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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A DIRICHLET PROBLEM INVOLVING PERTURBED $p(x)$-LAPLACIAN OPERATOR

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Abstract. In this article we study the existence of solutions for the Dirichlet problem

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{q(x)-2}u = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, V is a given function in a generalized Lebesgue space $L^{s(x)}(\Omega)$ and $f(x,u)$ is a Carathéodory function which satisfies some growth condition. Using variational arguments based on "Fountain theorem" and "Dual Fountain theorem", we shall prove under appropriate conditions on the above nonhomogeneous quasilinear problem the existence of two sequences of weak solutions for this problem.

1. Introduction

In this work we study the existence of multiple solutions for a nonlinear Dirichlet problem involving the $p(x)$-Laplacian operator,

$$\begin{gathered} -\Delta_{p(x)}u + V(x)|u|^{q(x)-2}u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p, q, s : \overline{\Omega} \to \mathbb{R}$ are continuous functions, $V \in L^{s(x)}(\Omega)$ and $f(x,u)$ is a Carathéodory function. Here, the $p(x)$-Laplacian operator is given by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a generalization of the usual p-Laplacian operator.

Nonlinear boundary value problems with variable exponent have received considerable attention in recent years. This is partly due to their frequent appearance in applications such as the modeling of electrorheological fluids [22, 24, 28], elastic mechanics, flow in porous media and image processing [6], but these problems are very interesting from a purely mathematical point of view as well. The main interest in studying such problems arises from the presence of the $p(x)$-Laplacian operator which is a natural extension of the classical p-Laplacian operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case where $p(x) \equiv p$ is a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplacian operator possesses a more complicated

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structure than the p-Laplacian operator, for example, it is inhomogeneous. Many authors have studied problems with variable exponent, we refer for example to the works in [4, 12, 13, 15, 17, 18, 19, 21] and references therein.

When $p(x) \equiv p$ is a constant and $V \equiv 0$, Dinca et al. [7], using variational and topological methods, proved the existence and multiplicity of weak solutions for the following Dirichlet problem with p-Laplacian

$$-\Delta_p u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $f(x,u)$ is a Carathéodory function which satisfies some growth condition. The main tool in their work was the well known "Mountain Pass theorem" of Ambrosetti and Rabinowitz.

Fan and Zhang [13] studied the variable exponent case with $V \equiv 0$

$$-\Delta_{p(x)} u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $f(x,u)$ is a Carathéodory function which satisfies some subcritical growth condition. By the "Mountain Pass lemma", the authors showed that the considered problem admits at least one nontrivial weak solution and, by the "Fountain theorem", the infinite many pairs of weak solutions.

In [17], Iliaş considered the Dirichlet problem as in [13] under some more general conditions on the Carathéodory function. Using "Fountain theorem" and "Dual Fountain theorem", the existence of two different sequences of weak solutions was proved.

Chabrowski and Fu [4] established in the superlinear and sublinear cases the existence of nontrivial nonnegative weak solutions for the Dirichlet problem

$$-\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $a(x)$ and $b(x)$ are positive functions in $L^\infty(\Omega)$, the continuous function $p(x)$ satisfies $1 < p(x) < N$ and the Carathéodory function $f(x,u)$ satisfies two different subcritical growth conditions. Their argument was based on the "Mountain Pass theorem".

Recently, Liang et al. [21] studied the $p(x)$-Laplacian equation

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In this problem, contrarily to all previous, the Carathéodory function $f(x,u)$ satisfies some critical growth condition, which is rare in the literature. The existence of infinitely many pairs of weak solutions was proved by applying the "Fountain theorem" and the "Dual Fountain theorem".

Inspired by the works as in [4, 7, 12, 17, 21], we want to study the nonlinear Dirichlet problem (1.1) with a perturbation term $V(x)|u|^{q(x)-2}u$ in the left-hand side of the first equation of (1.1), where the function $V(x)$ has an indefinite sign and belongs to the generalized Lebesgue space $L^{s(x)}(\Omega)$, the nonlinearity $f(x,u)$ is superlinear and satisfies some subcritical growth condition. The discussions on the existence of multiple weak solutions will be based on the theory of generalized Lebesgue and Sobolev spaces with variable exponents by using the critical points theory, the "Fountain Theorem" and the "Dual Fountain Theorem". The results obtained here generalize some well known other results established in [7, 13, 17].

The remainder of this paper is organized as follows, in section 2 we introduce some technical results and formulate the required hypotheses in order to solve our problem, finally, in section 3 we state some auxiliary results and prove the main results of this work.

2. Preliminaries and hypotheses

To study the Dirichlet problem (1.1), we need to recall some properties of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, respectively, which will be used later. We refer to [9, 14, 20] for the basic properties of these spaces.

