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SELFADJOINT SINGULAR DIFFERENTIAL OPERATORS OF FIRST ORDER AND THEIR SPECTRUM

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Dedicated to Prof. E. S. Panakhov on his 60-th birthday

Abstract. Based on Calkin-Gorbachuk method, we describe all selfadjoint extensions of the minimal operator generated by linear multipoint singular symmetric differential-operator, as a direct sum of weighted Hilbert space of vector-functions. Another approach to the investigation of this problem has been done by Everitt, Zettl and Markus. Also we study the structure of spectrum of these extensions.

1. Introduction

The first works devoted to the general theory of selfadjoint extensions of symmetric operators having equal deficiency indexes in any Hilbert space belong to von Neumann [12]. Generalization of this theory to normal extensions has been done Coddington [3]. Applications of this theory to two-point differential operators in Hilbert spaces can be found in works of many mathematicians; see for example [4, 6, 7, 13].

It is known that for the existence of selfadjoint extension of any linear closed densely defined symmetric T in a Hilbert space $\mathcal{H}$ the necessary and sufficient condition is a equality of deficiency indexes $m(T) = n(T)$, where $m(T) = \dim\ker(T^* + iE)$, $n(T) = \dim\ker(T^* - iE)$.

In the multipoint cases it may be faced with different views. Let T_1 and T_2 be minimal operators generated by the linear singular differential expression

$$l(u) = itu'(t)$$

in the weighted Hilbert $L^2_\alpha(-\infty, a)$ and $L^2_\alpha(b, \infty)$, $\alpha(t) = t$, $a, b \in \mathbb{R}$ of functions, respectively. In this case it is clear that

$$(m(T_1), n(T_1)) = (0, 1), \quad (m(T_2), n(T_2)) = (1, 0)$$

Consequently, the operators T_1 and T_2 are maximal symmetric. Hence they are not any selfadjoint extensions. However, direct sum $T = T_1 \oplus T_2$ of operators T_1 and T_2 in the direct sum $\mathcal{H} = L^2_\alpha(-\infty, a) \oplus L^2_\alpha(b, \infty)$ have an equal defect numbers (1,1). Then by the general theory it has at least one selfadjoint extension [12].

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Indeed it can be easily shown that the multipoint differential expression

$$l(u) = (itu_1'(t), itu_2'(t)), \quad u = (u_1, u_2), \quad u_1 \in D(T_1^*), \quad u_2 \in D(T_2^*)$$

with boundary condition $u_2(b) = u_1(a)$ generates selfadjoint extension of $T_1 \oplus T_2$.

There are different methods for the description all selfadjoint extensions of minimal symmetric operators. For example, in mathematical literature the Eweritt-Zettl-Markus and Calkin-Gorbachuk methods are very important of them.

The models of many physical and technical phenomena are expressed as differential operators. Therefore operator theory plays an exceptionally important role in modern mathematics, especially in the modelling of processes of multi-particle quantum mechanics, quantum field theory, the multipoint boundary value problems for differential equations [1, 15].

Although the first studies of the theory multipoint differential operators were performed at the beginning of twentieth century, most of them which are about the investigation of the theory and application to spectral problems, have been found since 1950 [5, 8, 9, 10, 15, 16].

In the article by Everitt and Zettl [5] in the scalar case, all selfadjoint extensions of the minimal operator generated by Lagrange-symmetric any order quasi-differential expression with equal deficiency indexes in terms of boundary conditions are described by Glazman-Krein-Naimark method for regular and singular cases in the direct sum of corresponding Hilbert spaces of functions.

2. Statement of the problem

Let H be a separable Hilbert space and let $a, b \in \mathbb{R}$. In the Hilbert space $L^2_{1/\alpha}(H, (-\infty, a)) \oplus L^2_{1/\beta}(H, (b, +\infty))$ of a vector-functions we consider the following linear multipoint differential-operator expression for first order in a form

$$l(w) = (k(u), m(v)), \tag{2.1}$$

where $w = (u, v)$, $k(u) = i\alpha(t)u'(t) + Au(t)$, $m(v) = i\beta(t)v'(t) + Bv(t)$, $\alpha \in C(-\infty, a)$, $\beta \in C(b, +\infty)$ and there exist positive numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\alpha_1 \le \alpha(t) \le \alpha_2$ for any $t < a$, $\beta_1 \le \beta(t) \le \beta_2$ for any $t > b$ and for simplicity assumed that A and B are linear bounded selfadjoint operators in H.

