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EXISTENCE AND CONTINUATION OF SOLUTIONS FOR CAPUTO TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we consider a fractional differential equation (FDE) with Caputo derivative and study the existence and continuation of its solution. Firstly, we prove a theorem on the existence of local solutions. Then we extend the continuation theorems for ODEs to those FDEs. Also several global existence results for FDE are obtained.

1. Introduction

Recently, fractional differential equations (FDEs) have been the center of attention of many studies and played a vital role due to emergence in various applications and exact description of nonlinear phenomena. It has been found that models using mathematical tools from fractional calculus can describe various phenomena such as viscoelasticity, electrochemistry, control, porous media, and many other branches of sciences [12, 14, 16, 31]. However, the development of existence and uniqueness of solution of FDEs are very slow. Some contributions about existence of solution of FDEs can be found in [14, 15, 20, 26].

Many authors [1, 5, 7, 6, 8, 10, 11, 17, 19, 22, 27, 28, 29, 30, 33, 34, 35], studied the existence-uniqueness of solution for FDEs on the finite interval $[0,T]$. But few researchers [2, 3, 4, 21] present results about the global existence-uniqueness of solution FDEs on the half axis $[0,+\infty)$. As far as we know, we cannot find directly the existence of global solution of FDEs by using the results from local existence because, yet continuation theorems for FDEs have not been derived. Recently, Kou, et al. [18] found the existence and continuation theorems for Riemann-Liouville type FDEs. Motivated by that work, a natural question is, do there also exist local existence, continuation theorems and global existence for Caputo type FDEs? In this paper, we give an active answer.

In this article, we consider the fractional order initial value problems (IVPs) of the form

$$\begin{gathered} {}_{C}D^{\alpha}_{0,t}\, x(t) = f(t,x), \quad 0<\alpha<1,\ t\in(0,+\infty), \\ x(t)|_{t=0} = x_0, \quad x\in\mathbb{R}. \end{gathered} \tag{1.1}$$

To ensure the existence of a unique solution to (1.1) we always assume that f satisfies Lipschitz condition with respect to the second variable, that is, $|f(t,x_1)-f(t,x_2))| \le L|x_1-x_2|$, where $L>0$.

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For the system of equations

$$\begin{gathered}
{}_CD^{\alpha}_{0,t}x_1(t) = f_1(t, x_1, x_2, \dots, x_n), \quad 0 < \alpha < 1,\ t \in (0, +\infty), \\
{}_CD^{\alpha}_{0,t}x_2(t) = f_2(t, x_1, x_2, \dots, x_n), \quad x \in \mathbb{R}^n, \\
\cdots \\
{}_CD^{\alpha}_{0,t}x_n(t) = f_n(t, x_1, x_2, \dots, x_n), \\
x_i(t)|_{t=0} = x_0, \quad i = 1, 2, \dots, n,
\end{gathered} \tag{1.2}$$

we assume that $f_n(t, x_1, x_2, \dots, x_n)$ satisfy the Lipschitzian conditions,

$$|f_k(t, x_1, x_2, \dots, x_n) - f_k(t, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)| \le \sum_{k=1}^{n} L_k |x_k - \tilde{x}_k|,$$

($L_k > 0,\ k = 1, 2, \dots, n$), where ${}_CD^{\alpha}_{0,t}$ is the Caputo derivative, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ in the IVP (1.1) and $f_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ in IVP (1.2) have weak singularities with respect to t respectively. In this paper, we establish the local existence for IVP ((1.1) and IVP (1.2). Then we extend the continuation theorems for ODEs to those of FDEs. Furthermore, we present global existence of solutions for IVP (1.1).

The rest of this article is organized as follows: In Section 2, we introduce some basic definitions and previously known results that will be used in our main results. A new local existence theorem for IVP (1.1) is given in Section 3. In Section 4 we present two new continuation theorems for IVP (1.1) which are generalization of the continuation theorems for ODEs. Concluding remarks and comments are included in the last section.

2. Preliminaries

In this section, we introduce some basic definitions and lemmas [15, 20, 23, 24, 26, 25] from the theory of fractional calculus which are used later. Let $C[a, b]$ be the Bannach space of all continuous functions mapping $[a, b]$ into $\mathbb{R}$ where the norm $\|x\|_{[a,b]} = \max_{t\in[a,b]} |x(t)|$

Definition 2.1. The Riemann-Liouville integral of function $f(t)$ with order $\alpha > 0$ is defined as

$${}_{RL}D^{-\alpha}_{0,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)ds, \quad t > 0.$$

Definition 2.2. The Riemann-Liouville derivative of function $f(t)$ with order $\alpha > 0$ is defined as

$${}_{RL}D^{\alpha}_{0,t} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} f(s)ds,\ t > 0,$$

where $n - 1 < \alpha < n \in \mathbb{Z}^+$.

