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EXISTENCE OF BOUNDED SOLUTIONS FOR QUASILINEAR PARABOLIC SYSTEMS WITH QUADRATIC GROWTH

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Abstract. Assuming the existence of an upper and a lower solution, we prove the existence of at least one bounded solution of a quasilinear parabolic systems, with nonlinear second member having a quadratic growth with respect to the gradient of the solution.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, with boundary $\partial\Omega$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Consider the quasilinear parabolic system

$$\begin{gathered}\frac{\partial u^1}{\partial t} - \operatorname{div}(A(u)\nabla u^1) = G^1(u,\nabla u) + F(u,\nabla u)\cdot\nabla u^1 \quad \text{in } Q,\\ \frac{\partial u^2}{\partial t} - \operatorname{div}(A(u)\nabla u^2) = G^2(u,\nabla u) + F(u,\nabla u)\cdot\nabla u^2 \quad \text{in } Q,\\ u^1(x,t) = u^2(x,t) = 0 \quad \text{on} \quad \Sigma = \partial\Omega\times(0,T),\\ u^1(x,0) = u_0^1(x), \quad u^2(x,0) = u_0^2(x) \quad \text{in } \Omega.\end{gathered} \tag{1.1}$$

where $u_0^1(x), u_0^2(x) \in L^\infty(\Omega)$ and

$$\operatorname{div}(A(u)\nabla u^\gamma) = \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(A_{i,j}(u)\frac{\partial u^\gamma}{\partial x_j}\Big), \quad \gamma = 1,2$$

with $A_{i,j} : Q\times\mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions which satisfy the following assumptions

$$\exists\alpha > 0,\ \forall\xi\in\mathbb{R}^N,\ \forall s\in\mathbb{R}^2, \sum_{i,j=1}^N A_{i,j}(x,t,s)\xi_i\xi_j \ge \alpha|\xi|^2 \text{ a.e. } (x,t)\in Q; \tag{1.2}$$

and there exists $\varrho > 0$ such that for all $s \in \mathbb{R}^2$,

$$|A_{i,j}(x,t,s)| \le \varrho \quad \text{a.e. } (x,t)\in Q. \tag{1.3}$$

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The functions $G^1, G^2 : Q \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are Carathéodory functions which satisfy the quadratic growth assumptions: for all $s \in \mathbb{R}^2$ and all $\xi = (\xi^1, \xi^2) \in \mathbb{R}^{2N}$,

$$|G^1(x,t,s,\xi)| \le C_0 + C_2|\xi^1|^2 + \eta|\xi^2|^2 \quad \text{a.e. } (x,t) \in Q; \tag{1.4}$$

$$|G^2(x,t,s,\xi)| \le C_0 + C_2[|\xi^1|^2 + |\xi^2|^2] \quad \text{a.e. } (x,t) \in Q. \tag{1.5}$$

$F : Q \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}^N$ is a Carathéodory function satisfying the sublinearity assumption: for all $s \in \mathbb{R}^2$ and all $\xi = (\xi^1, \xi^2) \in \mathbb{R}^{2N}$,

$$|F(x,t,s,\xi)| \le C_3 + C_4|\xi| \quad \text{a.e. } (x,t) \in Q. \tag{1.6}$$

where C_0, C_2, C_3, C_4 and η are positive constants, η being small enough. Several papers concern mainly the regularity properties of solution of elliptic and parabolic system; see e.g. [[10], [16]–[23], [33], [38], [40]–[42]]. In the elliptic case with quadratic growth, in [35], the author studies a unilalteral problem for L^1-data, in which the truncate function is used instead of upper and lower solutions. In [5]–[7], the authors study renormalized or entropic parabolic systems to overcame the lake of regularities of solutions.

Others articles study the existence and regularity of a solution using as a main tool some regularity arguments and strong maximum principles see e.g. [12]–[14]. Others extend the weak maximum principles to a special class of systems of parabolic equations, the so-called weakly coupled systems(it is coupled only through the terms which are not differentiated, each equation containing derivatives of just one component) see e.g. [11, 28, 34]. Others articles study the existence of a solution using monotony arguments (the method of upper and lower solutions) see e.g. [3, 9, 24, 26, 30, 32, 37] and the book [25] and the references therein. In [1, 2, 26] the authors have extended the method of classical upper-lower methods for elliptic and parabolic systems without the assumption of quasi-monotonicity. The Growing conditions (1.4) and (1.5) imposed on G^1 and G^2 and the growth condition (1.6) imposed on F are sufficient to have an uniform estimate of u in the space $\big(L^2(0,T,H_0^1(\Omega))\big)^2$. Note that if the condition (1.4) is the same as (1.5) we need to add a condition of type $C_2\|u\|_\infty < \alpha$ to have a uniform estimate of u in the space $\big(L^2(0,T,H_0^1(\Omega))\big)^2$ and also an uniform estimation in space $\mathcal{C}^\delta$ see e.g. [21] for more detailed on this subject.

When $\eta = 0$, the system has a triangular structure with respect to the quadratic terms, and the system can be decoupled. When η is positive and small, in [31] the author establishes the existence of solutions of the associated elliptic systems.

2. Statement of the main result

Theorem 2.1. *Under hypotheses* (1.2)–(1.6) *and the smallness condition* (3.20) *for η, there exists at least one solution u of system* (1.1).

A solution to (1.1) must be interpreted in the weak sense:

$$u \in \big(L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)\big)^2, \quad \frac{\partial u}{\partial t} \in \big(L^2(0,T;H^{-1}(\Omega)) + L^1(Q)\big)^2,$$

such that for all $v \in (L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)^2$, $\frac{\partial v}{\partial t} = \beta_1 + \beta_2$ where $\beta_1 \in \big(L^2(0,T;H^{-1}(\Omega))\big)^2$ and $\beta_2 \in \big(L^1(Q)\big)^2$,

$$\int_\Omega u^\gamma(T)v^\gamma(T)dx - \int_\Omega u^\gamma(0)v^\gamma(0)dx - \int_0^T \langle \beta_1^\gamma, u^\gamma \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} dt$$

$$
\begin{aligned}
&-\int_Q \beta_2^\gamma u^\gamma dx\,dt + \int_Q A(u)\nabla u^\gamma \cdot \nabla v^\gamma dx\,dt \\
&= \int_Q G^\gamma(u,\nabla u) v^\gamma dx\,dt + \int_Q F(u,\nabla u)\cdot \nabla u^\gamma v^\gamma dx\,dt, \quad \text{for } \gamma = 1,2
\end{aligned}
$$

The proof of Theorem 2.1] will be performed in three steps: firstly prove that the approximated system of (1.1) admits at least one bounded solution, denoted u_ε; secondly we prove an $(L^2(0,T;H_0^1(\Omega)))^2$-estimate for u_ε, then the strong convergence in $(L^2(0,T;H_0^1(\Omega)))^2$ of u_ε; and finally we pass to the limit in the approximated system of (1.1).

3. Approximation

According to [8, 31, 35], we regularize the nonlinear terms to be bounded, for that we consider now the approximated system of (1.1):

$$
\begin{gathered}
\frac{\partial u_\varepsilon^1}{\partial t} - \operatorname{div}(A(u_\varepsilon)\nabla u_\varepsilon^1) = G_\varepsilon^1(u_\varepsilon,\nabla u_\varepsilon) + F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^1 \quad \text{in } Q,\\
\frac{\partial u_\varepsilon^2}{\partial t} - \operatorname{div}(A(u_\varepsilon)\nabla u_\varepsilon^2) = G_\varepsilon^2(u_\varepsilon,\nabla u_\varepsilon) + F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^2 \quad \text{in } Q,\\
u_\varepsilon^1(x,t) = 0,\ u_\varepsilon^2(x,t) = 0 \quad \text{on } \Sigma,\\
u_\varepsilon^1(x,0) = u_0^1(x),\ u_\varepsilon^2(x,0) = u_0^2(x) \quad \text{in } \Omega,
\end{gathered} \tag{3.1}
$$

where $\varepsilon > 0$, and $G_\varepsilon^1(x,t,s,\xi),\ G_\varepsilon^2(x,t,s,\xi) : Q\times\mathbb{R}^2\times\mathbb{R}^{2N} \to \mathbb{R}$ and $F_\varepsilon(x,s,\xi) : Q\times\mathbb{R}^2\times\mathbb{R}^{2N}\to\mathbb{R}^N$ are Carathéodory functions such that:

$$
\begin{gathered}
G_\varepsilon^1(x,t,s,\xi) = \frac{G^1(x,t,s,\xi)}{1+\varepsilon|G^1(x,t,s,\xi)|},\\
G_\varepsilon^2(x,t,s,\xi) = \frac{G^2(x,t,s,\xi)}{1+\varepsilon|G^2(x,t,s,\xi)|},\\
F_\varepsilon(x,t,s,\xi) = \frac{F(x,t,s,\xi)}{1+\varepsilon|F(x,t,s,\xi)||\xi|}.
\end{gathered} \tag{3.2}
$$

Noting that the functions G_ε^1, G_ε^2 and F_ε satisfy the following conditions: for all $s\in\mathbb{R}^2$, all $\xi = (\xi^1,\xi^2)\in\mathbb{R}^{2N}$, and for a.e. $(x,t)\in Q$, we have

$$
|G_\varepsilon^1(x,t,s,\xi)| \le \frac{1}{\varepsilon},\quad |G_\varepsilon^2(x,t,s,\xi)| \le \frac{1}{\varepsilon},\quad |F_\varepsilon(x,t,s,\xi)\xi^\gamma| \le \frac{1}{\varepsilon}, \tag{3.3}
$$

$$
|G_\varepsilon^1(x,t,s,\xi)| \le |G^1(x,t,s,\xi)|,\quad |G_\varepsilon^2(x,t,s,\xi)| \le |G^2(x,t,s,\xi)|, \tag{3.4}
$$

$$
|F_\varepsilon(x,t,s,\xi)| \le |F(x,t,s,\xi)|. \tag{3.5}
$$

Since the right hand side of each equation in (3.1) is bounded by $\frac{2}{\varepsilon}$, then by applying the De Giorgi iteration technique [43, theorem 4.2.1], for each $\gamma = 1,2$, we have

$$
\sup_Q u_\varepsilon^\gamma \le \sup_{\partial Q} u_\varepsilon^\gamma + C\|H_\varepsilon^\gamma\|_{L^\infty(Q)} \tag{3.6}
$$

where $H_\varepsilon^\gamma = G_\varepsilon^\gamma(u_\varepsilon,\nabla u_\varepsilon) + F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^\gamma$ and C is a constant depending only on N and Ω. Hence

$$
\|u_\varepsilon^\gamma\|_{L^\infty(Q)} \le \|u_0^\gamma\|_{L^\infty(Q)} + \frac{2C}{\varepsilon} = C_\varepsilon. \tag{3.7}
$$

Unfortunately the estimate (3.7) is not sufficient to obtain a uniform estimate of a possible solution of the regularization system (3.1) in the space $\big(L^2(0,T;H_0^1(\Omega))\big)^2$ (see the proof of Lemma 4.1). In fact, we will need the uniform estimate

$$\|u_\varepsilon^\gamma\|_{L^\infty(Q)} \le M \quad \text{for } \gamma = 1, 2,$$

where M is positive constant independent of ε, this is the main goal of the next step. We will define the upper and lower solution of regularization system (3.1) and we will consider an auxiliary modified system whose solution is between the upper and lower solutions and satisfy the system (3.1). First, we need to state some notations. For given $v = (v^1, v^2)$ and $w = (w^1, w^2)$, we say $v \le w$ if $v^1 \le w^1$ and $v^2 \le w^2$. We consider also this notation $[v]_w^1 = (w^1, v^2)$ and $[v]_w^2 = (v^1, w^2)$.