In similar way in [7] the minimal $K_0(M_0)$ and maximal $K(M)$ operators associated with differential expression $k(m)$ in $L^2_{1/\alpha}(H, (-\infty, a))(L^2_{1/\beta}(H, (b, +\infty)))$ can be constructed.

The operators $L_0 = K_0 \oplus M_0$ and $L = K \oplus M$ in the Hilbert space $\mathcal{H} = L^2_{1/\alpha}(H, (-\infty, a)) \oplus L^2_{1/\beta}(H, (b, +\infty))$ are called minimal and maximal operators associated with differential expression (2.1), respectively. It is clear that operator L_0 is a symmetric and $L_0^* = L$ in $\mathcal{H}$. The minimal operator L_0 is not maximal. Indeed, differential expression (2.1) with boundary condition $u(a) = v(b)$ generates a selfadjoint extension of L_0.

The main goal in this article is to describe all selfadjoint extensions of minimal operator L_0 in $\mathcal{H}$ in terms of boundary values (Sec. 3). In section 4 the structure of spectrum of these extensions will be investigated.

3. Description of Selfadjoint Extensions

In this section using by Calkin-Gorbachuk method will be investigated the general representation of selfadjoint extensions of minimal operator L_0. Firstly, prove the following proposition.

Lemma 3.1. *The deficiency indexes of the operators K_0 and M_0 are of the form*

$$(m(K_0), n(K_0)) = (\dim H, 0), \quad (m(M_0), n(M_0)) = (0, \dim H).$$

Proof. From the conditions under the weight functions $\alpha(\cdot)$ and $\beta(\cdot)$ imply that

$$\frac{1}{\alpha_2} \le \frac{1}{\alpha(t)} \le \frac{1}{\alpha_1},\ t < a \quad \text{and} \quad \frac{1}{\beta_2} \le \frac{1}{\beta(t)} \le \frac{1}{\beta_1},\ t > b\,.$$

From these relations we have

$$\int_{-\infty}^{a} \frac{dt}{\alpha(t)} = \int_{b}^{\infty} \frac{dt}{\beta(t)} = +\infty$$

On the other hand, it is clear that the general solutions of differential equations

$$i\alpha(t)u_{\pm}' \pm iu_{\pm} = 0,\ t < a \quad \text{and} \quad i\beta(t)v_{\pm}' \pm iv_{\pm} = 0,\ t > b$$

in the $L^2_{1/\alpha}(H, (-\infty, a))$ and $L^2_{1/\beta}(H, (b, +\infty))$ are

$$u_{\pm}(t) = \exp\Big(\mp \int_{-\infty}^{t} \frac{ds}{\alpha(s)}\Big) f, \quad f \in H,\ t < a$$

and

$$v_{\pm}(t) = \exp\Big(\pm \int_{t}^{\infty} \frac{ds}{\beta(s)}\Big) g, \quad g \in H,\ t > b$$

respectively From these representations we have

$$\begin{aligned}
\|u_+\|^2_{L^2_{1/\alpha}(H,(-\infty,a))} &= \int_{-\infty}^{a} \frac{1}{\alpha(t)} \|u_+(t)\|^2_H dt \\
&= \int_{-\infty}^{a} \frac{1}{\alpha(t)} \exp\Big(-2\int_{-\infty}^{t} \frac{ds}{\alpha(s)}\Big) dt \|f\|^2_H \\
&= \int_{-\infty}^{a} \exp\Big(-2\int_{-\infty}^{t} \frac{ds}{\alpha(s)}\Big) d\Big(\int_{-\infty}^{t} \frac{ds}{\alpha(s)}\Big) \|f\|^2_H \\
&= \frac{1}{2}\|f\|^2_H < \infty
\end{aligned}$$

Consequently,

$$\dim\ker(K + iE) = \dim H$$

On the other hand it is clear that for any $f \in H$ the solutions

$$u_-(t) = \exp\Big(\int_{-\infty}^{t} \frac{ds}{\alpha(s)}\Big) f \notin L^2_{1/\alpha}(H, (-\infty, a)).$$

Hence $\dim\ker(K - iE) = 0$.