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let us denote by

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for every } x \in \overline{\Omega}\},$$
$$p^- = \min_{x\in\overline{\Omega}} p(x), \quad p^+ = \max_{x\in\overline{\Omega}} p(x), \quad \text{for } p \in C_+(\overline{\Omega}),$$
$$M = \{u : \Omega \to \mathbb{R} : u \text{ is a measurable real-valued function}\}.$$

Definition 2.1. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \big\{u \in M : \int_\Omega |u|^{p(x)}dx < +\infty\big\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(x)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u}{\lambda}|^{p(x)}dx \le 1\big\}.$$

Remark 2.2. Variable exponent Lebesgue spaces resemble to classical Lebesgue spaces in many respects, they are separable Banach spaces and the Hölder inequality holds. The inclusions between Lebesgue spaces are also naturally generalized, that is, if $0 < mes(\Omega) < \infty$ and p, q are variable exponents such that $p(x) < q(x)$ a.e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

Definition 2.3. The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\big\},$$

with the norm

$$\|u\|_{1,p(x)} = \inf\big\{\lambda > 0 : \int_\Omega \Big(|\frac{\nabla u}{\lambda}|^{p(x)} + |\frac{u}{\lambda}|^{p(x)}\Big)dx \le 1\big\},$$
$$\|u\|_{1,p(x)} = ||\nabla u||_{p(x)} + |u|_{p(x)},$$

where $|\nabla u| = \sqrt{\sum_{i=1}^N \big(\frac{\partial u}{\partial x_i}\big)^2}$.

Proposition 2.4 (Fan and Zhao [14])**.** *Both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Proposition 2.5 (Fan and Zhao [14])**.** *The Hölder inequality holds, namely*

$$\int_\Omega |uv|dx \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)},$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Moreover, if $h_1, h_2, h_3 : \overline{\Omega} \to (1,\infty)$ are Lipschitz continuous functions such that $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$, $w \in L^{h_3(x)}(\Omega)$, the following inequality holds (see [15, Proposition 2.5])

$$\int_\Omega |uvw|dx \le \big(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\big)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \tag{2.1}$$

The modular is an important tool in studying generalized Lebesgue-Sobolev spaces, which is a mapping $\varphi_p : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_p(u) = \int_\Omega |u|^{p(x)}dx.$$

The statements below establish the connection between φ_p and $|\cdot|_{p(x)}$.

Proposition 2.6 (Fan and Zaho [14]). *We have the relations:*

(1) *The following assertions are equivalent*
  (i) $|u|_{p(x)} < (>,=)1 \Longleftrightarrow \varphi_p(u) < (>,=)1$,
  (ii) $|u|_{p(x)} = \alpha \Longleftrightarrow \varphi_p(u) = \alpha$ *when* $\alpha \neq 0$,
  (iii) $|u|_{p(x)} \to 0 \Longleftrightarrow \varphi_p(u) \to 0$,
  (iv) $|u|_{p(x)} \to \infty \Longleftrightarrow \varphi_p(u) \to \infty$.

(2) $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \le \varphi_p(u) \le \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.
  Let $u_n,\ u \subset L^{p(x)}(\Omega)$, *with* $n = 1, 2\cdots$.

(3) *The following assertions are equivalent*
  (i) $\lim_{n\to+\infty} |u_n - u|_{p(x)} = 0$,
  (ii) $\lim_{n\to+\infty} \varphi_p(u_n - u) = 0$,
  (iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty} \varphi_p(u_n) = \varphi_p(u)$.

(4) $\lim_{n\to+\infty} |u_n|_{p(x)} = +\infty$ *if and only if* $\lim_{n\to+\infty} \varphi_p(u_n) = +\infty$.

Proposition 2.7 (Edmunds and Rákosník [10]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \le ||u|^{p(x)}|_{q(x)} \le |u|_{p(x)q(x)}^{p^-},$$
$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \le ||u|^{p(x)}|_{q(x)} \le |u|_{p(x)q(x)}^{p^+}.$$

In particular if $p(x) = p$ is a constant, then

$$|\,|u|^p|_{q(x)} = |u|_{pq(x)}^p.$$

Definition 2.8. For $p \in C_+(\overline{\Omega})$, let us define the so-called critical Sobolev exponent of p by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N, \end{cases}$$

for every $x \in \overline{\Omega}$.

We also define the space $W_0^{1,p(x)}(\Omega)$ as the closure of the space of C^∞-functions with compact support in Ω $C_0^\infty(\Omega)$ in the space $W^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\| = ||\nabla u||_{p(x)}.$$

The dual space of $W_0^{1,p(x)}(\Omega)$ is denoted by $W^{-1,p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for every $x \in \overline{\Omega}$.

Next, we recall some embedding results regarding variable exponent Lebesgue-Sobolev spaces.

