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POINCARE INEQUALITY AND CAMPANATO ESTIMATES FOR WEAK SOLUTIONS OF PARABOLIC EQUATIONS

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Abstract. We shall show that the Poincaré type inequality holds for the weak solution of a parabolic equation. The key is to control the L^p norm of the first derivative of the weak solution with respect to the time variable. The inequality is necessary to get an estimate in the Campanato space $\mathcal{L}^{p,\mu}$ for general parabolic equations.

1. Introduction

Let $B_r(x_0) \subset \mathbb{R}^n$ be a ball with center x_0 and the radius $r > 0$. Let $u \in W^{1,p}(B_r(x_0))$ and define

$$u_{x_0,r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u dx.$$

Then we have the well known Poincaré inequality

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^p dx \le Cr^p \int_{B_r(x_0)} |\nabla u|^p dx, \tag{1.1}$$

where the constant C depends on n and p, but is independent of r and u (see, for example, Chen and Wu [1, Appendix I Corollary 3.1]). This type estimate is used in the Campanato space approach to obtain the Hölder regularity of weak solutions for elliptic or quasilinear elliptic system with degeneracies. For example, for the elliptic system in divergence form, see Giaquinta [7], and the regularity of solutions for the quasilinear elliptic system with degeneracy as p-Laplacian has been treated recently in Giacomoni et al. [4, 5, 6] who essentially used the technique of the Campanato estimates of the Lieberman [9, p. 1211] and [7, p. 45], using the Poincaré inequality (1.1). See, for example, [6, Theorem A.1 in the Appendix].

We are interested in the Hölder regularity of weak solutions for parabolic equations. To apply the Campanato estimate in this case, it suffices to use the following Poincaré type inequality. To explain precisely, let $z_0 = (x_0, t_0) \in Q_T = \Omega \times (0, T)$. If we put

$$u_{z_0,r} = \frac{1}{|Q_r(z_0)|} \iint_{Q_r(z_0)} u \, dx \, dt$$

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for any cylinder $Q_r(z_0) = B_r(x_0) \times (t_0, t_0 + r^2] \subset Q_T$, the following inequality

$$\iint_{Q_r(z_0)} |u - u_{z_0,r}|^p \, dx \, dt \le C r^p \iint_{Q_r(z_0)} |\nabla u|^p \, dx \, dt, \tag{1.2}$$

where ∇u is the gradient of u with respect to the space variable x does not hold for a general function $u(x,t) \in L^p(0,T; W^{1,p}(Q_r(z_0))$. This is the fundamental difference from the elliptic theory. However, when u is a weak solution of a parabolic equation, by using the equation and combining the Poincaré inequality (1.1) with respect to the space variable, we shall show that the inequality (1.2) holds.

Such inequality is used in the Campanato space $\mathcal{L}^{p,\mu}$ estimates for weak solutions of a parabolic equation. In fact, Yin [11] used the inequality for $p = 2$ and conducted us to $\mathcal{L}^{2,\mu}$-estimate for weak solution of parabolic equations. We are convinced that we can use the general inequality (1.2) for $\mathcal{L}^{p.\mu}$-estimates for parabolic equations. It will appear in the future work.

2. Main results

In this section, we give the main theorem of this paper. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$, and define $Q_T = \Omega \times (0,T)$ with $T > 0$. We consider the parabolic equation

$$u_t - \sum_{i,j=1}^{n} \big(a_{ij}(x,t) u_{x_j}\big)_{x_i} = 0 \quad \text{in } Q_T \tag{2.1}$$

where $a_{ij} \in L^\infty(Q_T)$ satisfies the ellipticity condition: there exist constants a_0 and A_0 with $0 < a_0 \le A_0 < \infty$ such that

$$a_0 |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t) \xi_i \xi_j \le A_0 |\xi|^2$$

for all $(x,t) \in Q_T$ and $\xi \in \mathbb{R}^n$.

We shall consider the weak solution of (2.1).