In a similar way we can show that

$$m(M_0) = \dim\ker(M + iE) = 0 \quad \text{and} \quad n(M_0) = \dim\ker(M - iE) = \dim H$$

This completes the proof of theorem. $\square$

From last Lemma 3.1 we have

$$m(L_0) = m(K_0) + m(M_0) = \dim H \quad \text{and} \quad n(L_0) = n(K_0) + n(M_0) = \dim H$$

Consequently, $(m(L_0), n(L_0)) = (\dim H, \dim H)$. Consequently, the minimal operator L_0 has a selfadjoint extensions; see [12].

Definition 3.2 ([7])**.** Let $\mathbf{H}$ be any Hilbert space and $S : D(S) \subset \mathbf{H} \to \mathbf{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathbf{H}$ having equal finite or infinite deficiency indexes. A triplet $(\mathcal{B}, \gamma_1, \gamma_2)$, where $\mathcal{B}$ is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathcal{B}$, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$(S^* f, g)_{\mathbf{H}} - (f, S^* g)_{\mathbf{H}} = (\gamma_1(f), \gamma_2(g))_{\mathcal{B}} - (\gamma_2(f), \gamma_1(g))_{\mathcal{B}}$$

while for any $F_1, F_2 \in \mathcal{B}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

It is known that for any symmetric operator with equal deficiency indexes have at least one space of boundary values [7].

Lemma 3.3. *If $u \in D(K)$ and $v \in D(M)$, then $\lim_{t\to-\infty} u(t) = 0$, $\lim_{t\to+\infty} v(t) = 0$ and $u(a), v(b) \in H$.*

Proof. Under the assumptions for $\alpha(\cdot)$ and $\beta(\cdot)$ we have

$$\frac{1}{\alpha_2} \le \frac{1}{\alpha(t)} \le \frac{1}{\alpha_1},\ t < a,$$
$$\frac{1}{\beta_2} \le \frac{1}{\beta(t)} \le \frac{1}{\beta_1},\ t > b\,.$$

Then for any pair of functions $x(\cdot) \in L^2(H, (-\infty, a))$, $y(\cdot) \in L^2(H, (b, +\infty))$, from the above relations,

$$0 \le \int_{-\infty}^{a} \frac{1}{\alpha(t)} \|x(t)\|_H^2 dt \le \frac{1}{\alpha_1} \int_{-\infty}^{a} \|x(t)\|_H^2 dt,$$
$$0 \le \int_{b}^{\infty} \frac{1}{\beta(t)} \|y(t)\|_H^2 dt \le \frac{1}{\beta_1} \int_{b}^{\infty} \|y(t)\|_H^2 dt$$

Consequently,

$$L^2(H, (-\infty, a)) \subset L^2_{1/\alpha}(H, (-\infty, a)) \quad \text{and} \quad L^2(H, (b, +\infty)) \subset L^2_{1/\beta}(H, (b, +\infty))\,.$$

Similarly, if $x(\cdot) \in L^2_{1/\alpha}(H, (-\infty, a))$ and $y(\cdot) \in L^2_{1/\beta}(H, (b, +\infty))$, then from the conditions on the weight functions we have

$$\frac{1}{\alpha_2} \int_{-\infty}^{a} \|x(t)\|_H^2 dt \le \int_{-\infty}^{a} \frac{1}{\alpha(t)} \|x(t)\|_H^2 dt,$$
$$\frac{1}{\beta_2} \int_{b}^{\infty} \|y(t)\|_H^2 dt \le \int_{b}^{\infty} \frac{1}{\beta(t)} \|y(t)\|_H^2 dt$$

Hence,

$$L^2_{1/\alpha}(H, (-\infty, a)) \subset L^2(H, (-\infty, a)) \quad \text{and} \quad L^2_{1/\beta}(H, (b, +\infty)) \subset L^2(H, (b, +\infty)).$$

Therefore it is obtained that

$$L^2_{1/\alpha}(H, (-\infty, a)) = L^2(H, (-\infty, a)) \quad \text{and} \quad L^2_{1/\beta}(H, (b, +\infty)) = L^2(H, (b, +\infty)).$$

On the other hand, from the relations

$$D(K) \subset L^2(H,(-\infty,a)) \quad \text{and} \quad D(M) \subset L^2(H,(b,+\infty)),$$

we have $\lim_{t\to-\infty} u(t)=0$ and $\lim_{t\to+\infty} v(t)=0$ for any $u\in D(K)$ and $v\in D(M)$. Now we assume that $u\in D(K)$. Then

$$u\in L^2_{1/\alpha}(H,(-\infty,a)) \quad \text{and} \quad \alpha u' \in L^2_{1/\alpha}(H,(-\infty,a))$$