Proposition 2.9 (Fan and Zhao [14]). *The following statements hold:*

(i) *The space* $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ *is a separable and reflexive Banach space.*
(ii) *If* $p, q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for every* $x \in \overline{\Omega}$*, then there is a compact and continuous embedding*
$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$
(iii) *There is a constant* $C > 0$ *such that*
$$|u|_{p(x)} \leq C||\nabla u||_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.10. Using the result of Fan and Zhao in Proposition 2.9 (iii), the norm $\|u\|_{1,p(x)} = ||\nabla u||_{p(x)} + |u|_{p(x)}$ is equivalent to the norm $\|u\| = ||\nabla u||_{p(x)}$ in $W_0^{1,p(x)}(\Omega)$. Hence from now, we will consider the space $W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|u\| = ||\nabla u||_{p(x)}$.

Remark 2.11. If $p, q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for every $x \in \overline{\Omega}$, then the embedding from $W_0^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact.

As in the case $p(x) \equiv p$ (constant), we consider the $p(x)$-Laplacian operator
$$-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$
defined by
$$\langle -\Delta_{p(x)}u, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx, \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Proposition 2.12 (Fan and Zhang [13]). *We have the following properties:*

(i) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a homeomorphism.*
(ii) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a stictly monotone operator, that is,*
$$-\langle \Delta_{p(x)}u - \Delta_{p(x)}v, u - v\rangle > 0, \quad \text{for all } u \neq v \in W_0^{1,p(x)}(\Omega).$$
(iii) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a mapping of type* (S_+)*, that is,*
$$\text{if } u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega) \text{ and } \limsup_{n\to\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle \leq 0,$$
then $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$.

Proposition 2.13 (Chang [5]). *The functional* $\Psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *defined by*
$$\Psi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$$
is continuously Fréchet differentiable and $\Psi'(u) = -\Delta_{p(x)}u$*, for all* $u \in W_0^{1,p(x)}(\Omega)$.

We recall now some basic results concerning the Nemytskii operator. Note that, if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $u \in M$, then the function $N_f u : \Omega \to \mathbb{R}$ defined by $(N_f u)(x) = f(x, u(x))$ for $x \in \Omega$ is measurable in Ω. Thus, the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ generates an operator $N_f : M \to M$, which is called the Nemytskii operator. The propositions below give the properties of N_f.

Proposition 2.14 (Zhao and Fan [26]). *Suppose that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)|\le c|t|^{\frac{\alpha(x)}{\beta(x)}}+h(x),\quad \text{for every } x\in\Omega,\ t\in\mathbb{R},$$

where $\alpha,\beta\in C_+(\overline{\Omega})$, $c\ge 0$ is constant and $h\in L^{\beta(x)}(\Omega)$. Then $N_f(L^{\alpha(x)}(\Omega))\subseteq L^{\beta(x)}(\Omega)$. Moreover, N_f is continuous from $L^{\alpha(x)}(\Omega)$ into $L^{\beta(x)}(\Omega)$ and maps bounded set into bounded set.

Proposition 2.15 (Zhao and Fan [26]). *Suppose that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)|\le c|t|^{\alpha(x)-1}+h(x),\quad \text{for every } x\in\Omega,\ t\in\mathbb{R},$$

where $c\ge 0$ is constant, $\alpha,\beta\in C_+(\overline{\Omega})$, $h\in L^{\beta(x)}(\Omega)$ with β the conjugate exponent of α, i.e., $\beta(x)=\frac{\alpha(x)}{\alpha(x)-1}$. Let $F:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$F(x,t)=\int_0^t f(x,s)ds.$$

Then

(i) *F is a Carathéodory function and there exist a constant $c_1\ge 0$ and $\sigma\in L^1(\Omega)$ such that*
$$|F(x,t)|\le c_1|t|^{\alpha(x)}+\sigma(x),\quad \text{for all } x\in\Omega,\ t\in\mathbb{R}.$$
(ii) *The functional $\Phi:L^{\alpha(x)}(\Omega)\to\mathbb{R}$ defined by $\Phi(u)=\int_\Omega F(x,u(x))dx$ is continuously Fréchet differentiable and $\Phi'(u)=N_f(u)$, for all $u\in L^{\alpha(x)}(\Omega)$.*

Remark 2.16. In Proposition 2.15 if we take $\alpha\in C_+(\overline{\Omega})$ with $\alpha(x)<p^*(x)$ for every $x\in\overline{\Omega}$, the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ is compact. Hence the diagram

$$W_0^{1,p(x)}(\Omega)\overset{I}{\hookrightarrow}L^{\alpha(x)}(\Omega)\overset{N_f}{\to}L^{\beta(x)}(\Omega)\overset{I^*}{\hookrightarrow}W^{-1,p'(x)}(\Omega)$$

shows that $N_f:W_0^{1,p(x)}(\Omega)\to W^{-1,p'(x)}(\Omega)$ is strongly continuous on $W_0^{1,p(x)}(\Omega)$. Moreover, using the same argument, the functional $\Phi:W_0^{1,p(x)}(\Omega)\to\mathbb{R}$ defined by $\Phi(u)=\int_\Omega F(x,u(x))dx$ is strongly continuous on $W_0^{1,p(x)}(\Omega)$.