Definition 2.3. The Caputo derivative of function $f(t)$ with order $\alpha > 0$ is defined as

$${}_CD^{\alpha}_{0,t} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{\alpha-1} f^{(n)}(s)ds,\ t > 0,$$

where $n - 1 < \alpha < n \in \mathbb{Z}^+$.

Lemma 2.4. *Suppose that $f(t,x)$ is a continuous function. Then the initial value problem* (1.1) *is equivalent to the nonlinear Volterra integral equation of the second kind*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds. \tag{2.1}$$

In other words, every solution of the Volterra integral equation (2.1) *is also the solution of our original IVP* (1.1) *and vise versa.*

Lemma 2.5. *Let M be a subset of $C[0,T]$. Them M is precompact if and only if the following conditions hold:*

(1) *$\{x(t) : x \in M\}$ is uniformly bounded,*
(2) *$\{x(t) : x \in M\}$ is equicontinuous on $[0,T]$.*

Lemma 2.6 (Schauder fixed point theorem)**.** *Let U be a closed bounded convex subset of Bannach space X. Suppose that $T : U \to U$ is completely continuous. Then T has a fixed point in U.*

3. Local existence theorems

In this section, we study the existence of local solutions for (1.1). Suppose that $f(t,x)$ in (1.1) and $f_i(t,x_i)$, $i = 1,2,\dots,n$ in (1.2) have some weak singularity with respect to t respectively. By applying Schauder fixed point theorem, a new local existence theorem is obtained. For this, we make the following hypothesis for our discussion.

(H1) Let $f : R^+ \times R \to R$ in (1.1) be a continuous function then there exists a constant $0 \le \delta < 1$ such that $(Ax)(t) = t^\delta f(t,x)$ is a continuous bounded map from $C[0,T]$ into $C[0,T]$ where T is positive.
(H2) Let $f_i : R^+ \times R^n \to R$ in (1.2) be continuous functions then there exist constants $0 \le \delta_i < 1$, such that $(A_i x_i)(t) = t^{\delta_i} f_i(t,x_1,x_2,\dots,x_n)$, $i = 1,2,\dots,n$ are continuous bounded maps from $C[0,T]$ into $C[0,T]$ where T is positive.

Theorem 3.1. *Suppose that condition* (H1) *is satisfied. Then IVP* (1.1) *has at least one solution $x \in C[0,h]$ for some $(T \ge)\ h > 0$.*

Proof. Let

$$E = \{x \in C[0,T] : \|x - x_0\|_{C[0,T]} = \sup_{0\le t\le T} |x - x_0| \le b\},$$

where $b > 0$ is a constant. Since operator A is bounded then there exists a constant $M > 0$ such that

$$\sup\{|(Ax)(t)| : t \in [0,T], x \in E\} \le M.$$

Again let

$$D_h = \big\{x : x \in C[0,h],\ \sup_{0\le t\le h} |x - x_0| \le b\big\},$$

where $h = \min\{(\frac{b\Gamma(\alpha+1-\delta)}{M\ \Gamma(1-\alpha)})^{\frac{1}{\alpha-\delta}}, T\}$, $\alpha > \delta$.

It is clear that $D_h \subseteq C[0,h]$ is nonempty, bounded closed and convex subset. Note that $h \le T$, define an operator B as follows

$$(Bx)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds, \quad t \in [0,h]. \tag{3.1}$$

By (3.1), for any $x \in C[0,h]$ we have

$$|(Bx)(t) - x_0| \le \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\delta} ds \le \frac{M\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta)} h^{\alpha-\delta} \le b,$$

which shows that $BD_h \subset D_h$.

Next we show that B is continuous. Let $x_n,\ x \in D_h$ such that $\|x_n - x\|_{C[0,h]} \to 0$ as $n \to +\infty$. In the continuity of A we have $\|Ax_n - Ax\|_{[0,h]} \to 0$ as $n \to +\infty$. Now

$$\begin{aligned}
&|(Bx_n)(t) - (Bx)(t)| \\
&= |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_n(s)) - f(s, x(s))| ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\delta} |(Ax_n)(s) - (Ax)(s)| ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\delta} ds \|(Ax_n)(s) - (Ax)(s)\|_{[0,h]}.
\end{aligned}$$

We have

$$\|(Bx_n)(s) - (Bx)(s)\|_{[0,h]} \le \frac{\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta)} h^{\alpha-\delta} \|(Ax_n)(s) - (Ax)(s)\|_{[0,h]}.$$

Then $\|(Bx_n)(s) - (Bx)(s)\|_{[0,h]} \to 0$ as $n \to +\infty$. Thus B is continuous.