Definition 3.1. Let φ and $\phi \in \big(L^\infty(0,T;W^{1,\infty}(\Omega))\big)^2$ such that $\frac{\partial\varphi}{\partial t}$ and $\frac{\partial\phi}{\partial t} \in \big(L^2(0,T;H^{-1}(\Omega))\big)^2$. Then φ and ϕ are called ordered coupling weak upper and lower solution of (3.1), if $\phi \le \varphi$ and for all $v \in \big(L^2(0,T;H_0^1(\Omega))\big)^2$ such that $\phi \le v \le \varphi$, a.e. in Q, they satisfy

$$\begin{gathered}
\frac{\partial\varphi^1}{\partial t} - \operatorname{div}(A([v]_\varphi^1)\nabla\varphi^1) \ge G_\varepsilon^1([v]_\varphi^1, \nabla[v]_\varphi^1) + F_\varepsilon([v]_\varphi^1, \nabla[v]_\varphi^1)\cdot\nabla\varphi^1 \quad \text{in } Q,\\
\frac{\partial\varphi^2}{\partial t} - \operatorname{div}(A([v]_\varphi^1)\nabla\varphi^2) \ge G_\varepsilon^2\big([v]_\varphi^2, \nabla[v]_\varphi^2\big) + F_\varepsilon([v]_\varphi^2, \nabla[v]_\varphi^2)\cdot\nabla\varphi^2 \quad \text{in } Q,\\
\varphi \ge 0, \quad \text{on } \Sigma,\\
\varphi(0) \ge u_0, \quad \text{in } \Omega,
\end{gathered} \tag{3.8}$$

and

$$\begin{gathered}
\frac{\partial\phi^1}{\partial t} - \operatorname{div}(A([v]_\phi^1)\nabla\phi^1) \le G_\varepsilon^1([v]_\phi^1, \nabla[v]_\phi^1) + F_\varepsilon([v]_\phi^1, \nabla[v]_\phi^1)\cdot\nabla\phi^1 \quad \text{in } Q,\\
\frac{\partial\phi^2}{\partial t} - \operatorname{div}(A([v]_\phi^2)\nabla\phi^2) \le G_\varepsilon^2([v]_\phi^2, \nabla[v]_\phi^2) + F_\varepsilon([v]_\phi^2, \nabla[v]_\phi^2)\cdot\nabla\phi^2 \quad \text{in } Q,\\
\phi \le 0, \quad \text{on } \Sigma,\\
\phi(0) \le u_0, \quad \text{in } \Omega.
\end{gathered} \tag{3.9}$$

Remark 3.2. For applications to stochastic differential games (see for example, the [14, condition (2.7)]) we can assume that G^1 and G^2 satisfy the following conditions: There exists $K > 0$ such that for all $s \in \mathbb{R}^2$ and all $\xi = (\xi^1, \xi^2) \in \mathbb{R}^{2N}$, we have

$$|G^1(x,t,s,\xi)|_{\xi^1=0} \le K, \quad |G^2(x,t,s,\xi)|_{\xi^2=0} \le K \quad \text{a.e. } (x,t) \in Q. \tag{3.10}$$

According to this conditions, it is easy to see that we can take as upper and lower solution the constant functions with respect to the space variable $\varphi = (\varphi^1, \varphi^2) = (Kt + \|u_0^1\|_{L^\infty(\Omega)}, Kt + \|u_0^2\|_{L^\infty(\Omega)})$ and $\phi = (\phi^1, \phi^2) = (-Kt - \|u_0^1\|_{L^\infty(\Omega)}, -Kt - \|u_0^2\|_{L^\infty(\Omega)})$.

Let us extend the solution of problem (3.1) by continuity as

$$\bar{u}_\varepsilon = (\bar{u_\varepsilon}^1, \bar{u_\varepsilon}^2), \quad \bar{u_\varepsilon}^\gamma = \begin{cases} \varphi^\gamma & \text{if } u_\varepsilon^\gamma \ge \varphi^\gamma,\\ u_\varepsilon^\gamma & \text{if } \phi^\gamma \le u_\varepsilon^\gamma \le \varphi^\gamma, \\ \phi^\gamma & \text{if } u_\varepsilon^\gamma \le \phi^\gamma. \end{cases} \quad \gamma = 1, 2. \tag{3.11}$$

Now we consider the approximated and the extended system

$$\begin{gathered}
\frac{\partial u_\varepsilon^1}{\partial t} - \operatorname{div}(\widehat{A}(u_\varepsilon)\nabla u_\varepsilon^1) = \widehat{G}_\varepsilon^1(u_\varepsilon, \nabla u_\varepsilon) + \widehat{F}_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot \nabla \bar{u_\varepsilon}^1 \quad \text{in } Q,\\
\frac{\partial u_\varepsilon^2}{\partial t} - \operatorname{div}(\widehat{A}(u_\varepsilon)\nabla u_\varepsilon^2) = \widehat{G}_\varepsilon^2(u_\varepsilon, \nabla u_\varepsilon) + \widehat{F}_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot \nabla \bar{u_\varepsilon}^2 \quad \text{in } Q,\\
u_\varepsilon^1(x,t) = 0,\ u_\varepsilon^2(x,t) = 0 \quad \text{on } \Sigma,\\
u_\varepsilon^1(x,0) = u_0^1(x),\ u_\varepsilon^2(x,0) = u_0^2(x) \quad \text{in } \Omega,
\end{gathered} \tag{3.12}$$

where

$$\widehat{G}_\varepsilon^\gamma(u_\varepsilon, \nabla u_\varepsilon) = G_\varepsilon^\gamma(\bar{u_\varepsilon}, \nabla \bar{u_\varepsilon}), \quad \widehat{F}_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) = F_\varepsilon(\bar{u_\varepsilon}, \nabla \bar{u_\varepsilon}), \quad \widehat{A}(u_\varepsilon) = A(\bar{u_\varepsilon}). \tag{3.13}$$

3.1. **Existence of solutions of system (3.16).**

Theorem 3.3. *If there exist an upper and lower solutions φ and ϕ of system* (3.1)*, then for all $\varepsilon > 0$, there exists at least one solution u_ε of system* (3.12) *which satisfies*

$$\begin{gathered}
u_\varepsilon \in \left(L^2(0,T;H_0^1(\Omega))\right)^2, \quad \frac{\partial u_\varepsilon}{\partial t} \in \left(L^2(0,T;H^{-1}(\Omega))\right)^2,\\
\phi \le u_\varepsilon \le \varphi, \quad \text{a.e. in } Q.
\end{gathered}$$

Proof of Theorem 3.3. In view of (3.3) and (3.13), an application of Schauder's fixed point theorem implies that system 3.12 has at least one solution for $\varepsilon > 0$ given. Let now u_ε be a solution of system (3.12) and let us show that $u_\varepsilon^\gamma \le \varphi^\gamma$ a.e. in Q for all $\gamma = 1, 2$. Using (3.8) and (3.12) (with $v = \bar{u_\varepsilon}$) for all $\gamma = 1, 2$ we obtain

$$\begin{gathered}
\frac{\partial(u_\varepsilon^\gamma - \varphi^\gamma)}{\partial t} - \operatorname{div}\Big(\widehat{A}(u_\varepsilon)\nabla(u_\varepsilon^\gamma - \varphi^\gamma)\Big) - \operatorname{div}\Big(\Big[\widehat{A}(u_\varepsilon) - A([\bar{u_\varepsilon}]_\varphi^\gamma)\Big]\nabla\varphi^\gamma\Big)\\
+\big[G_\varepsilon^\gamma([\bar{u_\varepsilon}]_\varphi^\gamma, \nabla[\bar{u_\varepsilon}]_\varphi^\gamma) - \widehat{G}_\varepsilon^\gamma(u_\varepsilon, \nabla u_\varepsilon)\big]\\
+\big[F_\varepsilon([\bar{u_\varepsilon}]_\varphi^\gamma, \nabla[\bar{u_\varepsilon}]_\varphi^\gamma)\cdot\nabla\varphi^\gamma - \widehat{F}_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla\bar{u_\varepsilon}^\gamma\big] \le 0 \quad \text{in } Q,\\
u_\varepsilon^\gamma - \varphi^\gamma \le 0, \quad \text{on } \Sigma,\\
(u_\varepsilon^\gamma - \varphi^\gamma)(x,0) \le 0, \quad \text{in } \Omega.
\end{gathered} \tag{3.14}$$

We multiply by $(u_\varepsilon^\gamma - \varphi^\gamma)^+$, using [8, lemma 2.4] we obtain

$$\begin{aligned}
&\frac{1}{2}\|(u_\varepsilon^\gamma - \varphi^\gamma)^+(T)\|_{L^2(\Omega)}^2 + \int_Q \widehat{A}(u_\varepsilon)\nabla(u_\varepsilon^\gamma - \varphi^\gamma)\cdot\nabla(u_\varepsilon^\gamma - \varphi^\gamma)^+ dx\,dt\\
&+\int_Q \Big[\widehat{A}(u_\varepsilon) - A([\bar{u_\varepsilon}]_\varphi^\gamma)\Big]\nabla\varphi^\gamma\cdot\nabla(u_\varepsilon^\gamma - \varphi^\gamma)^+ dx\,dt\\
&+\int_Q \Big[G_\varepsilon^\gamma([\bar{u_\varepsilon}]_\varphi^\gamma, \nabla[\bar{u_\varepsilon}]_\varphi^\gamma) - \widehat{G}_\varepsilon^\gamma(u_\varepsilon, \nabla u_\varepsilon)\Big](u_\varepsilon^\gamma - \varphi^\gamma)^+ dx\,dt\\
&+\int_Q \big[F_\varepsilon([\bar{u_\varepsilon}]_\varphi^\gamma, \nabla[\bar{u_\varepsilon}]_\varphi^\gamma)\cdot\nabla\varphi^\gamma - \widehat{F}_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla\bar{u_\varepsilon}^\gamma\big](u_\varepsilon^\gamma - \varphi^\gamma)^+ dx\,dt \le 0.
\end{aligned} \tag{3.15}$$