Throughout this work, we make the following assumptions on the Dirichlet problem (1.1):

(A1) $p,q,s\in C_+(\overline{\Omega})$ such that $1<q(x)<p(x)\le N<s(x)$ for every $x\in\overline{\Omega}$ and $V\in L^{s(x)}(\Omega)$.
(A2) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition and the following subcritical growth condition
$$|f(x,t)|\le c|t|^{\beta(x)-1}+h(x),\quad \text{for all } x\in\Omega,\ t\in\mathbb{R},$$
where $c\ge 0$ is constant, $\beta\in C_+(\overline{\Omega})$ with $\beta(x)<p^*(x)$, for every $x\in\overline{\Omega}$, $h\in L^{\beta'(x)}(\Omega)$ where β' is the conjugate exponent of β.
(A3) There exist $\theta\in(p^+,(p^*)^-)$ and $M>0$ such that
$$0<\theta F(x,s)\le sf(x,s),\quad \text{for } x\in\Omega,\ s\in\mathbb{R} \text{ with } |s|\ge M,$$
where $F(x,s)=\int_0^s f(x,t)dt$.
(A4) $f(x,-t)=-f(x,t)$ for $x\in\Omega$, $s\in\mathbb{R}$.
(A5) $\beta^->p^+$.

3. Proofs of main results and auxiliary results

In this section, we investigate some auxiliary results which allow us to prove our main results. Here and henceforth, we denote by X the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|\cdot\|$, X^* its dual space, $s'(x)$ the conjugate exponent of the function $s(x)$ and we define a continuous function

$$\alpha(x) = \frac{s(x)q(x)}{s(x)-q(x)}.$$

By assumptions (A1), (A2) on the functions p, q, s and β, a straightforward computation gives

$$q(x) < p^*(x),\ \alpha(x) < p^*(x),\quad s'(x)q(x) < p^*(x) \text{ and } \beta(x) < p^*(x),$$

for every $x \in \overline{\Omega}$. Then, we have the following remark.

Remark 3.1. From Proposition 2.9 (i), the embeddings $X \hookrightarrow L^{q(x)}(\Omega)$, $X \hookrightarrow L^{\alpha(x)}(\Omega)$, $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\beta(x)}(\Omega)$ are compact and continuous. Therefore, there exists a positive constant C such that

$$|u|_{q(x)} \le C\|u\|,\quad |u|_{\alpha(x)} \le C\|u\|,\quad |u|_{s'(x)q(x)} \le C\|u\|,\quad |u|_{\beta(x)} \le C\|u\|, \tag{3.1}$$

for all $u \in X$. Without any loss of generality, we can suppose that $C > 1$.

By a solution of problem (1.1), we mean a weak solution which satisfies the following condition.

Definition 3.2. We say that $u \in X$ is a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega V(x)|u|^{q(x)-2}uv dx = \int_\Omega f(x,u)v dx,\ \forall\ v \in X. \tag{3.2}$$

Let us consider the Euler-Lagrange functional or the energy functional $H : X \to \mathbb{R}$ associated with problem (1.1) defined by

$$H(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega F(x,u)dx.$$

Let us introduce the functionals $\Psi, J, \Phi : X \to \mathbb{R}$ defined by

$$\Psi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx,\quad J(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx,\quad \Phi(u) = \int_\Omega F(x,u)dx.$$

Then, the energy functional H can be written as

$$H(u) := \Psi(u) + J(u) - \Phi(u).$$

The functional J is well defined. Indeed, using Hölder inequality and Proposition 2.7, for all $u \in X$, we have

$$|J(u)| \le \frac{2}{q^-}|V|_{s(x)}||u|^{q(x)}|_{s'(x)} \le \frac{2}{q^-}|V|_{s(x)} \max\{|u|^{q^-}_{s'(x)q(x)}, |u|^{q^+}_{s'(x)q(x)}\}.$$

We have the following result concerning the regularity of the functional H.

Proposition 3.3. *The functional $H \in C^1(X,\mathbb{R})$, i.e., H is continuously Fréchet differentiable. Moreover, $u \in X$ is a critical point of H if and only if u is a weak solution of* (1.1).