Furthermore, we prove that operator BD_h is continuous. Let $x \in D_h$ and $0 \le t_1 \le t_2 \le h$. For any $\epsilon > 0$, note that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\delta} ds = \frac{\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta)} t^{\alpha-\delta} \to 0, \text{ as } t \to 0^+,$$

where $0 \le \delta < 1$. There exists a $\tilde{\delta} > 0$ such that for $t \in [0,h]$,

$$\frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\delta} ds < \epsilon$$

holds. In this case, for $t_1, t_2 \in [0, \tilde{\delta}]$ one has

$$\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds\Big| \\
&\le \frac{M}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} s^{-\delta} ds + \frac{M}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} s^{-\delta} ds < \epsilon.
\end{aligned} \tag{3.2}$$

In this case for $t_1, t_2 \in [\frac{\tilde{\delta}}{2}, h]$ one gets

$$\begin{aligned}
&|(Bx)(t_1) - (Bx)(t_2)| \\
&= \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds\Big| \\
&\le \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] f(s, x(s)) ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds\Big|.
\end{aligned} \tag{3.3}$$

Now, from the first term on the right hand side of (3.3) one has

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]f(s,x(s))ds\Big| \\
&\le \frac{M}{\Gamma(\alpha)}\int_0^{t_1}|[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]s^{-\delta}|ds \\
&\le \frac{M}{\Gamma(\alpha)}\int_0^{\tilde{\delta}/2}|[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]s^{-\delta}|ds \\
&\quad + \frac{M(\frac{\tilde{\delta}}{2})^{-\delta}}{\Gamma(\alpha)}\int_{\frac{\tilde{\delta}}{2}}^{t_1}|[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]|ds \\
&\le \frac{2M}{\Gamma(\alpha)}\int_0^{\frac{\delta_1}{2}}\big(\frac{\tilde{\delta}}{2}-s\big)^{\alpha-1}s^{-\delta}ds + \frac{M(\frac{\tilde{\delta}}{2})^{-\delta}}{\Gamma(\alpha)}[(t_2-t_1)^{\alpha} \\
&\quad + \big(t_1-\frac{\tilde{\delta}}{2}\big)^{\alpha} - \big(t_2-\frac{\tilde{\delta}}{2}\big)^{\alpha}] \\
&\le \epsilon + \frac{M(\frac{\tilde{\delta}}{2})^{-\delta}}{\Gamma(\alpha)}[(t_2-t_1)^{\alpha} + \big(t_1-\frac{\tilde{\delta}}{2}\big)^{\alpha} - \big(t_2-\frac{\tilde{\delta}}{2}\big)^{\alpha}].
\end{aligned} \tag{3.4}
$$

Next from the second term on the right hand side of (3.3), one has

$$
\begin{aligned}
\Big|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\Big| &\le \frac{M(\frac{\delta_1}{2})^{-\delta}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\le \frac{M(\frac{\delta_1}{2})^{-\delta}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.
\end{aligned} \tag{3.5}
$$

From the above discussion, there exists a $(\frac{\tilde{\delta}}{2} >)\tilde{\delta}_1 > 0$ such that for $t_1, t_2 \in [\frac{\tilde{\delta}}{2}, h]$ and $|t_1 - t_2| < \tilde{\delta}_1$,

$$
|(Bx)(t_1) - (Bx)(t_2)| < 2\epsilon. \tag{3.6}
$$

It follows from (3.2) and (3.6) that $\{(Bx)(t) : x \in D_h\}$ is equicontinuous. It is also clear that $\{(Bx)(t) : x \in D_h\}$ is uniformly bounded due to $BD_h \subset D_h$. So BD_h is precompact. Therefore B is completely continuous. By Schauder fixed point theorem and Lemma 2.4, IVP (1.1) has a local solution. The proof is thus completed. □

Theorem 3.2. *Suppose that condition* (H2) *is satisfied. Then IVP* (1.2) *has at least one solution* $x_i \in C[0,h]$ *for some* $(T \ge)\ h > 0$.

Proof. Let

$$
E = \big\{x_i \in C[0,T] : \|x_i - x_0\|_{C[0,T]} = \sup_{0\le t\le T}|x_i - x_0| \le b_i,\ i = 1,2,\dots,n\big\},
$$

where $b_i > 0$, $i = 1,2,\dots,n$ are constants. Since the operators A_i, $i = 1,2,\dots,n$ are bounded then there exist constants $M_i > 0$, $i = 1,2,\dots,n$ such that

$$
\sup\{|(A_i x_i)(t)| : t \in [0,T],\ x_i \in E\} \le M_i, \quad i = 1,2,\dots,n.
$$

Again let

$$
D_{ih} = \{x_i : x_i \in C[0,h],\ \sup_{0\le t\le h}|x_i - x_0| \le b_i,\ i = 1,2,\dots,n\},
$$

where

$$h=\min\big\{\big(\frac{b_1\Gamma(\alpha+1-\delta_1)}{M_1\ \Gamma(1-\alpha)}\big)^{\frac{1}{\alpha-\delta_1}},\big(\frac{b_2\Gamma(\alpha+1-\delta_2)}{M_2\ \Gamma(1-\alpha)}\big)^{\frac{1}{\alpha-\delta_2}},\dots,\\ \big(\frac{b_n\Gamma(\alpha+1-\delta_n)}{M_n\ \Gamma(1-\alpha)}\big)^{\frac{1}{\alpha-\delta_n}},T\big\},$$

$\alpha>\delta_i$, $i=1,2,\dots,n$.

