1.5) is the solution set of (1.3). Actually, define $\mathfrak{u} := \bar{u}$. It is clear that $\mathfrak{u} + c$ solves (1.3) for any constant $c \in \mathbb{R}$. Conversely, given a solution u of (1.3) one has $u - (\frac{1}{|\Omega|}\int_\Omega u\,dx) \in \mathscr{M}$ satisfies (2.1) and then is a critical point of $\mathcal{J}_p\big|_{\mathscr{M}}$. The uniqueness from step 3 implies $u = \mathfrak{u} + c$, with $c = \frac{1}{|\Omega|}\int_\Omega u\,dx$. The proof is complete. □

3. Proof of Theorem 1.2

Recall that the first and second eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ are

$$\lambda_1 = \inf_{u \in H_0^1(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx} \tag{3.1}$$

and

$$\lambda_2 = \inf_{u \in \mathscr{O},\, u \neq 0} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx}, \tag{3.2}$$

respectively, where

$$\mathscr{O} := \big\{u \in H_0^1(\Omega) \,:\, \int_\Omega u\varphi_1\,dx = 0\big\} \tag{3.3}$$

and $\varphi_1 > 0$ is the normalized eigenfunction associated with λ_1. Also one has $\lambda_2 > \lambda_1 > 0$ (see [4]).

Lemma 3.1. *In the Hilbert space $\mathscr{O}$ given by* (3.3) *the expression*

$$\|u\|_{\mathscr{O}} := \Big(\int_\Omega |\nabla u|^2\,dx - \lambda_1 \int_\Omega u^2\,dx\Big)^{1/2}$$

defines a norm equivalent to the usual norm in $H_0^1(\Omega)$.

Proof. Note that $\|\cdot\|_{\mathscr{O}}$ is induced by the inner product, in $\mathscr{O}$,

$$(u, v)_{\mathscr{O}} := \int_\Omega \nabla u \cdot \nabla v\,dx - \lambda_1 \int_\Omega uv\,dx.$$

Indeed, linearity and symmetry of $(\cdot,\cdot)_{\mathscr{O}}$ are trivial. For $u \in \mathscr{O}$, with $u \not\equiv 0$, by (3.1) and (3.2) one gets

$$(u,u)_{\mathscr{O}} > \int_\Omega |\nabla u|^2\,dx - \lambda_2 \int_\Omega u^2\,dx > 0;$$

that is, $(u,u)_{\mathscr{O}} = 0$ if and only if $u = 0$. Hence, $(\cdot,\cdot)_{\mathscr{O}}$ is an inner product and induces the norm $\|\cdot\|_{\mathscr{O}}$. Finally, the equivalence between the norms $\|\cdot\|_{\mathscr{O}}$ and the usual norm $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ follows from (3.1) and (3.2) since

$$\begin{aligned}
\|u\|^2_{H_0^1(\Omega)} \geq \|u\|^2_{\mathscr{O}} &= \int_\Omega |\nabla u|^2\,dx - \frac{\lambda_1}{\lambda_2}\Big[\lambda_2 \int_\Omega u^2\,dx\Big] \\
&\geq \int_\Omega |\nabla u|^2\,dx - \frac{\lambda_1}{\lambda_2}\Big[\int_\Omega |\nabla u|^2\,dx\Big] \\
&= [1 - \frac{\lambda_1}{\lambda_2}]\|u\|^2_{H_0^1(\Omega)},
\end{aligned}$$

where $1 - \frac{\lambda_1}{\lambda_2} > 0$. The proof is complete. □

Proof of Theorem 1.2. The proof will be given in two steps.

Step 1: $\mathscr{J}_p$ has the global minimizer geometry for all $p > 1$. Indeed, from Theorem 1.1 all critical points of $\mathscr{J}_p$ belong to the set

$$\big\{u \in W^{1,p}(\Omega) : u = \mathfrak{u} + c,\ c \in \mathbb{R}\big\}$$

of solutions to (1.3). Thus under condition (1.4), and thanks to $\mathfrak{u}$ being a global minimizer of $\mathscr{J}_p\big|_{\mathscr{M}}$, one obtains that for all $c, d \in \mathbb{R}$ and $v \in \mathscr{M}$,

$$\mathscr{J}_p(\mathfrak{u} + c) = \mathscr{J}_p(\mathfrak{u}) = \mathscr{J}_p\big|_{\mathscr{M}}(\mathfrak{u}) \leq \mathscr{J}_p\big|_{\mathscr{M}}(v) = \mathscr{J}_p(v + d).$$

Since $W^{1,p}(\Omega) = \mathbb{R} \oplus \mathscr{M}$ for $p > 1$ and v, c, d are arbitrary, we conclude that

$$\mathscr{J}_p(\mathfrak{u} + c) \leq \mathscr{J}_p(u)$$

for all $u \in W^{1,p}(\Omega)$; that is, all critical points of $\mathscr{J}_p$ are global minimizers. Thus $\mathscr{J}_p$ has the global minimizer geometry for all $p > 1$.

Step 2: $\mathcal{E}_2$ has the global minimizer geometry. First note that the set $\mathscr{O}$ given by (3.3) is a closed subspace of $H_0^1(\Omega)$ of codimension one. When restricted to $\mathscr{O}$, the functional $\mathcal{E}_2$ given by (1.7) can be expressed, using previous lemma, as

$$\mathcal{E}_2\big|_{\mathscr{O}}(u) = \|u\|_{\mathscr{O}} - \int_\Omega fu\,dx$$

for all $u \in \mathscr{O}$. That is, $\mathcal{E}_2\big|_{\mathscr{O}}$ is the sum of a norm with a continuous linear functional and thus all arguments used in steps 1 to 5 of the proof of Theorem 1.1 apply.

Then, like $\mathscr{J}_p\big|_{\mathscr{M}}$ one has $\mathcal{E}_2\big|_{\mathscr{O}}$ coercive and strictly convex, having a global minimizer $\tilde{\mathfrak{u}}$ on $\mathscr{O}$ which is a critical point of $\mathcal{E}_2$ in $H_0^1(\Omega)$. Also, the unbounded set

$$\big\{t\varphi_1 + \tilde{\mathfrak{u}} : t \in \mathbb{R}\big\}$$

is the set of critical points of $\mathcal{E}_2$. A similar procedure as in step 1 allows one to infer that all those critical points are global minimizers of $\mathcal{E}_2$; that is, $\mathcal{E}_2$ has the global minimizer geometry. The proof is complete. □

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