At the points where $(u_\varepsilon^\gamma - \varphi^\gamma)^+$ is not zero, we have in particular $u_\varepsilon^\gamma \ge \varphi^\gamma$, then $\widehat{G}_\varepsilon^\gamma(u_\varepsilon, \nabla u_\varepsilon) = G_\varepsilon^\gamma([\bar{u_\varepsilon}]_\varphi^\gamma, \nabla[\bar{u_\varepsilon}]_\varphi^\gamma)$ and $F_\varepsilon([\bar{u_\varepsilon}]_\varphi^\gamma, \nabla[\bar{u_\varepsilon}]_\varphi^\gamma)\cdot\nabla\varphi^\gamma = \widehat{F}_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla\bar{u_\varepsilon}^\gamma$ and $\widehat{A}(u_\varepsilon) = A([\bar{u_\varepsilon}]_\varphi^\gamma)$. On the other hand, $\nabla(u_\varepsilon^\gamma - \varphi^\gamma)^+ = 0$, a.e. on the set where $(u_\varepsilon^\gamma - \varphi^\gamma)^+ = 0$ and

$$\nabla(u_\varepsilon^\gamma - \varphi^\gamma)\cdot\nabla(u_\varepsilon^\gamma - \varphi^\gamma)^+ = \nabla(u_\varepsilon^\gamma - \varphi^\gamma)^+\cdot\nabla(u_\varepsilon^\gamma - \varphi^\gamma)^+ \quad \text{a.e. in } Q. \tag{3.16}$$

Using (1.2), (3.15) and (3.16) we obtain

$$\alpha\|(u_\varepsilon^\gamma-\varphi^\gamma)^+\|^2_{L^2(0,T;H_0^1(\Omega))}\le 0, \tag{3.17}$$

and so $(u_\varepsilon^\gamma-\varphi^\gamma)^+=0$ hence $u_\varepsilon^\gamma\le\varphi^\gamma$ a.e. in Q. In the same way we can show that $u_\varepsilon^\gamma\ge\phi^\gamma$ a.e. in Q for $\gamma=1,2$. Then, we have

$$\phi\le u_\varepsilon\le\varphi \quad \text{a.e. in } Q. \tag{3.18}$$

Finally, we have prove that (3.12) admits at least one solution u_ε satisfy (3.18). By the definition of $\widehat{G}_\varepsilon^\gamma(u_\varepsilon,\nabla u_\varepsilon)$, $\widehat{F}_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)$ and $\bar{u_\varepsilon}^\gamma$ for $\gamma=1,2$, it is clear that u_ε is solution of (3.1). Using (3.18) we obtain

$$\|u_\varepsilon^\gamma\|_{L^\infty(Q)}\le M, \quad \text{for } \gamma=1,2. \tag{3.19}$$

where M is positive constant independent of ε. We are now able to specify the smallness of the constant η which appears in the growth condition (1.4): we will assume that

$$0\le\eta\le\frac{C_2}{4\exp\left(\frac{64C_2M}{\alpha}\right)}. \tag{3.20}$$

□

4. $(L^2(0,T;H_0^1(\Omega)))^2$-ESTIMATE

We have show that the regularized system (3.1) admits at least one solution u_ε for all $\varepsilon>0$. In the next step we will establish the sufficient conditions which allow us to pass to the limit in system (3.1) to obtain a solution of (1.1). In this step we will need some needful lemmas. Firstly we consider the functions: $\varphi:\mathbb{R}\to\mathbb{R}$ defined by

$$\varphi(\tau)=\exp(\lambda\tau)+\exp(-\lambda\tau)-2, \quad \forall\tau\in\mathbb{R} \tag{4.1}$$

and $\psi:\mathbb{R}^2\to\mathbb{R}$ defined by

$$\psi(s^1,s^2)=\beta_1\varphi(s^1)+\beta_2\varphi(s^2) \quad \forall s=(s^1,s^2)\in\mathbb{R}^2, \tag{4.2}$$

and λ, μ and β are positive constants that we choose as

$$\lambda=\frac{2C_2}{\alpha}, \quad \beta_2=\frac{\beta_1}{2\exp(\lambda M)}, \quad \mu=\frac{C_3^2}{2\theta\alpha}+\frac{C_4^2}{2\theta\alpha} \tag{4.3}$$

with θ being a fixed number such that $0<\theta\le\frac{C_2^2\beta_1}{2\alpha\exp(\lambda M)}$.

Lemma 4.1. *For any u_ε, such that $|u_\varepsilon^\delta|\le M$ for any $\delta=1,2$, we have*

$$\rho_1=\alpha\beta_1\varphi''(u_\varepsilon^1)-C_2\beta_1|\varphi'(u_\varepsilon^1)|-C_2\beta_2|\varphi'(u_\varepsilon^2)|-\frac{\theta}{2}\ge\alpha_0, \tag{4.4}$$

$$\rho_2=\alpha\beta_2\varphi''(u_\varepsilon^2)-C_2\beta_2|\varphi'(u_\varepsilon^2)|-\eta\beta_1|\varphi'(u_\varepsilon^1)|-\frac{\theta}{2}\ge\alpha_0 \tag{4.5}$$

where α_0 is defined by

$$\alpha_0=\frac{C_2^2\beta_1}{4\alpha\exp(\lambda M)}. \tag{4.6}$$

Proof. Since

$$\forall\tau,\ |\tau|\le M, |\varphi'(\tau)|\le\lambda\exp(\lambda|\tau|)\le\lambda\exp(\lambda M),$$
$$\varphi''(\tau)=\lambda^2\left(\exp(\lambda\tau)+\exp(-\lambda\tau)\right)\ge\lambda^2\exp(\lambda|\tau|),$$

we obtain that, for any u_ε^δ with $|u_\varepsilon^\delta| \le M$, $\delta = 1, 2$,

$$\rho_1 \ge \alpha\beta_1\lambda^2 \exp(\lambda|u_\varepsilon^1|) - C_2\beta_1\lambda\exp(\lambda|u_\varepsilon^1|) - C_2\beta_2\lambda\exp(\lambda M) - \frac{\theta}{2} \tag{4.7}$$

Since, from (4.3), we have $\alpha\lambda > C_2$, the infinimum of the right hand side of (4.4) is achieved for $|u_\varepsilon^1| = 0$. We estimate from below the right hand side of (4.7) by

$$C_2\beta_1\lambda - C_2\beta_2\lambda\exp(\lambda M) - \frac{\theta}{2}. \tag{4.8}$$

In view of the values of λ, β_2 and θ given by (4.3), the right hand side of (4.8) is greater than

$$\frac{C_2^2\beta_1}{\alpha} - \frac{C_2^2\beta_1}{4\alpha\exp(\lambda M)} \ge \frac{C_2^2\beta_1}{4\alpha\exp(\lambda M)} = \alpha_0. \tag{4.9}$$

Inequality (4.5) can be proved by same way. $\square$

Lemma 4.2. *if $v \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ and $\frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$, then there exists a sequence w_j such that $w_j \in L^2(0,T;H_0^1(\Omega))$, $\frac{\partial w_j}{\partial t} \in L^2(0,T;H^1(\Omega))$, w_j bounded in $L^\infty(Q)$ and*

$$w_j \to v \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)), \tag{4.10}$$

$$\frac{\partial w_j}{\partial t} \to \frac{\partial v}{\partial t} \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega)), \tag{4.11}$$

$$w_j(0) \to v(0) = \quad \text{strongly in } L^2(\Omega). \tag{4.12}$$

The proof of the above Lemma is given by [8, lemma 2.2].

Proposition 4.3. *Assume that* (1.2)–(1.6) *and* (3.4) *hold. If the solutions u_ε of the approximated problem* (3.1) *satisfy* (3.19)*, then the solution u_ε remains bounded in $(L^2(0,T;H_0^1(\Omega)))^2$*

Proof. We consider the test functions

$$v_\varepsilon^\gamma = \beta_\gamma\varphi'(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)], \quad \text{for } \gamma = 1, 2.$$

Noting that

$$\nabla\psi(u_\varepsilon) = \sum_{\gamma=1}^{2} \beta_\gamma\varphi'(u_\varepsilon^\gamma)\nabla u_\varepsilon^\gamma. \tag{4.13}$$

We set

$$I_\varepsilon = \sum_{\gamma=1}^{2} \int_0^T \langle \frac{\partial u_\varepsilon^\gamma}{\partial t}, \beta_\gamma\varphi'(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)]\rangle dt.$$

We use v_ε^γ as test function in the γ-th equation of system (3.1) and sum up from $\gamma = 1$ to $\gamma = 2$, we obtain

$$\begin{aligned}
&I_\varepsilon + \sum_{\gamma=1}^{2}\int_Q A(u_\varepsilon)\nabla u_\varepsilon^\gamma \cdot \nabla u_\varepsilon^\gamma \beta_\gamma \varphi''(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)]dx\,dt \\
&+ \mu \sum_{\gamma=1}^{2}\int_Q A(u_\varepsilon)\nabla u_\varepsilon^\gamma \cdot \nabla \psi(u_\varepsilon)\beta_\gamma \varphi'(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)]dx\,dt \\
&= \sum_{\gamma=1}^{2}\int_Q G_\varepsilon^\gamma(u_\varepsilon,\nabla u_\varepsilon)\beta_\gamma \varphi'(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)]dx\,dt \\
&\quad + \sum_{\gamma=1}^{2}\int_Q F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot \nabla u_\varepsilon^\gamma \beta_\gamma \varphi'(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)]dx\,dt.
\end{aligned} \tag{4.14}$$

Firstly, we prove that

$$I_\varepsilon \geq -\frac{1}{\mu}\int_\Omega \exp\left[\mu\psi(u_0)\right]dx \tag{4.15}$$

Since (for $\gamma = 1,2$) $u_\varepsilon^\gamma \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ and $\frac{\partial u_\varepsilon^\gamma}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$, using lemma 4.2, there exists a sequence w_j^γ such that $w_j^\gamma \in L^2(0,T;H_0^1(\Omega))$, $\frac{\partial w_j^\gamma}{\partial t} \in L^2(0,T;H^1(\Omega))$, w_j^γ bounded in $L^\infty(Q)$ and

$$w_j^\gamma \to u_\varepsilon^\gamma \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)), \tag{4.16}$$

$$\frac{\partial w_j^\gamma}{\partial t} \to \frac{\partial u_\varepsilon^\gamma}{\partial t} \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega)), \tag{4.17}$$

$$w_j^\gamma(0) \to u_\varepsilon^\gamma(0) = u_0^\gamma \quad \text{strongly in } L^2(\Omega). \tag{4.18}$$

Then

$$\begin{aligned}
&\sum_{\gamma=1}^{2}\int_0^T \langle \frac{\partial w_j^\gamma}{\partial t}, \beta_\gamma \varphi'(w_j^\gamma)\exp[\mu\psi(w_j)]\rangle dt \\
&= \frac{1}{\mu}\int_\Omega \exp\left[\mu\psi(w_j(T))\right]dx - \frac{1}{\mu}\int_\Omega \exp\left[\mu\psi(w_j(0))\right]dx \\
&\geq -\frac{1}{\mu}\int_\Omega \exp\left[\mu\psi(w_j(0))\right]dx,
\end{aligned} \tag{4.19}$$

then, we obtain (4.15) by letting $j \to \infty$ in (4.19).