Proof. By Proposition 2.13 and Proposition 2.15, we know that Ψ respectively Φ are of class $C^1(X,\mathbb{R})$ and their derivative functions are given by

$$\langle d\Psi(u),v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx \text{ and } \langle d\Phi(u),v\rangle = \int_\Omega f(x,u)v dx,$$

for all $u,v\in X$. It is also well known (see [2, 19]) that the functional J is of class $C^1(X,\mathbb{R})$ and its derivative is given by

$$\langle dJ(u),v\rangle = \int_\Omega V(x)|u|^{q(x)-2}uv dx, \quad \text{for all } u,v\in X.$$

Therefore, the functional $H\in C^1(X,\mathbb{R})$ and its derivative function is given by

$$\langle dH(u),v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_\Omega V(x)|u|^{q(x)-2}uv dx - \int_\Omega f(x,u)v dx,$$

for all $u,v\in X$. Now, let u be a critical point of H, then we have $dH(u)=0_{X^*}$, which implies that

$$\langle dH(u),v\rangle = 0, \text{ for all } v\in X.$$

Consequently,

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_\Omega V(x)|u|^{q(x)-2}uv dx = \int_\Omega f(x,u)v dx, \ \forall\ v\in X.$$

It follows that u is a weak solution of (1.1). On the other hand, if u is a weak solution of (1.1), by definition, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_\Omega V(x)|u|^{q(x)-2}uv dx = \int_\Omega f(x,u)v dx, \ \forall\ v\in X,$$

which implies that

$$\langle dH(u),v\rangle = 0, \text{ for all } v\in X.$$

So, $dH(u)=0_{X^*}$ and hence u is a critical point of H. The proof is complete. $\square$

Remark 3.4 (see [27])**.** As the Sobolev space $X=W_0^{1,p(x)}(\Omega)$ is a reflexive and separable Banach space, there exist $(e_n)_{n\in\mathbb{N}^*}\subseteq X$ and $(f_n)_{n\in\mathbb{N}^*}\subseteq X^*$ such that $f_n(e_m)=\delta_{nm}$ for any $n,m\in\mathbb{N}^*$ and

$$X=\overline{\operatorname{span}\{e_n : n\in\mathbb{N}^*\}}, \quad X^*=\overline{\operatorname{span}\{f_n : n\in\mathbb{N}^*\}}^{w^*}.$$

For $k\in\mathbb{N}^*$ denote by

$$X_k=\operatorname{span}\{e_k\},\ Y_k=\oplus_{j=1}^k X_j, \quad Z_k=\overline{\oplus_k^\infty X_j}.$$

Definition 3.5. We say that

(1) The C^1-functional H satisfies the Palais-Smale condition (in short (PS) condition) if any sequence $(u_n)_{n\in\mathbb{N}}\subseteq X$ for which, $(H(u_n))_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is bounded and $dH(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence.
(2) The C^1-functional H satisfies the Palais-Smale condition at the level c (in short $(PS)_c$ condition) for $c\in\mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}}\subseteq X$ for which, $H(u_n)\to c$ and $dH(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence.
(3) The C^1-functional H satisfies the $(PS)_c^*$ condition for $c\in\mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}}\subseteq X$ for which, $u_n\in Y_n$ for each $n\in\mathbb{N}$, $H(u_n)\to c$ and $d(H_{|Y_n})(u_n)\to 0$ as $n\to\infty$ with $Y_n,\ n\in\mathbb{N}$ as defined in Remark 3.4, has a subsequence convergent to a critical point of H.

Remark 3.6. It is easy to see that if H satisfies the (PS) condition, then H satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

Now, we state our main results of this work.

Theorem 3.7. *Under assumptions* (A1)–(A5), *problem* (1.1) *has a sequence of weak solutions* $(\pm u_n)_{n\in\mathbb{N}} \subseteq X$ *such that* $H(\pm u_n) \to +\infty$ *as* $n \to \infty$.

Theorem 3.8. *Under assumptions* (A1)–(A5), *problem* (1.1) *has a sequence of weak solutions* $(\pm u_n)_{n\in\mathbb{N}} \subseteq X$ *such that* $H(\pm u_n) \le 0$ *for each* $n \in \mathbb{N}$ *and* $H(\pm u_n) \to 0$ *as* $n \to \infty$.

The proofs of these above results will be based on a variational approach, using the critical points theory, we shall prove that the C^1-functional H has two different sequences of critical values. The main tools for this end are "Fountain theorem" and "Dual Fountain theorem" (see Willem [25, Theorem 6.5]) which we give below.

Theorem 3.9 ("Fountain theorem", [25])**.** *Let X be a reflexive and separable Banach space, $I \in C^1(X,\mathbb{R})$ be an even functional and the subspaces X_k, Y_k, Z_k as defined in remark 3.4. If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that*

1. $\inf_{x\in Z_k,\|x\|=r_k} I(x) \to \infty$ *as* $k \to \infty$,
2. $\max_{x\in Y_k,\|x\|=\rho_k} I(x) \le 0$,
3. *I satisfies the $(PS)_c$ condition for every $c > 0$.*

Then I has a sequence of critical values tending to $+\infty$.