This implies

$$\frac{1}{\sqrt{\alpha}}u\in L^2(H,(-\infty,a)) \quad \text{and} \quad \sqrt{\alpha}u'\in L^2(H,(-\infty,a))$$

Then for any $t<a$ it holds

$$(u,u)'_H(t)=(u',u)_H(t)+(u,u')_H(t)=\Big(\sqrt{\alpha}u',\frac{1}{\sqrt{\alpha}}u\Big)_H(t)+\Big(\frac{1}{\sqrt{\alpha}}u,\sqrt{\alpha}u'\Big)_H(t)$$

and from this,

$$\begin{aligned}
&\big|\|u(a)\|_H^2-\|u(-\infty)\|_H^2\big|\\
&\le \int_{-\infty}^a \Big|\Big(\sqrt{\alpha}u',\frac{1}{\sqrt{\alpha}}u\Big)_H(t)\Big|dt+\int_{-\infty}^a \Big|\Big(\frac{1}{\sqrt{\alpha}}u,\sqrt{\alpha}u'\Big)_H(t)\Big|dt\\
&\le 2\Big(\int_{-\infty}^a \|\sqrt{\alpha}u'\|_H^2(t)dt\Big)^{1/2}\Big(\int_{-\infty}^a \|\frac{1}{\sqrt{\alpha}}u\|_H^2(t)dt\Big)^{1/2}\\
&=2\|\frac{u}{\sqrt{\alpha}}\|_{L^2(H,(-\infty,a))}\|\sqrt{\alpha}u'\|_{L^2(H,(-\infty,a))}<\infty
\end{aligned}$$

So $u(a)\in H$. In a similar way we can show that $v(b)\in H$. $\square$

The following theorem defines the space of boundary condition.

Theorem 3.4. *The triplet (H,γ_1,γ_2), with*

$$\gamma_1: D(L)\to H,\ \gamma_1(w)=\frac{1}{i\sqrt{2}}(u(a)+v(b)),$$
$$\gamma_2: D(L)\to H,\ \gamma_2(w)=\frac{1}{\sqrt{2}}(u(a)-v(b)),\ w=(u,v)\in D(L),$$

is a space of boundary values of the minimal operator L_0 in $\mathcal{H}$.

Proof. In this case for any $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$ from $D(L)$ can be easy verified that

$$\begin{aligned}
(Lw_1,w_2)_{\mathcal{H}}-(w_1,Lw_2)_{\mathcal{H}}&=\big[(Ku_1,u_2)_{L^2_{1/\alpha}(H,(-\infty,a))}-(u_1,Ku_2)_{L^2_{1/\alpha}(H,(-\infty,a))}\big]\\
&\quad+\big[(Mv_1,v_2)_{L^2_{1/\beta}(H,(b,\infty))}-(v_1,Mv_2)_{L^2_{1/\beta}(H,(b,\infty))}\big]\\
&=(\gamma_1(w_1),\gamma_2(w_2))_H\ -\ (\gamma_2(w_1),\gamma_2(w_2))_H
\end{aligned}$$

Now let f and g be any elements from H. Find the function $w=(u,v)\in D(L)$ such that

$$\gamma_1(w)=\frac{1}{i\sqrt{2}}(u(a)+v(b)) \quad \text{and} \quad \gamma_2(w)=\frac{1}{\sqrt{2}}(u(a)-v(b)).$$

From this it is obtained that

$$u(a)=(if+g)/\sqrt{2} \quad \text{and} \quad v(b)=(if-g)/\sqrt{2}$$

If we choose the functions $u(\cdot)$ and $v(\cdot)$ in the form

$$u(t) = \frac{\sqrt{\alpha(t)}}{\sqrt{\alpha(a)}} \int_{-\infty}^{t} e^{s-a} ds (if+g)/\sqrt{2}, \quad t < a,$$
$$v(t) = \frac{\sqrt{\beta(t)}}{\sqrt{\beta(b)}} \int_{t}^{\infty} e^{b-t} ds (if-g)/\sqrt{2}, \quad t > b,$$

then it is clear that $(u, v) \in D(L)$ and $\gamma_1(w) = f$, $\gamma_2(w) = g$. □

Lastly, using the method in [7] it can be established the following results.