It is clear that $D_{ih}\subseteq C[0,h]$ are nonempty, bounded closed, and convex subsets. Note that $h\leq T$, define operators B_i as follows

$$\begin{gathered}
(B_1x_1)(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_1(s,x_1(s),x_2(s),\dots,x_n(s))ds,\\
(B_2x_2)(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_2(s,x_1(s),x_2(s),\dots,x_n(s))ds,\\
\dots\\
(B_nx_n)(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_n(s,x_1(s),x_2(s),\dots,x_n(s))ds,
\end{gathered}\tag{3.7}$$

for $t\in[0,h]$. By (3.7), for any $x_i\in C[0,h]$ we have

$$\begin{gathered}
|(B_1x_1)(t)-x_0|\leq\frac{M_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{-\delta_1}ds,\\
|(B_2x_2)(t)-x_0|\leq\frac{M_2}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{-\delta_2}ds,\\
\dots\\
|(B_nx_n)(t)-x_0|\leq\frac{M_n}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{-\delta_n}ds,
\end{gathered}$$

and

$$\begin{gathered}
|(B_1x_1)(t)-x_0|\leq\frac{M_1\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta_1)}h^{\alpha-\delta_1}\leq b_1,\\
|(B_2x_2)(t)-x_0|\leq\frac{M_2\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta_2)}h^{\alpha-\delta_2}\leq b_2,\\
\dots\\
|(B_nx_n)(t)-x_0|\leq\frac{M_n\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta_1)}h^{\alpha-\delta_n}\leq b_n,
\end{gathered}$$

which shows that, $B_iD_{ih}\subset D_{ih}$, $i=1,2,\dots,n$.

Next we show that operators B_i are continuous. Let x_m, $x_i\in D_{ih}$, $m>n$, $i=1,2,\dots,n$ such that $\|x_m-x_i\|_{C[0,h]}\to0$ as $m\to+\infty$. In view of continuity of operators A_i we have $\|A_ix_m-A_ix_i\|_{[0,h]}\to0$ as $m\to+\infty$. Now

$$\begin{aligned}
&|(B_ix_m)(t)-(B_ix_i)(t)|\\
&=\big|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_i(s,x_m(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_i(s,x_i(s))ds\big|\\
&\leq\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f_i(s,x_m(s))-f_i(s,x_i(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\delta_i}|(A_ix_m)(s)-(A_ix_i)(s)|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\delta_i}ds\|(A_ix_m)(s)-(A_ix_i)(s)\|_{[0,h]}.
\end{aligned}
$$

We have

$$
\|(B_ix_m)(s)-(B_ix_i)(s)\|_{[0,h]} \le \frac{\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta_i)}h^{\alpha-\delta_i}\|(A_ix_m)(s)-(A_ix_i)(s)\|_{[0,h]}.
$$

Then $\|(B_ix_m)(s)-(B_ix_i)(s)\|_{[0,h]} \to 0$ as $m \to +\infty$. Thus B_i are continuous. Furthermore, we prove that operators B_iD_{ih} are continuous. Let $x_i \in D_{ih}$ and $0 \le t_1 \le t_2 \le h$. For any $\epsilon > 0$, note that

$$
\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\delta_i}ds = \frac{\Gamma(1-\alpha)}{\Gamma(\alpha+1-\delta_i)}t^{\alpha-\delta_i} \to 0, \quad \text{as } t \to 0^+,
$$

where $0 \le \delta_i < 1$. There exists $\tilde{\delta}_i > 0$ such that for $t \in [0,h]$,

$$
\frac{2M_i}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\delta_i}ds < \epsilon.
$$

In this case, for $t_1, t_2 \in [0, \tilde{\delta}_i]$, one has

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f_i(s,x_i(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f_i(s,x_i(s))ds\Big| \\
&\le \frac{M_i}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}s^{-\delta_i}ds + \frac{M_i}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}s^{-\delta_i}ds < \epsilon.
\end{aligned} \tag{3.8}
$$