Theorem 3.10 (Dual Fountain theorem [25])**.** *Let X be a reflexive and separable Banach space, $I \in C^1(X,\mathbb{R})$ be an even functional and the subspaces X_k, Y_k, Z_k as defined in remark 3.4. Assume that there is a $k_0 \in \mathbb{N}^*$ such that for each $k \in \mathbb{N}^*$, $k \ge k_0$, there exist $\rho_k > r_k > 0$ such that*

1. $\inf_{x\in Z_k,\ \|x\|=\rho_k} I(x) \ge 0$,
2. $b_k = \max_{x\in Y_k,\ \|x\|=r_k} I(x) < 0$,
3. $d_k = \inf_{x\in Z_k,\ \|x\|\le\rho_k} I(x) \to 0$ *as* $k \to \infty$,
4. *I satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.*

Then H has a sequence of negative critical values converging to 0.

We first prove that the functional H satisfies (PS) and $(PS)_c^*$ conditions.

Lemma 3.11. *Under assumptions* (A1)–(A3), *the functional H satisfies the (PS) condition.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq X$ be a (PS) sequence for H, i.e., $(H(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $dH(u_n) \to 0$ as $n \to \infty$. Then, there exists a positive constant $k \in \mathbb{R}$ such that

$$|H(u_n)| \le k, \text{ for every } n \in \mathbb{N}. \tag{3.3}$$

For $n \in \mathbb{N}$, we denote by $\Omega_n = \{x \in \Omega : |u_n(x)| \ge M\}$ and $\Omega'_n = \Omega\backslash\Omega_n$, with M as in assumption (A3). Without any loss of generality, we can suppose that $M \ge 1$. By Proposition 2.15 (i), there exist $c_1 \ge 0$ and $\sigma \in L^1(\Omega)$ such that

$$F(x, u_n(x)) \le c_1|u_n(x)|^{\beta(x)} + \sigma(x) \le c_1 M^{\beta^+} + \sigma(x),$$

for every $x \in \Omega_n'$. Hence,

$$\begin{aligned}
\int_{\Omega_n'} F(x, u_n(x))dx &\le \int_{\Omega_n'} (c_1 M^{\beta^+} + \sigma(x))dx \\
&\le \int_{\Omega} (c_1 M^{\beta^+} + \sigma(x))dx \\
&= c_1 M^{\beta^+} \operatorname{meas}(\Omega) + \int_{\Omega} \sigma(x)dx = k_1.
\end{aligned} \tag{3.4}$$

Using hypothesis (A3),

$$F(x, u_n(x)) \le \frac{1}{\theta} f(x, u_n(x))u_n(x), \quad \text{for all } x \in \Omega_n,$$

which gives

$$\begin{aligned}
&\int_{\Omega_n} F(x, u_n(x))dx \\
&\le \frac{1}{\theta} \int_{\Omega_n} f(x, u_n(x))u_n(x)dx \\
&= \frac{1}{\theta}\Big(\int_{\Omega} f(x, u_n(x))u_n(x)dx - \int_{\Omega_n'} f(x, u_n(x))u_n(x)dx\Big).
\end{aligned} \tag{3.5}$$

Using the growth condition in (A2),

$$\begin{aligned}
\Big|\int_{\Omega_n'} f(x, u_n(x))u_n(x)dx\Big| &\le \int_{\Omega_n'} (c|u_n(x)|^{\beta(x)} + h(x)|u_n(x)|)dx \\
&\le cM^{\beta^+} \operatorname{meas}(\Omega_n') + M \int_{\Omega_n'} h(x)dx \\
&\le cM^{\beta^+} \operatorname{meas}(\Omega) + M \int_{\Omega} |h(x)|dx = k_2,
\end{aligned}$$

which yields

$$-\frac{1}{\theta} \int_{\Omega_n'} f(x, u_n(x))u_n(x)dx \le \frac{k_2}{\theta}. \tag{3.6}$$

For $n \in \mathbb{N}$, using Hölder inequality, Proposition 2.7 and inequality 3.1, we can deduce that

$$\begin{aligned}
\int_{\Omega} |V(x)||u_n|^{q(x)}dx &\le 2|V|_{s(x)} \max\{|u_n|^{q^-}_{s'(x)q(x)}, |u_n|^{q^+}_{s'(x)q(x)}\} \\
&\le 2|V|_{s(x)} \max\{C^{q^-}\|u_n\|^{q^-}, C^{q^+}\|u_n\|^{q^+}\},
\end{aligned} \tag{3.7}$$

where $C > 1$ is a constant which appears in (3.1).

Let us show that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X. By contradiction, assume that $\|u_n\| \to +\infty$ as $n \to \infty$. For each $n \in \mathbb{N}$ with $\|u_n\| > 1$, using inequalities (3.3), (3.4), (3.5), (3.6) and (3.7), the following holds