Using (4.15), the coercivity condition (1.2) and the growth conditions (3.4), (1.4) and (1.5) on G_ε^1, G_ε^2 we obtain:

$$\begin{aligned}
&\alpha \sum_{\gamma=1}^{2}\int_Q |\nabla u_\varepsilon^\gamma|^2 \beta_\gamma \varphi''(u_\varepsilon^\gamma)\exp[\mu\psi(u_\varepsilon)]dx\,dt \\
&+ \alpha\mu \int_Q |\nabla\psi(u_\varepsilon)|^2 \exp[\mu\psi(u_\varepsilon)]dx\,dt \\
&\leq C_0 \sum_{\gamma=1}^{2}\int_Q \beta_\gamma |\varphi'(u_\varepsilon^\gamma)|\exp[\mu\psi(u_\varepsilon)]dx\,dt \\
&\quad + \int_Q F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi(u_\varepsilon)\exp[\mu\psi(u_\varepsilon)]dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ C_2 \int_Q |\nabla u_\varepsilon^1|^2 \beta_1 |\varphi'(u_\varepsilon^1)| \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&+ \eta \int_Q |\nabla u_\varepsilon^2|^2 \beta_1 |\varphi'(u_\varepsilon^1)| \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&+ C_2 \int_Q |\nabla u_\varepsilon^1|^2 \beta_2 |\varphi'(u_\varepsilon^2)| \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&+ C_2 \int_Q |\nabla u_\varepsilon^2|^2 \beta_2 |\varphi'(u_\varepsilon^2)| \exp[\mu\psi(u_\varepsilon)] dx\, dt + \frac{1}{\mu} \int_\Omega \exp\left[\mu\psi(u_0)\right] dx.
\end{aligned} \tag{4.20}
$$

We estimate the second integral of the right hand side of (4.20) by using the growth conditions (1.6) and (3.4) on F_ε and Young's inequality, we obtain

$$
\begin{aligned}
&\int_Q F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\psi(u_\varepsilon) \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&\leq \int_Q \left[C_3 + C_4 |\nabla u_\varepsilon|\right] |\nabla\psi(u_\varepsilon)| \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&\leq \int_Q \big[\frac{\theta}{2} + \frac{C_3^2}{2\theta} |\nabla\psi(u_\varepsilon)|^2 + \frac{\theta}{2} |\nabla u_\varepsilon|^2 + \frac{C_4^2}{2\theta} |\nabla\psi(u_\varepsilon)|^2\big] \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&= \int_Q \big[\frac{\theta}{2} + (\frac{C_3^2}{2\theta} + \frac{C_4^2}{2\theta}) |\nabla\psi(u_\varepsilon)|^2 + \frac{\theta}{2} |\nabla u_\varepsilon|^2\big] \exp[\mu\psi(u_\varepsilon)] dx\, dt,
\end{aligned} \tag{4.21}
$$

using the hypothesis (4.3) on μ and (4.21), the inequality (4.20) becomes

$$
\begin{aligned}
&\int_Q |\nabla u_\varepsilon^1|^2 \exp[\mu\psi(u_\varepsilon)] \underbrace{[\alpha\beta_1\varphi''(u_\varepsilon^1) - C_2\beta_1|\varphi'(u_\varepsilon^1)| - C_2\beta_2|\varphi'(u_\varepsilon^2)| - \frac{\theta}{2}]}_{=\rho_1} dx\, dt \\
&+ \int_Q |\nabla u_\varepsilon^2|^2 \exp[\mu\psi(u_\varepsilon)] \underbrace{[\alpha\beta_2\varphi''(u_\varepsilon^2) - C_2\beta_2|\varphi'(u_\varepsilon^2)| - \beta_1\eta|\varphi'(u_\varepsilon^1)| - \frac{\theta}{2}]}_{=\rho_2} dx\, dt \\
&+ \int_Q |\nabla\psi(u_\varepsilon)|^2 \underbrace{[\alpha\mu - \frac{C_3^2}{2\theta} - \frac{C_4^2}{2\theta}]}_{\geq 0} \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&\leq C_0 \sum_{\gamma=1}^{2} \int_Q \beta_\gamma |\varphi'(u_\varepsilon^\gamma)| \exp[\mu\psi(u_\varepsilon)] dx\, dt + \int_Q \frac{\theta}{2} \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&\quad + \frac{1}{\mu} \int_\Omega \exp\left[\mu\psi(u_0)\right] dx.
\end{aligned} \tag{4.22}
$$

Employing (4.22) and the lemma (4.1) yields

$$
\begin{aligned}
&\alpha_0 \sum_{\gamma=1}^{2} \int_Q |\nabla u_\varepsilon^\gamma|^2 \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&\leq \int_Q \frac{\theta}{2} \exp[\mu\psi(u_\varepsilon)] dx\, dt + C_0 \sum_{\gamma=1}^{2} \int_Q \beta_\gamma |\varphi'(u_\varepsilon^\gamma)| \exp[\mu\psi(u_\varepsilon)] dx\, dt \\
&\quad + \frac{1}{\mu} \int_\Omega \exp\left[\mu\psi(u_0)\right] dx,
\end{aligned} \tag{4.23}
$$

using the facts that $\exp[\mu\psi(u_\varepsilon)] \geq 1$ and that u_ε satisfy (3.19) (which implies that $\psi(u_\varepsilon)$, is bounded in $L^\infty(Q)$) and $u_0 \in (L^\infty(\Omega))^2$, implies that u_ε is bounded in $(L^2(0,T;H_0^1(\Omega)))^2$. $\square$

Since by proposition 4.3 u_ε remains bounded in $(L^2(0,T;H_0^1(\Omega)))^2$, we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } (L^2(0,T;H_0^1(\Omega)))^2. \tag{4.24}$$

Proposition 4.4. *The sequence u_ε is relatively compact in $(L^2(Q))^2$.*

Proof. For $s > 0$ large enough, we have $L^1(\Omega) \subset H^{-s}(\Omega)$. Then $\frac{\partial u_\varepsilon}{\partial t}$ is bounded in $(L^1(0,T;H^{-s}(\Omega))^2$ and u_ε is bounded in $(L^2(0,T;H_0^1(\Omega)))^2$.

As $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega)$, the injection being compact, proposition 4.4 follows from a compacticity lemma of Aubin's type. Such a lemma can be found for example in [39, p. 271] or in [36, section 8, corollary 4)]. Therefore we can extract a subsequence, still denoted by u_ε such that if $\varepsilon \to 0$

$$u_\varepsilon \rightharpoonup u \quad \text{in } (L^2(0,T;H_0^1(\Omega)))^2. \tag{4.25}$$

By possibly extracting a subsequence, we can suppose, without loss of generality using proposition 4.4, that

$$u_\varepsilon \to u \quad \text{in } (L^2(Q))^2, \tag{4.26}$$

$$u_\varepsilon \to u \quad \text{a.e. in } Q \tag{4.27}$$

$\square$

5. Strong convergence in $(L^2(0,T;H_0^1(\Omega)))^2$

To pass to the limit as $\varepsilon \to 0$ in the nonlinearities $G_\varepsilon^1(u_\varepsilon, \nabla u_\varepsilon)$, $G_\varepsilon^2(u_\varepsilon, \nabla u_\varepsilon)$ and $F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ in system (3.1), we need the strong convergence of $u_\varepsilon \to u$ in $(L^2(0,T;H_0^1(\Omega)))^2$. This is our goal in this step.

Proposition 5.1. *Assume that* (1.2)–(1.6) *and* (3.4) *hold true. If the solutions u_ε of the approximated problem* (3.1) *satisfy* (3.19) *and* (4.25)–(4.27) *then u_ε converges strongly to u in $(L^2(0,T;H_0^1(\Omega)))^2$.*

We consider the functions $\bar\varphi : \mathbb{R} \to \mathbb{R}$ and $\bar\psi : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$\bar\varphi(\tau) = e^{\bar\lambda\tau} + e^{-\bar\lambda\tau} - 2, \quad \forall \tau \in \mathbb{R},$$
$$\bar\psi(s) = \bar\beta_1\bar\varphi(s^1) + \bar\beta_2\bar\varphi(s^2), \quad \forall s \in \mathbb{R}^2,$$

where $\bar\lambda$, $\bar\mu$, $\bar\beta_1$ and $\bar\beta_2$ are positive constants defined by

$$\bar\lambda = \frac{16C_2}{\alpha}, \quad \bar\beta_2 = \frac{\bar\beta_1}{2\exp(2\bar\lambda M)}, \quad \bar\mu = \frac{2}{\alpha}\Big(\frac{C_3^2}{\bar\theta} + \frac{C_4^2}{\bar\theta}\Big) \tag{5.1}$$

with $\bar\theta$ a fixed number such that $0 < \bar\theta \leq \frac{4C_2^2\bar\beta_1}{\alpha\exp(2\bar\lambda M)}$.