Definition 2.1. Let $1 \le p < \infty$. We say $u \in L^p(0,T; W^{1,p}(\Omega))$ is a weak solution of (2.1) if the following equality holds.

$$\iint_{Q_T} \Big(-u v_t + \sum_{i,j=1}^{n} a_{ij} u_{x_j} v_{x_i}\Big) \, dx \, dt = 0 \tag{2.2}$$

for all $v \in W^{1,p'}(0,T; W_0^{1,p'}(\Omega))$ with $v(x,0) = v(x,T) = 0$ where p' is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Here $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the standard Sobolev spaces. We note that since $W^{1,p'}(0,T; W_0^{1,p'}(\Omega)) \subset C^0([0,T]; W_0^{1,p'}(\Omega))$ according to the Sobolev embedding theorem, the values $v(x,0)$ and $v(x,T)$ are meaningful. We use some standard notation. We will denote any point in Q_T by $z = (x,t)$, and for $z_1 = (x_1,t_1)$, $z_2 = (x_2,t_2) \in Q_T$, the parabolic distance is defined by

$$\mathrm{dist}(z_1,z_2) = \max\{|x_1 - x_2|, |t_1 - t_2|^{1/2}\}.$$

For $r > 0$ and $z_0 = (x_0,t_0) \in Q_T$, we write

$$B_r(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < r\}, \quad Q_r(z_0) = B_r(x_0) \times (t_0, t_0 + r^2).$$

We also write the average of a function u on $Q_r(z_0)$ by

$$u_{z_0,r} = \frac{1}{|Q_r(z_0)|} \iint_{Q_r(z_0)} u \, dx \, dt,$$

where $|Q_r(z_0)| = r^2 |B_r(x_0)|$ and $|B_r(x_0)|$ is the volume of $B_r(x_0)$.

We are in a position to state the main theorem.

Theorem 2.2. *Let $1 < p < \infty$, $u \in L^p(0, T; W^{1,p}(\Omega))$ be a weak solution of (2.1), and $z_0 = (x_0, t_0) \in Q_T$ with $Q_{2r}(z_0) \subset Q_T$. Then there exists a constant $C >$ independent of r and u such that*

$$\iint_{Q_r(z_0)} |u - u_{z_0,r}|^p dz \le C r^p \iint_{Q_{2r}(z_0)} |\nabla u|^p dz,$$

where ∇u denotes the gradient of u with respect to the space variable x.

3. Proof of Theorem 2.2

In this section, we use the technique based on [10, Lemmas 3 and 4]. We assume that $u \in L^p(0, T; W^{1,p}(\Omega))$ is a weak solution of (2.1) and $z_0 = (x_0, t_0) \in Q_T$ and $Q_{2r}(z_0) \subset Q_T$. By a translation, we assume that $z_0 = (0, 0)$, and we write $Q_r = Q_r(0, 0)$ and $B_r = B_r(0)$ for the brevity of notation. Choose a smooth cut-off function $\sigma(x)$ such that

$$\sigma(x) = \begin{cases} 1 & \text{if } |x| \le r, \\ 0 & \text{if } |x| \ge 2r, \end{cases} \tag{3.1}$$

$0 \le \sigma(x) \le 1$, $|\nabla \sigma(x)| \le 2/r$, and $\sigma(x) = \sigma(|x|)$ is monotone decreasing with respect to $|x|$. For $0 < s \le t < r^2$, let $\chi_{[s,t]}(\tau)$ be the characteristic function of $[s, t]$ and define

$$u_r^\sigma = \frac{\iint_{Q_{2r}} u\sigma dz}{\iint_{Q_{2r}} \sigma dz} = \frac{\iint_{Q_{2r}} u\sigma dz}{(2r)^2 \int_{B_{2r}} \sigma \, dx},$$

and

$$u_{r,t}^\sigma = \frac{\int_{Q_{2r}(t)} u\sigma \, dx}{\int_{Q_{2r}(t)} \sigma \, dx},$$

where $Q_r(t) = \{(x, t); |x| < r\}$. Define

$$\phi(x, \tau) = \sigma(x) \chi_{[s,t]}(\tau) \operatorname{sign}(u_{r,t}^\sigma - u_{r,s}^\sigma) |u_{r,t}^\sigma - u_{r,s}^\sigma|^{p-1},$$

where

$$\operatorname{sign}(a) = \begin{cases} 1 & \text{for } a > 0, \\ 0 & \text{for } a = 0, \\ -1 & \text{for } a < 0. \end{cases}$$

Using the Steklov averaging to approximate ϕ and then taking the limit, we can use $\phi(x, \tau)$ as a test function of (2.2) (cf. [10] or [11]), so formally

$$\iint_{Q_T} \Big(-u\phi_\tau + \sum_{i,j=1}^n a_{ij}(x, \tau) u_{x_j} \phi_{x_i} \Big) \, dx \, d\tau = 0. \tag{3.2}$$

We use the following two lemmas. The first is a weighted Poincaré inequality with respect to the space variable.