$$\begin{aligned}
k + 1 &\ge H(u_n) - \frac{1}{\theta}\langle dH(u_n), u_n\rangle + \frac{1}{\theta}\langle dH(u_n), u_n\rangle \\
&= \int_{\Omega} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\Omega} \frac{V(x)}{q(x)}|u_n|^{q(x)}dx - \int_{\Omega} F(x, u_n)dx \\
&\quad - \frac{1}{\theta}\Big[\int_{\Omega} |\nabla u_n|^{p(x)}dx + \int_{\Omega} V(x)|u_n|^{q(x)}dx - \int_{\Omega} f(x, u_n)u_ndx\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\theta}\langle dH(u_n),u_n\rangle\\
&=\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx+\int_\Omega \frac{V(x)}{q(x)}|u_n|^{q(x)}dx-\int_{\Omega_n'}F(x,u_n)dx\\
&\quad-\int_{\Omega_n}F(x,u_n)dx-\frac{1}{\theta}\int_\Omega |\nabla u_n|^{p(x)}dx-\frac{1}{\theta}\int_\Omega V(x)|u_n|^{q(x)}dx\\
&\quad+\frac{1}{\theta}\int_\Omega f(x,u_n)u_ndx+\frac{1}{\theta}\langle dH(u_n),u_n\rangle\\
&\geq \frac{1}{p^+}\int_\Omega |\nabla u_n|^{p(x)}dx-\frac{1}{q^-}\int_\Omega |V(x)||u_n|^{q(x)}dx-\int_{\Omega_n'}F(x,u_n)dx\\
&\quad-\frac{1}{\theta}\int_\Omega |\nabla u_n|^{p(x)}dx-\frac{1}{\theta}\int_\Omega |V(x)||u_n|^{q(x)}dx+\frac{1}{\theta}\int_{\Omega_n'}f(x,u_n)u_ndx\\
&\quad+\frac{1}{\theta}\langle dH(u_n),u_n\rangle\\
&\geq \big(\frac{1}{p^+}-\frac{1}{\theta}\big)\varphi_p(\nabla u_n)-2C^{q^+}\big(\frac{1}{q^-}+\frac{1}{\theta}\big)|V|_{s(x)}\|u_n\|^{q^+}\\
&\quad-\frac{1}{\theta}\|dH(u_n)\|_{X^*}\|u_n\|-k_1-\frac{k_2}{\theta}\\
&\geq \big(\frac{1}{p^+}-\frac{1}{\theta}\big)\|u_n\|^{p^-}-2C^{q^+}\big(\frac{1}{q^-}+\frac{1}{\theta}\big)|V|_{s(x)}\|u_n\|^{q^+}\\
&\quad-\frac{1}{\theta}\|dH(u_n)\|_{X^*}\|u_n\|-k_3,
\end{aligned}
$$

where $k_3=k_1+\frac{k_2}{\theta}$. Since $\theta>p^+>q^+$, letting $n\to\infty$ in the last inequality we obtain a contradiction. Therefore, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X. Consequently, we can extract a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ weakly convergent to some u in X. Using the compact embedding $X\hookrightarrow L^{\alpha(x)}(\Omega)$, we deduce that the subsequence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in $L^{\alpha(x)}(\Omega)$. To prove the strong convergence of $(u_n)_{n\in\mathbb{N}}$ in X, we need the following proposition.

Proposition 3.12. *If $(u_n)_{n\in\mathbb{N}}$ converges weakly to u in X, then*

$$\lim_{n\to\infty}\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)dx=0.$$

Proof.

$$
\begin{aligned}
\Big|\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)dx\Big| &\leq c_0|V|_{s(x)}||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{\alpha(x)}\\
&\leq c_0|V|_{s(x)}|u_n|^{k_0}_{q(x)}|u_n-u|_{\alpha(x)},
\end{aligned}
$$

where c_0 and $k_0\in\{q^--1,q^+-1\}$ are positive constants. Using the compact embeddings $X\hookrightarrow L^{q(x)}(\Omega)$, $X\hookrightarrow L^{\alpha(x)}(\Omega)$ and the inequality $||u_n|_{q(x)}-|u|_{q(x)}|\leq |u_n-u|_{q(x)}$, we obtain $|u_n-u|_{q(x)}\to 0$ in $L^{q(x)}(\Omega)$, $|u_n-u|_{\alpha(x)}\to 0$ in $L^{\alpha(x)}(\Omega)$ and $|u_n|_{q(x)}\to |u|_{q(x)}$. The proof is complete. □

Since $dH(u_n)\to 0$ as $n\to\infty$, $(u_n)_{n\in\mathbb{N}}$ is bounded in X and

$$
\begin{aligned}
|\langle dH(u_n),u_n-u\rangle| &\leq |\langle dH(u_n),u_n\rangle|+|\langle dH(u_n),u\rangle|\\
&\leq \|dH(u_n)\|_{X^*}\|u_n\|+\|dH(u_n)\|_{X^*}\|u\|,
\end{aligned}
$$

we infer that

$$\lim_{n\to\infty}\langle dH(u_n), u_n - u\rangle = 0. \tag{3.8}$$

The Nemytskii operator N_f being strongly continuous, so $\lim_{n\to\infty} N_f(u_n) = N_f(u)$ in X^*, combine this fact and the weak convergence $u_n \rightharpoonup u$ in X, it follows that

$$\lim_{n\to\infty}\langle N_f(u_n), u_n - u\rangle = 0. \tag{3.9}$$

By Proposition 3.12, expressions (3.8) and (3.9), we can conclude that

$$\lim_{n\to\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle = 0.$$

Now, by Proposition 2.12 (iii), it is clear that the subsequence $u_n \to u$ in X strongly, since $-\Delta_{p(x)}$ is a mapping of type (S_+). The proof of Lemma 3.11 is complete. $\square$