In this case, for $t_1, t_2 \in [\frac{\tilde{\delta}_i}{2}, h]$, one gets

$$
\begin{aligned}
&|(B_ix_i)(t_1)-(B_ix_i)(t_2)| \\
&= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f_i(s,x_i(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f_i(s,x_i(s))ds\Big| \\
&\le \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]f_i(s,x_i(s))ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f_i(s,x_i(s))ds\Big|.
\end{aligned} \tag{3.9}
$$

Now, from the first term on the right hand side of (3.9) one has

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]f_i(s,x_i(s))ds\Big| \\
&\le \frac{M_i}{\Gamma(\alpha)}\int_0^{t_1}|[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]s^{-\delta_i}|ds \\
&\le \frac{M_i}{\Gamma(\alpha)}\int_0^{\frac{\tilde\delta_i}{2}}|[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]s^{-\delta_i}|ds \\
&\quad + \frac{M_i(\frac{\tilde\delta_i}{2})^{-\delta_i}}{\Gamma(\alpha)}\int_{\frac{\tilde\delta_i}{2}}^{t_1}|[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]|ds \qquad (3.10)\\
&\le \frac{2M_i}{\Gamma(\alpha)}\int_0^{\tilde\delta_i/2}\big(\frac{\tilde\delta_i}{2}-s\big)^{\alpha-1}s^{-\delta_i}ds + \frac{M_i(\frac{\tilde\delta_i}{2})^{-\delta_i}}{\Gamma(\alpha)}\Big[(t_2-t_1)^{\alpha} \\
&\quad + \big(t_1-\frac{\tilde\delta_i}{2}\big)^{\alpha}-\big(t_2-\frac{\tilde\delta_i}{2}\big)^{\alpha}\Big] \\
&\le \epsilon + \frac{M_i(\frac{\tilde\delta_i}{2})^{-\delta_i}}{\Gamma(\alpha)}\big[(t_2-t_1)^{\alpha}+\big(t_1-\frac{\tilde\delta_i}{2}\big)^{\alpha}-\big(t_2-\frac{\tilde\delta_i}{2}\big)^{\alpha}\big].
\end{aligned}
$$

Next from the second term on the right hand side of (3.3), one has

$$
\begin{aligned}
\Big|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f_i(s,x_i(s))ds\Big| &\le \frac{M_i(\frac{\tilde\delta_i}{2})^{-\delta_i}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\le \frac{M_i(\frac{\tilde\delta_i}{2})^{-\delta_i}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.
\end{aligned}
$$

So from the above discussion, there exist $(\frac{\tilde\delta_i}{2} >)\ \lambda > 0$ such that for $t_1, t_2 \in [\frac{\tilde\delta_i}{2}, h]$ and $|t_1 - t_2| < \lambda$,

$$
|(B_ix_i)(t_1)-(B_ix_i)(t_2)| < 2\epsilon. \tag{3.11}
$$

It follows from (3.2) and (3.6) that $\{(B_ix_i)(t) : x_i \in D_{ih}\}$ are equicontinuous. It is also clear that $\{(B_ix_i)(t) : x_i \in D_{ih}\}$ are uniformly bounded due to $B_iD_{ih} \subset D_{ih}$. So B_iD_{ih} are precompact. Therefore operators B_i are completely continuous. By Schauder fixed point theorem and Lemma 2.4, IVP (1.2) has a local solution. The proof is thus completed. □

4. Continuation theorems

In this section, we study the continuation of solution for IVP (1.1). The basic techniques may be applied to system (1.2), so we omit the detail here or leave to the interested readers. We extend the continuation theorem for ODEs to Caputo type FDEs. Initially, we give the following definition.

Definition 4.1 ([18]). Let $x(t)$ on $(0, \beta)$ and $\tilde x(t)$ on $(0, \tilde\beta)$ both are the solutions of (1.1). If $\beta < \tilde\beta$ and $x(t) = \tilde x(t)$ for $t \in (0, \beta)$, we say that $\tilde x(t)$ can be continued to $(0, \tilde\beta)$. A solution $x(t)$ is noncontinuable if it has no continuation. The existing interval of noncontinuable solution $x(t)$ is called the maximum existing interval of $x(t)$.

Theorem 4.2. *Assume that condition* (H1) *is satisfied. Then* $x = x(t)$, $t \in (0, \beta)$ *is noncontinuable if and only if for some* $\eta \in (0, \frac{\beta}{2})$ *and any bounded closed subset* $S \subset [\eta, +\infty) \times \mathbb{R}$ *there exists a* $t^* \in [\eta, \beta)$ *such that* $(t^*, x(t^*)) \notin S$.

Proof. The proof of this theorem is given in two steps. Suppose that there exists a compact subset $S \subset [\eta, +\infty) \times \mathbb{R}$ such that $\{(t, x(t)) : t \in [\eta, \beta)\} \subset S$. The compactness of S implies $\beta < +\infty$. By (H1) there exists a $K > 0$ such that $\sup_{(t,x)\in S} |f(t, x)| \le K$.