Lemma 3.1. *Let the function σ be as in (3.1), $u(x) \in W^{1,p}(B_{2r})$ for some $1 \le p < \infty$ and define*
$$u_{2r,\sigma} = \frac{\int_{B_{2r}} u\sigma\,dx}{\int_{B_{2r}} \sigma\,dx}.$$
Then we have
$$\int_{B_{2r}} |u - u_{2r,\sigma}|^p \sigma\,dx \le C(n,p) r^p \int_{B_{2r}} |\nabla u|^p \sigma\,dx.$$

For a proof of the above lemma, see Lieberman [8, Lemma 6.12].

Lemma 3.2. *Let $1 \le p < \infty$ and $u \in L^p(Q_r(z_0))$. Then we have*
$$\iint_{Q_r(z_0)} |u - u_{z_0,r}|^p dz \le 2^p \iint_{Q_r(z_0)} |u - L|^p dz$$
for any $L \in \mathbb{R}$.

Proof. By the triangle inequality, for any $L \in \mathbb{R}$, we have
$$\begin{aligned}
&\Big(\iint_{Q_r(z_0)} |u - u_{z_0,r}|^p dz\Big)^{1/p} \\
&\le \Big(\iint_{Q_r(z_0)} |u - L|^p dz\Big)^{1/p} + \Big(\iint_{Q_r(z_0)} |u_{z_0,r} - L|^p dz\Big)^{1/p}.
\end{aligned}$$
By the definition of $u_{z_0,r}$ and the Hölder inequality,
$$\begin{aligned}
&\iint_{Q_r(z_0)} |u_{z_0,r} - L|^p dz = |u_{z_0,r} - L|^p |Q_r(z_0)| \\
&= \Big|\frac{1}{|Q_r(z_0)|} \iint_{Q_r(z_0)} (u - L) dz\Big|^p |Q_r(z_0)| \\
&\le |Q_r(z_0)|^{1-p} \Big[\Big(\iint_{Q_r(z_0)} |u - L|^p dz\Big)^{1/p} |Q_r(z_0)|^{1/p'}\Big]^p \\
&= |Q_r(z_0)|^{1-p+p/p'} \iint_{Q_r(z_0)} |u - L|^p dz \\
&= \iint_{Q_r(z_0)} |u - L|^p dz.
\end{aligned}$$
Here we used $1 - p + p/p' = 0$. Thus we get the conclusion. ☐

We shall estimate each term of (3.2). We have
$$\begin{aligned}
&\iint_{Q_T} -u\phi_\tau\,dx\,d\tau \\
&= \iint_{Q_T} u_\tau(x,\tau)\sigma(x)\chi_{[s,t]}(\tau)\operatorname{sign}(u^\sigma_{r,t} - u^\sigma_{r,s})|u^\sigma_{r,t} - u^\sigma_{r,s}|^{p-1} dx d\tau \\
&= \int_s^t \int_\Omega u_\tau(x,\tau)\sigma(x) dx d\tau \operatorname{sign}(u^\sigma_{r,t} - u^\sigma_{r,s})|u^\sigma_{r,t} - u^\sigma_{r,s}|^{p-1} \\
&= \Big[\int_{Q_{2r}(t)} u(x,t)\sigma\,dx - \int_{Q_{2r}(s)} u(x,s)\sigma\,dx\Big] \\
&\quad \times \operatorname{sign}(u^\sigma_{r,t} - u^\sigma_{r,s})|u^\sigma_{r,t} - u^\sigma_{r,s}|^{p-1}
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sigma\,dx\,(u^\sigma_{r,t} - u^\sigma_{r,s})\mathrm{sign}(u^\sigma_{r,t} - u^\sigma_{r,s})|u^\sigma_{r,t} - u^\sigma_{r,s}|^{p-1} \\
&= \int_\Omega \sigma\,dx\,|u^\sigma_{r,t} - u^\sigma_{r,s}|^p.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\iint_{Q_T} -u\phi_\tau\,dx\,d\tau &= \int_\Omega \sigma(x)dx|u^\sigma_{r,t} - u^\sigma_{r,s}|^p \\
&\ge \int_{B_r} \sigma(x)dx|u^\sigma_{r,t} - u^\sigma_{r,s}|^p \\
&\ge c_0 r^n |u^\sigma_{r,t} - u^\sigma_{r,s}|^p
\end{aligned} \tag{3.3}
$$

for some $c_0 > 0$.