Theorem 3.5. *If $\widetilde{L}$ is a selfadjoint extension of the minimal operator L_0 in $\mathcal{H}$, then it generates by the differential-operator expression* (2.1) *and boundary condition*

$$v(b) = Wu(a),$$

where $W : H \to H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_W$ and vice versa.

4. Spectrum of the selfadjoint extensions

In this section the structure of the spectrum of the selfadjoint extension L_W in $\mathcal{H}$ will be investigated. Firstly, prove the following results.

Theorem 4.1. *The point spectrum of selfadjoint extension L_W is empty, i.e.*

$$\sigma_p(L_W) = \emptyset.$$

Proof. Consider the eigenvalue problem

$$l(w) = \lambda w, \quad w = (u, v) \in \mathcal{H}, \ \lambda \in \mathbb{R}$$

with boundary condition $v(b) = Wu(a)$. From this it is obtained that

$$i\alpha(t)u'(t) + Au(t) = \lambda u(t), \ t < a,$$
$$i\beta(t)v'(t) + Bv(t) = \lambda v(t), \ t > b,$$
$$v(b) = Wu(a)$$

The general solutions of the above equations are

$$u(\lambda; t) = \exp\Big(-i(A-\lambda)\int_t^a \frac{ds}{\alpha(s)}\Big) f_\lambda, \quad f_\lambda \in H, \ t < a,$$
$$v(\lambda; t) = \exp\Big(i(B-\lambda)\int_b^t \frac{ds}{\beta(s)}\Big) g_\lambda, \quad g_\lambda \in H, \ t > b,$$
$$v(\lambda; b) = Wu(\lambda; a)$$

It is clear that for $f_\lambda \neq 0$ and $g_\lambda \neq 0$ the solutions $u(\lambda; \cdot) \notin L^2_{1/\alpha}(H, (-\infty, a))$ and $v(\lambda; \cdot) \notin L^2_{1/\beta}(H, (b, \infty))$. Therefore, for every unitary operator W we have $\sigma_p(L_W) = \emptyset$. Since the residual spectrum for any selfadjoint operator in a Hilbert space is empty, then we study the continuous spectrum of selfadjoint extensions L_W of the minimal operator L_0. On the other hand from the general theory of linear selfadjoint operators in Hilbert spaces for the resolvent set $\rho(L_W)$ of any selfadjoint extension L_W is true

$$\rho(L_W) \supset \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \neq 0\}.$$

□

For the continuous spectrum of selfadjoint extensions the following proposition is true.

Theorem 4.2. *The continuous spectrum of the selfadjoint extension L_W is of the form*
$$\sigma_c(L_W) = \mathbb{R}.$$

Proof. For $\lambda \in \mathbb{C}$ with $\lambda_i = \operatorname{Im}\lambda > 0$ and $f = (f_a, f_b) \in \mathcal{H}$ the norm of function $R_\lambda(L_W)f(t)$ in $\mathcal{H}$ satisfies

$$\begin{aligned}
\|R_\lambda(L_W))f(t)\|^2_{\mathcal{H}} &= \|\exp\Big(i(\lambda - A)\int_t^a \frac{ds}{\alpha(s)}\Big) f_\lambda \\
&\quad + i\int_t^a \exp\Big(i(A-\lambda)\int_s^t \frac{d\tau}{\alpha(\tau)}\Big)\frac{f_a(s)}{\alpha(s)} ds\|^2_{L^2_{1/\alpha}(H,(-\infty,a))} \\
&\quad + \|i\int_t^\infty \exp\Big(i(B-\lambda)\int_s^t \frac{d\tau}{\beta(\tau)}\Big)\frac{f_b(s)}{\beta(s)} ds\|^2_{L^2_{1/\beta}(H,(b,\infty))} \\
&\geq \|i\int_t^\infty \exp\Big(i(B-\lambda)\int_s^t \frac{d\tau}{\beta(\tau)}\Big)\frac{f_b(s)}{\beta(s)} ds\|^2_{L^2_{1/\beta}(H,(b,\infty))}
\end{aligned}$$