Lemma 5.2. *For any u_ε and u_ν, such that $|u_\varepsilon^\delta - u_\nu^\delta| \leq 2M$ for any $\delta = 1,2$, we have*

$$\begin{aligned}\bar\rho_1 &= \alpha\bar\beta_1\bar\varphi''(u_\varepsilon^1 - u_\nu^1)\\ &\quad - 4\underbrace{\big(2C_2\bar\beta_1|\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| + 2C_2\bar\beta_2|\bar\varphi'(u_\varepsilon^2 - u_\nu^2)| + \bar\theta\big)}_{=L_1(u_\varepsilon,u_\nu)} \geq \bar\alpha_0\end{aligned} \tag{5.2}$$

and

$$\bar\rho_2 = \alpha\bar\beta_2\bar\varphi''(u_\varepsilon^2-u_\nu^2) - 4\underbrace{\left(2C_2\bar\beta_2|\bar\varphi'(u_\varepsilon^2-u_\nu^2)| + 2\eta\bar\beta_1|\bar\varphi'(u_\varepsilon^1-u_\nu^1)| + \bar\theta\right)}_{=L_2(u_\varepsilon,u_\nu)} \ge \bar\alpha_0 \tag{5.3}$$

where

$$\bar\alpha_0 = \frac{16C_2^2\bar\beta_1}{\alpha\exp(2\lambda M)}. \tag{5.4}$$

The proof of the above lemma is the same of the proof of (4.1) where φ, λ, β_1, and β_2 are replaced by $\bar\varphi$, $\bar\lambda$, $\bar\beta_1$ and $\bar\beta_2$; and where C_2,η,θ and M are replaced by $8C_2, 8\eta, 8\theta$ and $2M$.

Proof of proposition 5.1. If ε and ν are two parameters, we write (3.1) as

$$\begin{gathered}
\begin{aligned}
&\frac{\partial(u_\varepsilon^1-u_\nu^1)}{\partial t} - \operatorname{div}(A(u_\varepsilon)\nabla(u_\varepsilon^1-u_\nu^1)) + \frac{\partial u_\nu^1}{\partial t} - \operatorname{div}(A(u_\varepsilon)\nabla u_\nu^1)\\
&= G_\varepsilon^1(u_\varepsilon,\nabla u_\varepsilon) + F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^1 \quad \text{in } Q,\\
&\frac{\partial(u_\varepsilon^2-u_\nu^2)}{\partial t} - \operatorname{div}(A(u_\varepsilon)\nabla(u_\varepsilon^2-u_\nu^2)) + \frac{\partial u_\nu^2}{\partial t} - \operatorname{div}(A(u_\varepsilon)\nabla u_\nu^2)\\
&= G_\varepsilon^2(u_\varepsilon,\nabla u_\varepsilon) + F_\varepsilon(u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^2 \quad \text{in } Q,
\end{aligned}\\
u_\varepsilon^1-u_\nu^1=0, \quad u_\varepsilon^2-u_\nu^2=0, \quad \text{on } \Sigma,\\
(u_\varepsilon^1-u_\nu^1)(0)=0, \quad (u_\varepsilon^2-u_\nu^2)(0)=0, \quad \text{in } \Omega.
\end{gathered} \tag{5.5}$$

We consider the test functions

$$\bar v_{\varepsilon\nu}^\gamma = \bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)], \quad \gamma=1,2$$

Noting that

$$\nabla\bar\psi(u_\varepsilon-u_\nu) = \sum_{\gamma=1}^2 \bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\nabla(u_\varepsilon^\gamma-u_\nu^\gamma) \tag{5.6}$$

Using $\bar v_{\varepsilon\nu}^\gamma$ as test function in the γ-th equation of system (5.5) and summing from $\gamma=1$ to $\gamma=2$, we obtain

$$\begin{aligned}
&\underbrace{\sum_{\gamma=1}^2\int_0^T\langle\frac{\partial(u_\varepsilon^\gamma-u_\nu^\gamma)}{\partial t}, \bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\rangle dt}_{=J_1(\varepsilon)}\\
&+\sum_{\gamma=1}^2\int_Q A(u_\varepsilon)\nabla(u_\varepsilon^\gamma-u_\nu^\gamma)\cdot\nabla(u_\varepsilon^\gamma-u_\nu^\gamma)\bar\beta_\gamma\bar\varphi''(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\\
&+\bar\mu\sum_{\gamma=1}^2\int_Q A(u_\varepsilon)\nabla(u_\varepsilon^\gamma-u_\nu^\gamma)\cdot\nabla\bar\psi(u_\varepsilon-u_\nu)\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\\
&+\sum_{\gamma=1}^2\int_0^T\langle\frac{\partial u_\nu^\gamma}{\partial t}, \bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\rangle dt\\
&+\sum_{\gamma=1}^2\int_Q A(u_\varepsilon)\nabla u_\nu^\gamma\cdot\nabla\left[\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\right]dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\gamma=1}^{2} \int_Q G_\varepsilon^\gamma(u_\varepsilon, \nabla u_\varepsilon)\bar\beta_\gamma \bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
&\quad + \int_Q F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla\bar\psi(u_\varepsilon - u_\nu)\exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
&\quad + \sum_{\gamma=1}^{2} \int_Q F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla u_\nu^\gamma \bar\beta_\gamma \bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
&= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)
\end{aligned}
\tag{5.7}
$$

We claim that the first term of (5.7) is nonnegative. Indeed as $(u_\varepsilon^\gamma - u_\nu^\gamma) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ and $\frac{\partial(u_\varepsilon^\gamma - u_\nu^\gamma)}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$ for $\gamma = 1,2$, then there exists a sequence w_j^γ such that $w_j^\gamma \in L^2(0,T;H_0^1(\Omega))$, $\frac{\partial w_j^\gamma}{\partial t} \in L^2(0,T;H^1(\Omega))$, w_j^γ bounded in $L^\infty(Q)$ and

$$
w_j^\gamma \to u_\varepsilon^\gamma - u_\nu^\gamma \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)), \tag{5.8}
$$

$$
\frac{\partial w_j^\gamma}{\partial t} \to \frac{\partial(u_\varepsilon^\gamma - u_\nu^\gamma)}{\partial t} \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega)), \tag{5.9}
$$

$$
w_j^\gamma(0) \to (u_\varepsilon^\gamma - u_\nu^\gamma)(0) = 0 \quad \text{strongly in } L^2(\Omega). \tag{5.10}
$$

By (5.8), (5.9) and the the continuous injection of the space

$$
W(0,T) = \{v \in L^2(0,T;H_0^1(\Omega)),\ \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\} \tag{5.11}
$$

in $\mathcal{C}([0,T];L^2(\Omega))$, we obtain

$$
\begin{gathered}
w_j^\gamma(T) \to (u_\varepsilon^\gamma - u_\nu^\gamma)(T) \quad \text{strongly in } L^2(\Omega), \\
\sum_{\gamma=1}^{2} \int_0^T \langle \frac{\partial w_j^\gamma}{\partial t}, \bar\beta_\gamma \varphi'(w_j^\gamma)\exp[\bar\mu\bar\psi(w_j)]\rangle dt \\
= \frac{1}{\bar\mu}\int_Q \exp[\bar\mu\bar\psi(w_j(T))]dx - \frac{1}{\bar\mu}\int_Q \exp[\bar\mu\bar\psi(w_j(0))]dx
\end{gathered}
$$

and letting $j \to +\infty$ shows that

$$
J_1(\varepsilon) = \frac{1}{\bar\mu}\int_Q \{\exp[\bar\mu\bar\psi((u_\varepsilon^\gamma - u_\nu^\gamma)(T))] - 1\}dx \ge 0. \tag{5.12}
$$

Secondly we estimate various terms of the right hand side of (5.7). For the third term we have by using the growth conditions (1.6), (3.4) on F_ε and Young's inequality:

$$
\begin{aligned}
I_3(\varepsilon) &\le \sum_{\gamma=1}^{2} \int_Q [C_3 + C_4|\nabla u_\varepsilon|]|\nabla u_\nu^\gamma|\bar\beta_\gamma|\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)|\exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
&\le C_3 \sum_{\gamma=1}^{2} \int_Q |\nabla u_\nu^\gamma|\bar\beta_\gamma|\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)|\exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
&\quad + C_4 \sum_{\gamma=1}^{2} \int_Q |\nabla u_\varepsilon||\nabla u_\nu^\gamma|\bar\beta_\gamma|\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)|\exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
&= I_3^1(\varepsilon) + I_3^2(\varepsilon)
\end{aligned}
\tag{5.13}
$$

Concerning the second term, we use (5.6) and the growth conditions (1.6), (3.4) on F_ε and Young's inequality, we obtain

$$
\begin{aligned}
I_2(\varepsilon) \le & \int_Q [C_3 + C_4|\nabla u_\varepsilon|]|\nabla\bar\psi(u_\varepsilon - u_\nu)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& \le \int_Q \big[\frac{\bar\theta}{2} + \frac{C_3^2}{2\bar\theta}|\nabla\bar\psi(u_\varepsilon - u_\nu)|^2 + \frac{\bar\theta}{2}|\nabla u_\varepsilon|^2 + \frac{C_4^2}{2\bar\theta}|\nabla\bar\psi(u_\varepsilon - u_\nu)|^2\big] \\
& \quad \times \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& \le \bar\theta \int_Q \Big(\frac{1}{2} + |\nabla u_\nu|^2\Big) \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& \quad + \Big(\frac{C_3^2}{2\bar\theta} + \frac{C_4^2}{2\bar\theta}\Big) \int_Q |\nabla\bar\psi(u_\varepsilon - u_\nu)|^2 \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& \quad + \bar\theta \int_Q |\nabla(u_\varepsilon - u_\nu)|^2 \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& = I_2^1(\varepsilon) + I_2^2(\varepsilon) + I_2^3(\varepsilon).
\end{aligned}
\tag{5.14}
$$

Then, we estimate the first term by using the growth conditions (1.4), (1.5) and (3.4) on G_ε^1, G_ε^2 to obtain

$$
\begin{aligned}
I_1(\varepsilon) \le & \, C_0 \sum_{\gamma=1}^{2} \int_Q \bar\beta_\gamma |\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& + C_2 \int_Q |\nabla u_\varepsilon^1|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& + \eta \int_Q |\nabla u_\varepsilon^2|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& + C_2 \int_Q |\nabla u_\varepsilon^1|^2 \bar\beta_2 |\bar\varphi'(u_\varepsilon^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& + C_2 \int_Q |\nabla u_\varepsilon^2|^2 \bar\beta_2 |\bar\varphi'(u_\varepsilon^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
= & \, I_1^1(\varepsilon) + I_1^2(\varepsilon) + I_1^3(\varepsilon) + I_1^4(\varepsilon) + I_1^5(\varepsilon).
\end{aligned}
\tag{5.15}
$$