On the other hand, we put

$$
I = -\sum_{i,j=1}^n \iint_{Q_T} a_{ij}(x,\tau)u_{x_j}(x,\tau)\phi_{x_i}(x,\tau)dxd\tau.
$$

Since $a_{ij} \in L^\infty(Q_T)$, we have

$$
|I| \le C\int_s^t \int_{B_{2r}} |\nabla u(x,\tau)||\nabla\sigma(x)||u^\sigma_{r,t} - u^\sigma_{r,s}|^{p-1}\,dx\,d\tau.
$$

If we write $|\nabla\sigma(x)| = |\nabla\sigma(x)|^{1/p}|\nabla\sigma(x)|^{1/p'}$, and apply the Hölder inequality, we have

$$
\begin{aligned}
|I| \le &\Big(\int_s^t \int_{B_{2r}} |\nabla u(x,\tau)^p|\nabla\sigma(x)|\,dx\,d\tau\Big)^{1/p} \\
&\times \Big(\int_s^t \int_{B_{2r}} |\nabla\sigma(x)||u^\sigma_{r,t} - u^\sigma_{r,s}|^p\,dx\,d\tau\Big)^{1/p'}.
\end{aligned}
$$

Here we use the Young inequality (cf. Evans [2, p. 706]):

$$
ab \le \delta a^{p'} + (\delta p')^{-p/p'}p^{-1}b^p
$$

for any $a, b \ge 0$ and any $\delta > 0$. Using this inequality, we have

$$
\begin{aligned}
|I| \le &C\delta\int_s^t \int_{B_{2r}} |\nabla\sigma(x)||u^\sigma_{r,t} - u^\sigma_{r,s}|^p\,dx\,d\tau \\
&+ C(p')^{-p/p'}p^{-1}\delta^{-p/p'}\int_s^t \int_{B_{2r}} |\nabla u(x,\tau)|^p|\nabla\sigma(x)|\,dx\,d\tau.
\end{aligned}
$$

If we put $\varepsilon = r\delta$, using $t - s \le r^2$, we have

$$
\begin{aligned}
|I| &\le 2C\varepsilon r^{-2}(t-s)|B_{2r}||u^\sigma_{r,t} - u^\sigma_{r,s}|^p \\
&\quad + 2C(p')^{-p/p'}p^{-1}\varepsilon^{-p/p'}r^{-1+p/p'}\int_s^t \int_{B2r} |\nabla u(x,\tau)|^p\,dx\,d\tau \\
&\le C'\varepsilon r^n|u^\sigma_{r,t} - u^\sigma_{r,s}|^p + C(p,\varepsilon)r^{p-2}\int_s^t \int_{B2r} |\nabla u(x,\tau)|^p\,dx\,d\tau
\end{aligned} \tag{3.4}
$$

From (3.2), (3.3) and (3.4), if we choose $\varepsilon > 0$ small enough, we have

$$
|u^\sigma_{r,t} - u^\sigma_{r,s}|^p \le Cr^{-n+p-2}\int_s^t \int_{B_{2r}} |\nabla u(x,\tau)|^p\,dx\,d\tau. \tag{3.5}
$$

By Lemma 3.2 and an elementary inequality, we have

$$\begin{aligned}\iint_{Q_r} |u-u_{z_0,r}|^p\,dx\,dt &\le 2^p \iint_{Q_r} |u-u_r^\sigma|^p\,dx\,dt\\ &\le 4^p \iint_{Q_r} |u-u_{r,t}^\sigma|^p\,dx\,dt + 4^p \iint_{Q_r} |u_{r,t}^\sigma - u_r^\sigma|^p\,dx\,dt.\end{aligned}$$

Since $\sigma \ge 0$ and $\sigma(x)=1$ on B_r, we have

$$\iint_{Q_r} |u-u_{r,t}^\sigma|^p\,dx\,dt = \int_0^{r^2}\int_{B_r} |u-u_{r,t}^\sigma|^p\,dx\,dt \le \iint_{Q_{2r}} |u-u_{r,t}^\sigma|^p \sigma\,dx\,dt$$