The vector functions $f^*(\lambda; t)$ in the form
$$f^*(\lambda; t) = \Big(0, \exp\Big(-i(\overline{\lambda} - B)\int_b^t \frac{ds}{\beta(s)}\Big) f\Big),$$
with $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda > 0$, $f \in H$ belong to $\mathcal{H}$. Indeed,
$$\begin{aligned}
\|f^*(\lambda; t)\|^2_{\mathcal{H}} &= \int_b^\infty \frac{1}{\beta(t)}\|\exp\Big(-i(\overline{\lambda} - B)\int_b^t \frac{ds}{\beta(s)}\Big) f\|^2_H dt \\
&= \int_b^\infty \frac{1}{\beta(t)}\exp\Big(-2\lambda_i\int_b^t \frac{ds}{\beta(s)}\Big) dt \|f\|^2_H \\
&= \frac{1}{2\lambda_i}\|f\|^2_H < \infty
\end{aligned}$$

For the such functions $f^*(\lambda; \cdot)$ we have
$$\begin{aligned}
&\|R_\lambda(L_W) f^*(\lambda; \cdot)\|^2_{\mathcal{H}} \\
&\geq \|i\int_t^\infty \frac{1}{\beta(s)}\exp\Big(i(B-\lambda)\int_s^t \frac{d\tau}{\beta(\tau)} - i(\overline{\lambda} - B)\int_b^s \frac{d\tau}{\beta(\tau)}\Big) f ds\|^2_{L^2_{1/\beta}(H,(b,\infty))} \\
&= \|\exp\Big(-i\lambda\int_b^t \frac{d\tau}{\beta(\tau)} + iB\int_b^t \frac{d\tau}{\beta(\tau)}\Big) \\
&\quad \times \int_t^\infty \frac{1}{\beta(s)}\exp\Big(-2\lambda_i\int_b^s \frac{d\tau}{\beta(\tau)}\Big) f ds\|^2_{L^2_{1/\beta}(H,(b,\infty))} \\
&= \|\exp\Big(\lambda_i\int_b^t \frac{d\tau}{\beta(\tau)}\Big)\int_t^\infty \frac{1}{\beta(s)}\exp\Big(-2\lambda_i\int_b^s \frac{d\tau}{\beta(\tau)}\Big) ds\|^2_{L^2_{1/\beta}(H,(b,\infty))}\|f\|^2_H \\
&= \|\frac{1}{2\lambda_i}\exp\Big(-\lambda_i\int_b^t \frac{d\tau}{\beta(\tau)}\Big)\|^2_{L^2_{1/\beta}(H,(b,\infty))}\|f\|^2_H \\
&= \frac{1}{4\lambda_i^2}\int_b^\infty \frac{1}{\beta(t)}\exp\Big(-2\lambda_i\int_b^t \frac{d\tau}{\beta(\tau)}\Big) dt \|f\|^2_H \\
&= \frac{1}{8\lambda_i^3}\|f\|^2_H\,.
\end{aligned}$$

From this,

$$\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathcal{H}} \geq \frac{\|f\|_H^2}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} = \frac{1}{2\lambda_i}\|f^*(\lambda;t)\|_{\mathcal{H}},$$

i.e., for $\lambda_i = \operatorname{Im}\lambda > 0$ and $f \neq 0$ is valid

$$\frac{\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathcal{H}}}{\|f^*(\lambda;t)\|_{\mathcal{H}}} \geq \frac{1}{2\lambda_i}$$

On the other hand it is clear that

$$\|R_\lambda(L_W)\| \geq \frac{\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathcal{H}}}{\|f^*(\lambda;t)\|_{\mathcal{H}}}, \quad f \neq 0$$

Consequently,

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i} \quad \text{for } \lambda \in \mathbb{C},\ \lambda_i = \operatorname{Im}\lambda > 0.$$

□

Using the above theorem, the spectrum of the singular differential operator generated by differential expression

$$l((u,v)) = \Big(i\frac{t^{2\alpha}+1}{t^{2\alpha}}u'(t,x) + xu(t,x), i\frac{t^{2\beta}+1}{t^{2\beta}}v'(t,x) + xv(t,x)\Big), \quad \alpha,\beta > 0$$

with boundary condition

$$u(1,x) = e^{i\varphi}u(-1,x),\ \varphi \in [0,2\pi)$$

in the direct sum $L^2_{(t^{2\alpha}+1)/t^{2\alpha}}((-\infty,-1)\times\mathbb{R}) \oplus L^2_{(t^{2\beta}+1)/t^{2\beta}}((1,\infty)\times\mathbb{R})$ is purely continuous and coincides with $\mathbb{R}$.

Note that another approach for the singular differential operators for nth order in the scalar case has been given in [14]. In special case of functions $\alpha(\cdot)$ and $\beta(\cdot)$ the analogous results have been obtained in [2].

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