Now we estimate the four last terms of the right hand side of the inequality (5.15); for what concerns the second term we have by using the Young's inequality

$$
\begin{aligned}
I_1^2(\varepsilon) \le & \, 2C_2 \int_Q |\nabla(u_\varepsilon^1 - u_\nu^1)|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& + 2C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt = I_{11}^2(\varepsilon) + I_{12}^2(\varepsilon)
\end{aligned}
$$

concerning the third term we have

$$
\begin{aligned}
I_1^3(\varepsilon) \le & \, 2\eta \int_Q |\nabla(u_\varepsilon^2 - u_\nu^2)|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
& + 2\eta \int_Q |\nabla u_\nu^2|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u_\varepsilon - u_\nu)]dx\,dt \\
= & \, I_{11}^3(\varepsilon) + I_{12}^3(\varepsilon)
\end{aligned}
\tag{5.16}
$$

we estimate the fourth term as

$$\begin{aligned} I_1^4(\varepsilon) &\le 2C_2\int_Q |\nabla(u_\varepsilon^1-u_\nu^1)|^2\bar\beta_2|\bar\varphi'(u_\varepsilon^2-u_\nu^2)|\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\\ &\quad + 2C_2\int_Q |\nabla u_\nu^1|^2\bar\beta_2|\bar\varphi'(u_\varepsilon^2-u_\nu^2)|\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\\ &= I_{11}^4(\varepsilon)+I_{12}^4(\varepsilon) \end{aligned} \tag{5.17}$$

for the last term we have

$$\begin{aligned} I_1^5(\varepsilon) &\le 2C_2\int_Q |\nabla(u_\varepsilon^2-u_\nu^2)|^2\bar\beta_2|\bar\varphi'(u_\varepsilon^2-u_\nu^2)|\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\\ &\quad + 2C_2\int_Q |\nabla u_\nu^2|^2\bar\beta_2|\bar\varphi'(u_\varepsilon^2-u_\nu^2)|\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\\ &= I_{11}^5(\varepsilon)+I_{12}^5(\varepsilon). \end{aligned} \tag{5.18}$$

Set afterwards

$$J_2(\varepsilon)=\sum_{\gamma=1}^2\int_Q A(u_\varepsilon)\nabla u_\nu^\gamma\nabla\left[\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\right]dx\,dt, \tag{5.19}$$

$$J_3(\varepsilon)=\sum_{\gamma=1}^2\int_0^T\langle\frac{\partial u_\nu^\gamma}{\partial t},\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\rangle dt \tag{5.20}$$

Now we consider the matrices

$$\begin{aligned} B_{\varepsilon\nu}^1 &= \bar\beta_1\bar\varphi''(u_\varepsilon^1-u_\nu^1)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]A(u_\varepsilon)\\ &\quad - L_1(u_\varepsilon,u_\nu)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]Id, \end{aligned} \tag{5.21}$$

$$\begin{aligned} B_{\varepsilon\nu}^2 &= \bar\beta_2\bar\varphi''(u_\varepsilon^2-u_\nu^2)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]A(u_\varepsilon)\\ &\quad - L_2(u_\varepsilon,u_\nu)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]Id, \end{aligned} \tag{5.22}$$

$$E_{\varepsilon\nu}=\bar\mu\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]A(u_\varepsilon)-\Big(\frac{C_3^2}{2\bar\theta}+\frac{C_3^2}{2\bar\theta}\Big)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]Id. \tag{5.23}$$

where Id denotes the identity matrix. Inequalities (5.7) and (5.12)–(5.23) give

$$\begin{aligned} &J_2(\varepsilon)+J_3(\varepsilon)+\int_Q B_{\varepsilon\nu}^1\nabla(u_\varepsilon^1-u_\nu^1)\cdot\nabla(u_\varepsilon^1-u_\nu^1)dx\,dt\\ &+\int_Q B_{\varepsilon\nu}^2\nabla(u_\varepsilon^2-u_\nu^2)\cdot\nabla(u_\varepsilon^2-u_\nu^2)dx\,dt+\int_Q E_{\varepsilon\nu}\nabla\bar\psi(u_\varepsilon-u_\nu)\cdot\nabla\bar\psi(u_\varepsilon-u_\nu)dx\,dt\\ &\le I_1^1(\varepsilon)+I_{12}^2(\varepsilon)+I_{12}^3(\varepsilon)+I_{12}^4(\varepsilon)+I_{12}^5(\varepsilon)+I_3^1(\varepsilon)+I_3^2(\varepsilon)+I_2^1(\varepsilon). \end{aligned} \tag{5.24}$$

Now we pass to the limit in (5.24) for $\varepsilon\to 0$ and ν fixed. Using the fact that $\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]$ remains bounded in $L^2(0,T;H_0^1(\Omega))$ and converges almost everywhere in Q towards $\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]$. We have for $\gamma=1,2$,

$$\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\to\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)] \tag{5.25}$$

weakly in $L^2(0,T;H_0^1(\Omega))$, and strongly in $L^2(Q)$.

Using (1.3), (4.27) and Lebesgue's dominated convergence theorem, we obtain

$$A(u_\varepsilon)\nabla u_\nu^\gamma\to A(u)\nabla u_\nu^\gamma\quad\text{strongly in }\left(L^2(Q)\right)^N. \tag{5.26}$$

By (5.25) and (5.26) we have

$$J_2(\varepsilon)=\sum_{\gamma=1}^{2}\int_Q A(u_\varepsilon)\nabla u_\nu^\gamma\cdot\nabla\big[\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\big]\,dx\,dt\to J_2 \tag{5.27}$$

where

$$J_2=\sum_{\gamma=1}^{2}\int_Q A(u)\nabla u_\nu^\gamma\cdot\nabla\big[\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]\big]\,dx\,dt, \tag{5.28}$$

$$J_3(\varepsilon)=\sum_{\gamma=1}^{2}\int_0^T\langle\frac{\partial u_\nu^\gamma}{\partial t},\bar\beta_\gamma\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]\rangle dt\to J_3, \tag{5.29}$$

where

$$J_3=\sum_{\gamma=1}^{2}\int_0^T\langle\frac{\partial u_\nu^\gamma}{\partial t},\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]\rangle dt. \tag{5.30}$$

The matrices $B_{\varepsilon\nu}^1$, $B_{\varepsilon\nu}^2$, and $E_{\varepsilon\nu}$ are positive definite, because of the coercivity condition (1.2), (5.2) and (5.3) imply that

$$B_{\varepsilon\nu}^1\ge\bar\alpha_0 Id,\quad B_{\varepsilon\nu}^2\ge\bar\alpha_0 Id. \tag{5.31}$$

Using (1.2) and the definition of $\bar\mu$ in (5.1), we obtain

$$E_{\varepsilon\nu}\ge\frac{\bar\mu}{2}Id. \tag{5.32}$$

Using (4.27) we obtain that the matrix $B_{\varepsilon\nu}^1$ converges a.e. in Q to the matrix B_ν^1 defined by

$$B_\nu^1=\bar\beta_1\bar\varphi''(u^1-u_\nu^1)\exp[\bar\mu\bar\psi(u-u_\nu)]A(u)-L_1(u,u_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]Id, \tag{5.33}$$

and the matrix $B_{\varepsilon\nu}^2$ converges a.e. in Q to the matrix B_ν^2 defined by

$$B_\nu^2=\bar\beta_2\bar\varphi''(u^2-u_\nu^2)\exp[\bar\mu\bar\psi(u-u_\nu)]A(u)-L_2(u,u_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]Id, \tag{5.34}$$

and the matrix $E_{\varepsilon\nu}$ converges a.e. in Q to the matrix E_ν defined by

$$E_\nu=\bar\mu\exp[\bar\mu\bar\psi(u-u_\nu)]A(u)-\Big(\frac{C_3^2}{2\bar\theta}+\frac{C_3^2}{2\bar\theta}\Big)\exp[\bar\mu\bar\psi(u-u_\nu)]Id. \tag{5.35}$$

Thanks to the positive definiteness of $B_{\varepsilon\nu}^1$, $B_{\varepsilon\nu}^2$ and $E_{\varepsilon\nu}$ we obtain

$$\liminf_{\varepsilon\to0}\int_Q B_{\varepsilon\nu}^1\nabla(u_\varepsilon^1-u_\nu^1)\cdot\nabla(u_\varepsilon^1-u_\nu^1)dx\,dt\ge\int_Q B_\nu^1(\nabla u^1-u_\nu^1)\cdot\nabla(u^1-u_\nu^1)dx\,dt$$
$$\liminf_{\varepsilon\to0}\int_Q B_{\varepsilon\nu}^2\nabla(u_\varepsilon^2-u_\nu^2)\cdot\nabla(u_\varepsilon^2-u_\nu^2)dx\,dt\ge\int_Q B_\nu^2(\nabla u^2-u_\nu^2)\cdot\nabla(u^2-u_\nu^2)dx\,dt$$
$$\begin{aligned}&\liminf_{\varepsilon\to0}\int_Q E_{\varepsilon\nu}\nabla\bar\psi(u_\varepsilon-u_\nu)\cdot\nabla\bar\psi(u_\varepsilon-u_\nu)dx\,dt\\&\ge\int_Q E_\nu\nabla\bar\psi(u-u_\nu)\cdot\nabla\bar\psi(u-u_\nu)dx\,dt.\end{aligned}$$

To pass to the limit in the right hand side of inequality (5.24), we use Lebesgue's dominated convergence theorem which implies that

$$I_3^1(\varepsilon)+I_2^1(\varepsilon)+I_1^1(\varepsilon)+\sum_{i=2}^{5}I_{12}^i(\varepsilon)$$

$$
\begin{aligned}
&= C_3 \sum_{\gamma=1}^{2} \int_Q |\nabla u_\nu^\gamma| \bar\beta_\gamma |\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \\
&\quad + \bar\theta \int_Q \Big(\frac{1}{2} + |\nabla u_\nu|^2\Big) \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \\
&\quad + C_0 \sum_{\gamma=1}^{2} \int_Q \bar\beta_\gamma |\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \\
&\quad + 2C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \\
&\quad + 2\eta \int_Q |\nabla u_\nu^2|^2 \bar\beta_1 |\bar\varphi'(u_\varepsilon^1 - u_\nu^1)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \\
&\quad + 2C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_2 |\bar\varphi'(u_\varepsilon^2 - u_\nu^2)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \\
&\quad + 2C_2 \int_Q |\nabla u_\nu^2|^2 \bar\beta_2 |\bar\varphi'(u_\varepsilon^2 - u_\nu^2)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] dx\, dt \to I_3^1 + I_2^1 + I_1^1 + \sum_{i=2}^{5} I_{12}^i,
\end{aligned}
$$

where

$$
\begin{aligned}
&I_3^1 + I_2^1 + I_1^1 + \sum_{i=2}^{5} I_{12}^i \\
&= C_3 \sum_{\gamma=1}^{2} \int_Q |\nabla u_\nu^\gamma| \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)| \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt \\
&\quad + \bar\theta \int_Q \Big(\frac{1}{2} + |\nabla u_\nu|^2\Big) \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt \\
&\quad + C_0 \sum_{\gamma=1}^{2} \int_Q \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)| \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt \\
&\quad + 2C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt \\
&\quad + 2\eta \int_Q |\nabla u_\nu^2|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt \\
&\quad + 2C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt \\
&\quad + 2C_2 \int_Q |\nabla u_\nu^2|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu \bar\psi(u - u_\nu)] dx\, dt.
\end{aligned}
$$