Step 1. We show that $\lim_{t\to\beta^-} x(t)$ exists. Let

$$J(t) = \int_0^\eta (t-s)^{\alpha-1} s^{-\delta} ds, \quad t \in [2\eta, \beta].$$

We can easily see that $J(t)$ is uniformly continuous on $[2\eta, \beta]$. For all $t_1, t_2 \in [2\eta, \beta)$, $t_1 < t_2$ we have

$$\begin{aligned}
&|x(t_1) - x(t_2)| \\
&= \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, x(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s)) ds\Big| \\
&\le \Big|\frac{1}{\Gamma(\alpha)} \int_0^{\eta} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] s^{-\delta} (Ax)(s) ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)} \int_\eta^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] f(s, x(s)) ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s)) ds\Big| \\
&\le \frac{\|Ax\|_{[0,\eta]}}{\Gamma(\alpha)} \int_0^\eta \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right] s^{-\delta} ds \\
&\quad + \frac{K}{\Gamma(\alpha)} \int_\eta^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds + \frac{K}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&\le |J(t_1) - J(t_2)| \frac{\|Ax\|_{[0,\eta]}}{\Gamma(\alpha)} + \frac{K}{\Gamma(\alpha)} [2(t_2 - t_1)^\alpha + (t_1 - \eta)^\alpha - (t_2 - \eta)^\alpha].
\end{aligned}$$

From the continuity of $J(t)$ and Cauchy convergence criterion, it follows that $\lim_{t\to\beta^-} x(t) = x^*$.

Step 2. Now we show that $x(t)$ is continuable. Since S is a closed subset, we have $(\beta, x^*) \in S$. Define $x(\beta) = x^*$. Then $x(t) \in C[0, \beta]$, we define operator D as follows

$$(Dy)(t) = x_1 + \frac{1}{\Gamma(\alpha)} \int_\beta^t (t-s)^{\alpha-1} f(s, y(s)) ds,$$

where

$$x_1 = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^\beta (t-s)^{\alpha-1} f(s, y(s)) ds, \quad y \in C[\beta, \beta+1], \quad t \in [\beta, \beta+1].$$

Let

$$E_b = \{(t, y) : \beta \le t \le \beta + 1, |y| \le \max_{\beta\le t\le\beta+1} |x_1(t)| + b\}.$$

In view of the continuation of f on E_b, denote $M = \max_{(t,y)\in E_b} |f(t, y)|$. Again let

$$E_h = \{y \in C[\beta, \beta + 1] : \max_{t\in[\beta,\beta+h]} |y(t) - x_1(t)| \le b, y(\beta) = x_1(\beta)\},$$

where $h=\min\big\{1,(\frac{\Gamma(\alpha+1)b}{M})^{\frac{1}{\alpha}}\big\}$. We can claim that D is completely continuous on E_b. Set $\{y_n\}\subseteq C[\beta,\beta+h]$, $\|y_n-y\|_{[\beta,\beta+h]}\to 0$ as $n\to+\infty$. Then we have

$$\begin{aligned}|(Dy_n)(t)-(Dy)(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_\beta^t (t-s)^{\alpha-1}[f(s,y_n(s))-f(s,y(s))]ds\Big| \\ &\le \frac{h^\alpha}{\Gamma(\alpha+1)}\|f(s,y_n(s))-f(s,y(s))\|_{[\beta,\beta+h]}.\end{aligned}$$

By the continuity of f we have $\|f(s,y_n(s))-f(s,y(s))\|_{[\beta,\beta+h]}\to 0$ as $n\to+\infty$. Therefore, $\|(Dy_n)(t)-(Dy)(t)\|_{[\beta,\beta+h]}\to 0$ as $n\to+\infty$, which implies that operator D is continuous.

Secondly, we prove that DE_h is equicontinuous. For any $y\in E_h$ we have $(Dy)(\beta)=x_1(\beta)$ and

$$\begin{aligned}|(Dy)(t)-x_1| &= \Big|\frac{1}{\Gamma(\alpha)}\int_\beta^t (t-s)^{\alpha-1}f(s,y(s))ds\Big| \\ &\le \frac{M(t-\beta)^\alpha}{\Gamma(\alpha+1)}\le\frac{Mh^\alpha}{\Gamma(\alpha+1)}\le b.\end{aligned}$$