According to Lemma 3.1,

$$\begin{aligned}\int_{B_r} |u(x,t)-u_{r,t}^\sigma|^p dx &= \int_{B_{2r}} \Big|u(x,t) - \frac{\int_{Q_{2r}(t)} u(y,t)\sigma(y)dy}{\int_{Q_{2r}(t)}\sigma(y)dy}\Big|^p \sigma(x)dx\\ &\le C(n,p)r^p \int_{B_{2r}} |\nabla u(x,t)|^p \sigma(x)dx.\end{aligned}$$

Hence we see that

$$\iint_{Q_r} |u-u_{r,t}^\sigma|^p\,dx\,dt \le Cr^p \iint_{Q_{2r}} |\nabla u(x,t)|^p\,dx\,dt. \tag{3.6}$$

Since

$$\begin{aligned}u_r^\sigma &= \frac{\iint_{Q_{2r}} u\sigma dz}{(2r)^2\int_{B_{2r}}\sigma\,dx}\\ &= \frac{\int_0^{(2r)^2}\int_{B_{2r}} u(x,s)\sigma(x)\,dx\,ds}{(2r)^2\int_{B_{2r}}\sigma\,dx}\\ &= \frac{\int_0^{(2r)^2}\int_{Q_{2r}(s)} u(x,s)\sigma(x)\,dx\,ds}{(2r)^2\int_{Q_{2r}(s)}\sigma\,dx}\\ &= \frac{1}{(2r)^2}\int_0^{(2r)^2} u_{r,s}^\sigma ds\\ &= \frac{1}{|Q_{2r}|}\iint_{Q_{2r}} u_{r,s}^\sigma\,dy\,ds,\end{aligned}$$

we have

$$u_{r,t}^\sigma - u_r^\sigma = \frac{1}{|Q_{2r}|}\iint_{Q_{2r}} (u_{r,t}^\sigma - u_{r,s}^\sigma)\,dy\,ds.$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned}|u_{r,t}^\sigma - u_r^\sigma|^p &= |Q_{2r}|^{-p}\Big|\iint_{Q_{2r}} (u_{r,t}^\sigma - u_{r,s}^\sigma)\,dy\,ds\Big|^p\\ &\le |Q_{2r}|^{-p}\Big[\Big(\iint_{Q_{2r}} |u_{r,t}^\sigma - u_{r,s}^\sigma|^p\,dy\,ds\Big)^{1/p}|Q_{2r}|^{1/p'}\Big]^p\\ &= |Q_{2r}|^{-p+p/p'}\iint_{Q_{2r}} |u_{r,t}^\sigma - u_{r,s}^\sigma|^p\,dy\,ds\\ &= |Q_{2r}|^{-1}\iint_{Q_{2r}} |u_{r,t}^\sigma - u_{r,s}^\sigma|^p\,dy\,ds.\end{aligned}$$

Thus using (3.5), we have

$$
\begin{aligned}
&\iint_{Q_r} |u^\sigma_{r,t} - u^\sigma_r|^p \,dx\,dt \\
&\le \iint_{Q_r} |Q_{2r}|^{-1} \iint_{Q_{2r}} |u^\sigma_{r,t} - u^\sigma_{r,s}|^p \,dy\,ds\,dx\,dt \\
&\le \iint_{Q_r} |Q_{2r}|^{-1} \iint_{Q_{2r}} r^{-n+p-2} \int_s^t \int_{B_{2r}} |\nabla u(x',\tau)|^p dx' d\tau \,dy\,ds\,dx\,dt \\
&\le |Q_{2r}|^{-1} \iint_{Q_r} \iint_{Q_{2r}} r^{-n+p-2} \int_s^t \int_{B_{2r}} |\nabla u(x',\tau)|^p dx' d\tau \,dy\,ds\,dx\,dt \\
&\le |Q_{2r}|^{-1} |Q_r| |Q_{2r}| r^{-n+p-2} \int_s^t \int_{B_{2r}} |\nabla u(x',\tau)|^p dx' d\tau \\
&= C r^p \int_{Q_{2r}} |\nabla u(x',\tau)|^p dx' d\tau.
\end{aligned}
$$

Here we used the fact that $|Q_r| = \omega_n r^{n+2}$ where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. So we get

$$
\iint_{Q_r} |u - u_{z_0,r}|^p \,dx\,dt \le C r^p \iint_{Q_{2r}} |\nabla u(x,t)|^p \,dx\,dt.
$$

This completes the proof of Theorem 2.2.

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