Now we pass to the limit in the remaining term of the right hand side of (5.24). Due to the fact that $|\nabla u_\varepsilon|$ is uniformly bounded in $L^2(Q)$, one can extract a subsequence which is not relabled such that

$$|\nabla u_\varepsilon| \rightharpoonup w \quad \text{weakly in } L^2(Q). \tag{5.36}$$

Applying Lebesgue's dominated convergence theorem we obtain

$$
\begin{aligned}
&|\nabla u_\nu^\gamma| \bar\beta_\gamma |\bar\varphi'(u_\varepsilon^\gamma - u_\nu^\gamma)| \exp[\bar\mu \bar\psi(u_\varepsilon - u_\nu)] \\
&\to |\nabla u_\nu^\gamma| \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)| \exp[\bar\mu \bar\psi(u - u_\nu)] \quad \text{strongly in } L^2(Q)\,.
\end{aligned} \tag{5.37}
$$

(5.36) and (5.37) yield

$$I_3^2(\varepsilon)=C_4\sum_{\gamma=1}^{2}\int_Q|\nabla u_\varepsilon||\nabla u_\nu^\gamma|\bar\beta_\gamma|\bar\varphi'(u_\varepsilon^\gamma-u_\nu^\gamma)|\exp[\bar\mu\bar\psi(u_\varepsilon-u_\nu)]dx\,dt\to I_3^2, \tag{5.38}$$

where

$$I_3^2=C_4\sum_{\gamma=1}^{2}\int_Q w|\nabla u_\nu^\gamma|\bar\beta_\gamma|\bar\varphi'(u^\gamma-u_\nu^\gamma)|\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt. \tag{5.39}$$

Passing to the limit infimum in the inequality (5.24) we obtain

$$\begin{aligned}
&J_2+J_3+\int_Q B_\nu^1\nabla(u^1-u_\nu^1)\cdot\nabla(u^1-u_\nu^1)dx\,dt\\
&+\int_Q B_\nu^2\nabla(u^2-u_\nu^2)\cdot\nabla(u^2-u_\nu^2)dx\,dt+\int_Q E_\nu\nabla\bar\psi(u-u_\nu)\cdot\nabla\bar\psi(u-u_\nu)dx\,dt\\
&\le I_1^1+I_{12}^2+I_{12}^3+I_{12}^4+I_{12}^5+I_3^1+I_3^2+I_2^1.
\end{aligned} \tag{5.40}$$

Multiplying the γ-th equation of system (3.1) relative to the parameter ν by $\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]$ and sum from $\gamma=1$ to $\gamma=2$ we obtain

$$\begin{aligned}
J_3&=\sum_{\gamma=1}^{2}\int_Q A(u_\nu)\nabla(u^\gamma-u_\nu^\gamma)\cdot\nabla\left[\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]\right]dx\,dt\\
&\quad-\sum_{\gamma=1}^{2}\int_Q A(u_\nu)\nabla u^\gamma\cdot\nabla\left[\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]\right]dx\,dt\\
&\quad+\sum_{\gamma=1}^{2}\int_Q G_\nu^\gamma(u_\nu,\nabla u_\nu)\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&\quad-\int_Q F_\nu(u_\nu,\nabla u_\nu)\cdot\nabla\bar\psi(u-u_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&\quad+\sum_{\gamma=1}^{2}\int_Q F_\nu(u_\nu,\nabla u_\nu)\cdot\nabla u^\gamma\bar\beta_\gamma\bar\varphi'(u^\gamma-u_\nu^\gamma)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&=K_1+K_2+K_3+K_4+K_5.
\end{aligned} \tag{5.41}$$

Using (5.41) and replacing the values of the matrix B_ν^1, B_ν^2 and E_ν in (5.40) we have

$$J_2+K_1+S_1+S_2+S_3+S_4\le I_1^1+I_3^1+I_3^2+I_2^1+\sum_{i=1}^{5}I_{12}^i-\sum_{i=2}^{5}K_i \tag{5.42}$$

where

$$\begin{gathered}
S_1=-\int_Q L_1(u,u_\nu)|\nabla(u^1-u_\nu^1)|^2\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt,\\
S_2=-\int_Q L_2(u,u_\nu)|\nabla(u^2-u_\nu^2)|^2\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt,\\
S_3=-\Big(\frac{C_3^2}{2\bar\theta}+\frac{C_4^2}{2\bar\theta}\Big)\int_Q|\nabla\bar\psi(u-u_\nu)|^2\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt,
\end{gathered}$$

$$
\begin{aligned}
S_4 &= \int_Q A(u)\nabla(u^1-u^1_\nu)\cdot\nabla(u-u^1_\nu)\bar\beta_1\bar\varphi''(u^1-u^1_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&\quad+\int_Q A(u)\nabla(u^2-u^2_\nu)\cdot\nabla(u-u^1_\nu)\bar\beta_2\bar\varphi''(u^2-u^2_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&\quad+\bar\mu\int_Q A(u)\nabla\bar\psi(u-u_\nu)\cdot\nabla\bar\psi(u-u_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&=\sum_{\gamma=1}^{2}\int_Q A(u)\nabla(u^\gamma-u^\gamma_\nu)\cdot\nabla\left[\bar\beta_\gamma\bar\varphi'(u^\gamma-u^\gamma_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]\right]dx\,dt\\
&=\sum_{\gamma=1}^{2}\int_Q A(u)\nabla u^\gamma\cdot\nabla\left[\bar\beta_\gamma\bar\varphi'(u^\gamma-u^\gamma_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]\right]dx\,dt\\
&\quad-\sum_{\gamma=1}^{2}\int_Q A(u)\nabla u^\gamma_\nu\cdot\nabla\left[\bar\beta_\gamma\bar\varphi'(u^\gamma-u^\gamma_\nu)\exp[\bar\mu\bar\psi(u-u_\nu)]\right]dx\,dt\\
&=K_6-J_2.
\end{aligned}
$$

Replacing the value of S_4 in (5.42) we obtain

$$
K_1+S_1+S_2+S_3\le I^1_1+I^1_3+I^2_3+I^1_2+\sum_{i=1}^{5}I^i_{12}-\sum_{i=2}^{6}K_i. \tag{5.43}
$$

By using the growth conditions (1.6) and (3.4) on F_ν and Young's inequality we obtain

$$
\begin{aligned}
-K_4 &\le \int_Q (C_3+C_4|\nabla u_\nu|)|\nabla\bar\psi(u-u_\nu)|\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&\le \int_Q\Big[\frac{\bar\theta}{2}+\frac{C_3^2}{2\bar\theta}|\nabla\bar\psi(u-u_\nu)|^2+\frac{\bar\theta}{2}|\nabla u_\nu|^2+\frac{C_4^2}{2\bar\theta}|\nabla\bar\psi(u-u_\nu)|^2\Big]\\
&\quad\times\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt\\
&\le -S_3+\bar\theta\int_Q|\nabla(u-u_\nu)|^2\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt+K^1_4,
\end{aligned}
\tag{5.44}
$$

where

$$
K^1_4=\int_Q\bar\theta big(\frac{1}{2}+|\nabla u|^2\big)\exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt. \tag{5.45}
$$

Employing the growth conditions (1.4), (1.5) and (3.4) on G_ν^γ, the term $-K_3$ can be estimated as

$$
\begin{aligned}
-K_3 &\le C_0 \sum_{\gamma=1}^{2} \int_Q \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + \eta \int_Q |\nabla u_\nu^2|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + C_2 \int_Q |\nabla u_\nu^1|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + C_2 \int_Q |\nabla u_\nu^2|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\le I_1^1 + \frac{1}{2}\sum_{i=2}^{5} I_{12}^2 .
\end{aligned}
\tag{5.46}
$$

Using Young's inequality the term $\sum_{i=2}^5 I_{12}^2$ can be controlled by

$$
\begin{aligned}
\sum_{i=2}^{5} I_{12}^2 &\le 4C_2 \int_Q |\nabla(u^1 - u_\nu^1)|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 4\eta \int_Q |\nabla(u^2 - u_\nu^2)|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 4C_2 \int_Q |\nabla(u^1 - u_\nu^1)|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 4C_2 \int_Q |\nabla(u^2 - u_\nu^2)|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt + \sum_{i=2}^{5} R_{12}^2,
\end{aligned}
\tag{5.47}
$$

where

$$
\begin{aligned}
\sum_{i=2}^{5} R_{12}^2 &= 4C_2 \int_Q |\nabla u^1|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 4\eta \int_Q |\nabla u^2|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 4C_2 \int_Q |\nabla u^1|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 4C_2 \int_Q |\nabla u^2|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt.
\end{aligned}
\tag{5.48}
$$

We use again Young's inequality to obtain

$$
\begin{aligned}
I_2^1 &\le 2\bar\theta \int_Q |\nabla(u-u_\nu)|^2 \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + \underbrace{\bar\theta \int_Q \Big(\frac{1}{2} + 2|\nabla u|^2\Big) \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt}_{=R_2^1}.
\end{aligned}
\tag{5.49}
$$

We employ the coercivity condition (1.2) to estimate the term K_1 from below by

$$
\begin{aligned}
K_1 \geq \alpha \int_Q |\nabla(u^1-u^1_\nu)|^2\bar{\beta}_1\bar{\varphi}''(u^1-u^1_\nu)\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt \\
+ \alpha \int_Q |\nabla(u^2-u^2_\nu)|^2\bar{\beta}_2\bar{\varphi}''(u^2-u^2_\nu)\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt \\
+ \alpha\bar{\mu}\int_Q |\nabla\bar{\psi}|^2\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt.
\end{aligned}
\tag{5.50}
$$