Thus $DE_h\subset E_h$. Set $I(t)=\frac{1}{\Gamma(\alpha)}\int_0^\beta (t-s)^{\alpha-1}f(s,x(s))ds$. We know that $I(t)$ is continuous on $[\beta,\beta+1]$. For all $y\in E_h$, $\beta\le t_1\le t_2\le\beta+h$, we have

$$\begin{aligned}&|(Dy)(t_1)-(Dy)(t_2)| \\ &\le \Big|\frac{1}{\Gamma(\alpha)}\int_0^\beta \big[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big]f(s,y(s))ds\Big| \\ &\quad+\frac{1}{\Gamma(\alpha)}\Big|\int_\beta^{t_1}\big[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big]f(s,y(s))ds\Big| \\ &\quad+\frac{1}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds\Big| \\ &\le |I(t_1)-I(t_2)|+\frac{M}{\Gamma(\alpha+1)}[2(t_2-t_1)^\alpha+(t_1-\beta)^\alpha-(t_2-\beta)^\alpha].\end{aligned}\tag{4.1}$$

In view of the uniform continuity of $I(t)$ on $[\beta,\beta+h]$ and (4.1), we conclude that $\{(Dy)(t):y\in E_h\}$ is equicontinuous. Therefore D is completely continuous. By Schauder fixed point theorem, operator D has a fixed point $\tilde{x}(t)\in E_h$, i.e.,

$$\begin{aligned}\tilde{x}(t) &= x_1+\frac{1}{\Gamma(\alpha)}\int_\beta^t (t-s)^{\alpha-1}f(s,\tilde{x}(s))ds, \\ &= x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,\tilde{x}(s))ds,\quad t\in[\beta,\beta+h],\end{aligned}\tag{4.2}$$

where

$$\tilde{x}(t)=\begin{cases}x(t), & t\in(0,\beta]\\ \tilde{x}(t), & t\in[\beta,\beta+h]\end{cases}$$

It follows that $\tilde{x}(t)\in C[0,\beta+h]$ and

$$\tilde{x}(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,\tilde{x}(s))ds.\tag{4.3}$$

Therefore, according to Lemma 2.4, $\tilde{x}(t)$ is a solution of (1.1) on $(0, \beta + h]$. This yields a contradiction (since $x(t)$ is noncontinuable). The proof is thus complete. $\square$

Remark 4.3. Theorem 4.2 is generalization of [9, Theorem C], which is the continuation theorem for the ODE. To see this (1.1) is reduced to an ODE if we set $\alpha = 1$.

Now we present another continuation theorem, which is more convenient for applications.

Theorem 4.4 ([Continuation Theorem II)**.** *Suppose that condition* (H1) *is satisfied. Then $x = x(t)$, $t \in (0, \beta)$ is noncontinuable if and only if*

$$\lim_{t \to \beta^-} \sup |K(t)| = +\infty, \tag{4.4}$$

where $K(t) = (t, x(t))$, $\|K(t)\| = (x^2(t) + t^2)^{\frac{1}{2}}$.

Proof. We prove this theorem by contradiction. Suppose that (4.4) is not true. Then there exist a sequence $\{t_n\}$ and a positive constant $L > 0$ such that $t_n < t_{n+1}$, $n \in \mathbb{N}$,

$$\lim_{n \to \infty} t_n = \beta, \quad |K(t_n)| \le L, \quad \text{i.e., } (x^2(t_n) + t_n^2) \le L^2 \tag{4.5}$$

Since $\{x(t_n)\}$ is a bounded convergent sub-sequence, one can let

$$\lim_{n \to \infty} x(t_n) = x^*. \tag{4.6}$$

Now we show that, for any given $\varepsilon > 0$ there exists $T \in (0, \beta)$, such that $|x(t) - x^*| < \varepsilon$, $t \in (T, \beta)$, i.e.,

$$\lim_{t \to \beta^-} x(t) = x^*. \tag{4.7}$$

For sufficiently small $\tau > 0$, let

$$E_1 = \big\{(t, x) : t \in [\tau, \beta], |x| \le \sup_{t \in [\tau, \beta)} |x(t)|\big\}.$$

Since f is continuous on E_1, we can denote $K = \max_{(t,y) \in E_1} |f(t, y)|$. It follows from (4.5) and (4.6) that there exists n_0 such that $t_{n_0} > \tau$ and for $n \ge n_0$ we have

$$|x(t_n) - x^*| \le \frac{\varepsilon}{2}.$$

If (4.7) is not true, then for $n \ge n_0$, there exists $\lambda_n \in (t_n, \beta)$ such that $|x(\lambda_n) - x^*| \ge \varepsilon$ and $|x(t) - x^*| < \varepsilon$, $t \in (t_n, \lambda_n)$. Thus

$$\begin{aligned}
\varepsilon &\le |x(\lambda_n) - x^*| \\
&\le |x(t_n) - x^*| + |x(\lambda_n) - x(t_n)| \\
&\le \frac{\varepsilon}{2} + \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} f(s, x(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{\lambda_n} (\lambda_n - s)^{\alpha-1} f(s, x(s)) ds\Big| \\
&\le \frac{\varepsilon}{2} + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{\tau} [(t_n - s)^{\alpha-1} - (\lambda_n - s)^{\alpha-1}] f(s, x(s)) ds\Big| \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big| \int_{\tau}^{t_n} [(t_n - s)^{\alpha-1} - (\lambda_n - s)^{\alpha-1}] f(s, x(s)) ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha)} \int_{t_n}^{\lambda_n} (\lambda_n - s)^{\alpha-1} f(s, x(s)) ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\varepsilon}{2} + \frac{\|Ax\|_{[0,\tau]}}{\Gamma(\alpha)}|I(t_n) - I(\lambda_n)| + \frac{M}{\Gamma(\alpha+1)}[2(\lambda_n - t_n)^\alpha \\
&\quad + (t_n - \tau)^\alpha - (\lambda_n - \tau)^\alpha].
\end{aligned}
$$

In view of continuity of $I(t)$ on $[t_{n_0}, \beta]$, for sufficiently large $n \ge n_0$, we have

$$
\varepsilon \le |x(\lambda_n) - x^*| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

This implies the contradiction that $\lim_{t\to\beta^-} x(t)$ exists. By the similar argument to the proof of Theorem 4.2, we can find a continuation of $x(t)$. The proof is ended. □

Remark 4.5. If f in (1.1) satisfies the global Lipschitz condition with the second variable, then its solution globally exists and it is unique.

5. Global existence theorems

In this section, we study the existence of a global solution for (1.1) which is based on the previously results. The basic techniques may be applied to system (1.2), so we omit the details here, and leave them for the interested readers. Applying Theorem 4.4, in a straight way we acquire the following conclusion about the existence of global solution of (1.1).

Theorem 5.1. *Suppose that condition* (H1) *is satisfied. Let $x(t)$ be a solution of* (1.1) *on $(0, \beta)$. If $x(t)$ is bounded on $[\tau, \beta)$ for some $\tau > 0$, then $\beta = +\infty$.*

Continuing our discussion, we firstly present the following lemma, which is useful in our analysis.

Lemma 5.2 ([13, 32])**.** *Let $v : [0, b] \to [0, +\infty)$ be a real function, and $w(\cdot)$ be a nonnegative, locally integrable function on $[0, b]$. Suppose that there exist $a > 0$ and $0 < \alpha < 1$ such that*

$$
v(t) \le w(t) + a\int_0^t \frac{v(s)}{(t-s)^\alpha} ds.
$$

Then there exists a constant $k = k(\alpha)$ such that for $t \in [0, b]$, we have

$$
v(t) \le w(t) + ka\int_0^t \frac{w(s)}{(t-s)^\alpha} ds.
$$

Theorem 5.3. *Suppose that condition* (H1) *is satisfied and there exist three nonnegative continuous functions $l(t)$, $m(t)$, $p(t) : [0, +\infty) \to [0, +\infty)$ such that $|f(t, x)| \le l(t)m(|x|) + p(t)$, where $m(r) \le r$ for $r \ge 0$. Then* (1.1) *has one solution in $C[0, +\infty)$.*

Proof. The existence of a local solution $x(t)$ of (1.1) can be concluded by Theorem 3.1. By Lemma 2.4, $x(t)$ satisfies the integral equation

$$
x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds.
$$

Suppose that the maximum existing interval of $x(t)$ is $[0, \beta)$ ($\beta < +\infty$). Then

$$
\begin{aligned}
|x(t)| &= \big|x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds\big| \\
&\le x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(l(s)m(|x|) + p(s)) ds
\end{aligned}
$$

$$\le x_0 + \frac{\|l\|_{[0,\beta]}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(m(|x|)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds.$$

We take $v(t) = |x(t)|,\ w(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds$, $a = \frac{\|l\|_{[0,\beta]}}{\Gamma(\alpha)}$. By Lemma 5.2, we know that $v(t) = |x(t)|$ is bounded on $[0, \beta)$. Thus for any $\tau \in (0, \beta)$, $x(t)$ is bounded on $[\tau, \beta)$. By theorem 5.1, IVP (1.1) has a solution $x(t)$ on $(0, +\infty)$. $\square$

The following result guarantees the existence and uniqueness of global solution of (1.1) on $\mathbb{R}^+$.

Theorem 5.4. *Suppose that* (H1) *is satisfied and there exists a non-negative continuous function $l(t)$ defined on $[0, \infty)$ such that $|f(t, x) - f(t, y)| \le l(t)|x - y|$. Then* (1.1) *has a unique solution in $C[0, +\infty)$.*

The existence of a global solution can be obtained by using the same arguments as above. From the Lipschitz-type condition and Lemma 5.2, we can conclude the uniqueness of global solution. The proof is omitted here.

Conclusion. In this article, we obtained a new local existence theorem for Caputo type general FDE which has a certain singularity. Then we derived two continuation theorems which have been never studied before. Next we established global existence theorems for the FDEs.

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