Using (5.44)–(5.50), inequality (5.43) becomes

$$
\begin{aligned}
&\sum_{\gamma=1}^{2}\int_Q |\nabla(u^\gamma-u^\gamma_\nu)|^2\left\{\alpha\bar{\beta}_\gamma\bar{\varphi}''(u^\gamma-u^\gamma_\nu)-4L_\gamma(u,u_\nu)\right\}\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt \\
&+\int_Q |\nabla\bar{\psi}|^2\left[\alpha\bar{\mu}-\left(\frac{C_3^2}{\bar{\theta}}+\frac{C_4^2}{\bar{\theta}}\right)\right]\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt \\
&\leq 2I_1^1+I_3^1+I_3^2+R_2^1-K_2+K_4^1-K_5-K_6+\frac{3}{2}\sum_{i=2}^{5}R_{12}^i \\
&= R_\nu+R_2^1+K_4^1.
\end{aligned}
\tag{5.51}
$$

Using (5.2), (5.3) and the definition of $\bar{\mu}$ in (5.1) we obtain from (5.51) that

$$
\begin{aligned}
&\underbrace{\bar{\alpha}_0\sum_{\gamma=1}^{2}\int_Q |\nabla(u^\gamma-u^\gamma_\nu)|^2\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt
+\frac{\alpha\bar{\mu}}{2}\int_Q |\nabla\bar{\psi}|^2\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt}_{\geq 0} \\
&\leq R_\nu+\bar{\theta}\int_Q\left(1+3|\nabla u|^2\right)\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt,
\end{aligned}
\tag{5.52}
$$

hence

$$
\begin{aligned}
&\bar{\alpha}_0\sum_{\gamma=1}^{2}\int_Q |\nabla(u^\gamma-u^\gamma_\nu)|^2\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt \\
&\leq R_\nu+\bar{\theta}\int_Q\left(1+3|\nabla u|^2\right)\exp[\bar{\mu}\bar{\psi}(u-u_\nu)]dx\,dt,
\end{aligned}
\tag{5.53}
$$

such that

$$
R_\nu = 2I_1^1+I_3^1+I_3^2-K_5-(K_2+K_6)+\frac{3}{2}\sum_{i=2}^{5}R_{12}^i, \tag{5.54}
$$

where

$$
\begin{aligned}
&2I_1^1 + I_3^1 + I_3^2 - K_5 \\
&= 2C_0 \sum_{\gamma=1}^{2} \int_Q \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)|^2 \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + C_3 \sum_{\gamma=1}^{2} \int_Q |\nabla u_\nu^\gamma| \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + C_4 \sum_{\gamma=1}^{2} \int_Q w |\nabla u_\nu^\gamma| \bar\beta_\gamma |\bar\varphi'(u^\gamma - u_\nu^\gamma)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad - \sum_{\gamma=1}^{2} \int_Q F_\nu(u_\nu, \nabla u_\nu) \cdot \nabla u^\gamma \bar\beta_\gamma \bar\varphi'(u^\gamma - u_\nu^\gamma) \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt,
\end{aligned}
\tag{5.55}
$$

such that

$$
\begin{aligned}
&-(K_2 + K_6) \\
&= \sum_{\gamma=1}^{2} \int_Q [A(u_\nu) - A(u)] \cdot \nabla u^\gamma \nabla \left[\bar\beta_\gamma \bar\varphi'(u^\gamma - u_\nu^\gamma) \exp[\bar\mu\bar\psi(u-u_\nu)]\right] dx\,dt,
\end{aligned}
\tag{5.56}
$$

and

$$
\begin{aligned}
\frac{3}{2}\sum_{i=2}^{5} R_{12}^i &= 6C_2 \int_Q |\nabla u^1|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 6\eta \int_Q |\nabla u^2|^2 \bar\beta_1 |\bar\varphi'(u^1 - u_\nu^1)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 6C_2 \int_Q |\nabla u^1|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt \\
&\quad + 6C_2 \int_Q |\nabla u^2|^2 \bar\beta_2 |\bar\varphi'(u^2 - u_\nu^2)| \exp[\bar\mu\bar\psi(u-u_\nu)]dx\,dt.
\end{aligned}
\tag{5.57}
$$

In view of

$$
\begin{gathered}
u_\nu \rightharpoonup u \quad \text{weakly in } (L^2(0,T;H_0^1(\Omega)))^m, \\
u_\nu \to u \quad \text{strongly in } (L^2(Q))^m, \\
u_\nu \to u \quad \text{a.e.in } Q,
\end{gathered}
\tag{5.58}
$$

and the fact that $u - u_\nu$ is uniformly bounded in $L^\infty(Q)$, for all $\gamma = 1,2$, it holds

$$
\bar\beta_\gamma \bar\varphi'(u^\gamma - u_\nu^\gamma) \exp[\bar\mu\bar\psi(u-u_\nu)] \to 0 \quad \text{strongly in } L^2(Q), \tag{5.59}
$$

$$
\bar\beta_\gamma \bar\varphi'(u^\gamma - u_\nu^\gamma) \exp[\bar\mu\bar\psi(u-u_\nu)] \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{5.60}
$$

this follows from the fact that $\bar\beta_\gamma \bar\varphi'(u^\gamma - u_\nu^\gamma) \exp[\bar\mu\bar\psi(u-u_\nu)]$ remains bounded in $L^2(0,T;H_0^1(\Omega))$ and converges almost everywhere in Q towards 0. By (5.58) and employing Lebesgue's dominated convergence theorem, we obtain

$$
\frac{3}{2}\sum_{i=2}^{5} R_{12}^i \to 0 \quad \text{as } \nu \to 0. \tag{5.61}
$$

By (5.59), Young's inquality and employing Lebesgue's dominated convergence theorem, we obtain

$$2I_1^1 + I_3^1 + I_3^2 \to 0 \quad \text{in } \nu \to 0, \tag{5.62}$$

(5.58) and Lebesgue's dominated convergence yield

$$[A(u_\nu) - A(u)]\, \nabla u^\gamma \to 0 \quad \text{strongly in } \left(L^2(Q)\right)^N. \tag{5.63}$$

Therefore

$$- \operatorname{div}\left([A(u_\nu) - A(u)]\, \nabla u^\gamma\right) \to 0 \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega)), \tag{5.64}$$

(5.64) and (5.60) imply that

$$\begin{aligned} &-(K_2 + K_6) \\ &= \sum_{\gamma=1}^{2} \langle -div\left([A(u_\nu) - A(u)]\, \nabla u^\gamma\right); \bar{\beta}_\gamma \bar{\varphi}'(u^\gamma - u_\nu^\gamma) \exp[\bar{\mu}\bar{\psi}(u - u_\nu)]\rangle \to 0. \end{aligned} \tag{5.65}$$

For the rest terms of (5.55), using the growth conditions (1.6) and (3.4) on F_ν, and the fact that u_ν is uniformly bounded in $(L^2(Q))^2$, by applying the Young's inequality and afterwards Lebesgue's theorem one can prove that

$$- K_5 \to 0, \quad \text{when } \nu \to 0. \tag{5.66}$$

Finally we have $R_\nu \to 0$, when $\nu \to 0$, similarly we have

$$\bar{\theta} \int_Q \left(1 + 4|\nabla u|^2\right) \exp[\bar{\mu}\bar{\psi}(u - u_\nu)]dx\, dt \to \bar{\theta} \int_Q \left(1 + 4|\nabla u|^2\right) dx\, dt. \tag{5.67}$$

Since $\exp[\bar{\mu}\bar{\psi}(u - u_\nu)] \geq 1$, we deduce from (5.53) that

$$\limsup_{\varepsilon \to 0} \bar{\alpha}_0 \sum_{\gamma=1}^{2} \int_Q |\nabla(u^\gamma - u_\nu^\gamma)|^2 dx\, dt \leq \bar{\theta} \int_Q \left(1 + 4|\nabla u|^2\right) dx\, dt. \tag{5.68}$$

Since $\bar{\alpha}_0$ is independent of $\bar{\theta}$ this implies that $u_\nu \to u$ strongly in $(L^2(0,T;H_0^1(\Omega)))^2$.

5.1. **Passing to the limit.** As u_ε converges strongly to u in $\left(L^2(0,T;H_0^1(\Omega))\right)^2$, for all $\gamma = 1, 2$ we have

$$G^\gamma(u_\varepsilon, \nabla u_\varepsilon) + F(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^\gamma \to G^\gamma(u, \nabla u) + F(u, \nabla u) \cdot \nabla u^\gamma \tag{5.69}$$

strongly in $L^1(Q)$. Therefore, applying Vitali's theorem,

$$G_\varepsilon^\gamma(u_\varepsilon, \nabla u_\varepsilon) + F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^\gamma \to G^\gamma(u, \nabla u) + F(u, \nabla u) \cdot \nabla u^\gamma \tag{5.70}$$

strongly in $L^1(Q)$.

On the other hand,

$$- \operatorname{div}\left(A(u_\varepsilon)\nabla u_\varepsilon^\gamma\right) \to - \operatorname{div}\left(A(u)\nabla u^\gamma\right) \tag{5.71}$$

strongly in $L^2(0,T;H^{-1}(\Omega))$.

Thus $\frac{\partial u_\varepsilon^\gamma}{\partial t}$ converges strongly to $\frac{\partial u^\gamma}{\partial t}$ in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$ which implies

$$\frac{\partial u^\gamma}{\partial t} \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q),$$

and

$$\frac{\partial u^1}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{i,j}(u) \frac{\partial u^1}{\partial x_j} = G^1(u, \nabla u) + F(u, \nabla u) \cdot \nabla u^1 \quad \text{in } Q,$$

$$\frac{\partial u^2}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{i,j}(u)\frac{\partial u^2}{\partial x_j} = G^2(u, \nabla u) + F(u, \nabla u) \cdot \nabla u^2 \quad \text{in } Q.$$

For s large enough, for all $(\gamma = 1, 2)$, we have $\frac{\partial u_\varepsilon^\gamma}{\partial t}$ converges strongly to $\frac{\partial u^\gamma}{\partial t}$ in $L^1(0, T; H^{-s}(\Omega))$. Then u_ε^γ converges strongly to u^γ in $\mathcal{C}([0, T]; H^{-s}(\Omega))$, hence $u_\varepsilon^\gamma(0)$ converges strongly to $u^\gamma(0)$ in $H^{-s}(\Omega)$. Therefore, u satisfies (1.1). We thus have proved the existence of at least one solution of system (1.1). this solution u is bounded because u_ε is uniformly bounded in $\left(L^\infty(Q)\right)^2$ and $u_\varepsilon \to u$ a.e. in Q. This completes the proof. $\square$

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