Lemma 3.13. *Under assumptions* (A1)–(A3)*, the functional H satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}^*} \subseteq X$ be a $(PS)_c^*$ sequence for H with $c \in \mathbb{R}$, i.e., $u_n \in Y_n$ for each $n \in \mathbb{N}^*$, $H(u_n) \to c$ and $d(H_{|Y_n})(u_n) \to 0$ as $n \to \infty$. In a similar way to the proof of Lemma 3.11, we obtain the boundedness of the sequence $(u_n)_{n\in\mathbb{N}^*} \subseteq X$. Consequently, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}^*}$ of $(u_n)_{n\in\mathbb{N}^*}$ weakly convergent to some u in X. The space X can be written as $X = \overline{\cup_{n\in\mathbb{N}^*} Y_n}$, then we can choose a sequence $(v_n)_{n\in\mathbb{N}^*}$ such that $v_n \in Y_n$ for each $n \in \mathbb{N}^*$ and $\lim_{n\to\infty} v_n = u$ in X. We have the following expression

$$\langle dH(u_{n_k}), u_{n_k} - u\rangle = \langle dH(u_{n_k}), u_{n_k} - v_{n_k}\rangle + \langle dH(u_{n_k}), v_{n_k} - u\rangle. \tag{3.10}$$

As $d(H_{|Y_{n_k}})(u_{n_k}) \to 0$ as $k \to \infty$, $u_{n_k} - v_{n_k} \rightharpoonup 0$ in Y_{n_k} and $v_{n_k} \to u \in X$, we deduce that

$$\langle dH(u_{n_k}), u_{n_k} - v_{n_k}\rangle \to 0 \text{ and } \langle dH(u_{n_k}), v_{n_k} - u\rangle \to 0 \text{ as } k \to \infty. \tag{3.11}$$

Hence, (3.10) and (3.11) give us

$$\langle dH(u_{n_k}), u_{n_k} - u\rangle \to 0 \text{ as } k \to \infty. \tag{3.12}$$

We have seen that the Nemytskii operator $N_f : X \to X^*$ is strongly continuous while the $p(x)$-Laplacian operator is a mapping of type (S_+). These facts combine with Proposition 3.12, yield that $dH : X \to X^*$ is a mapping of type (S_+). Since the subsequence $(u_{n_k})_{k\in\mathbb{N}^*}$ converges weakly to u in X, from (3.12) it is clear that $\lim_{k\to\infty} u_{n_k} = u$ in X. Next, we show that u is a critical point of H. Choosing an arbitrary $w_n \in Y_n$, for any $n_k \ge n$, we can write

$$\begin{aligned}\langle dH(u), w_n\rangle &= \langle dH(u) - dH(u_{n_k}), w_n\rangle + \langle dH(u_{n_k}), w_n\rangle\\ &= \langle dH(u) - dH(u_{n_k}), w_n\rangle + \langle d(H_{|Y_{n_k}})(u_{n_k}), w_n\rangle.\end{aligned}$$

Since $H \in C^1(X, \mathbb{R})$ and $\lim_{k\to\infty} u_{n_k} = u$ in X, it follows that $\lim_{k\to\infty} dH(u_{n_k}) = dH(u)$. Therefore, (3.13) letting $k \to \infty$ we deduce that $\langle dH(u), w_n\rangle = 0$ for all $w_n \in Y_n$, hence $dH(u) = 0$. In conclusion, H satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. The proof is complete. $\square$

Now, we state several Lemmas that will be useful in the sequel.

Lemma 3.14 (see [13]). *If $\alpha \in C_+(\overline{\Omega})$ with $\alpha(x) < p^*(x)$, for every $x \in \overline{\Omega}$, for each $k \in \mathbb{N}^*$ denote*

$$\beta_k = \sup\{|u|_{\alpha(x)} : u \in Z_k,\ \|u\| = 1\}.$$

Then, $\beta_k < \infty$ and $\lim_{k\to\infty} \beta_k = 0$.

Lemma 3.15 (see [16]). *Assume that $\Theta : X \to \mathbb{R}$ is a strongly continuous functional and $\Theta(0) = 0$, for each $\gamma > 0$ and $k \in \mathbb{N}^*$ denote*

$$\alpha_k = \sup\{|\Theta(u)| : u \in Z_k,\ \|u\| \le \gamma\}.$$

Then, $\alpha_k < \infty$ and $\lim_{k\to\infty} \alpha_k = 0$.

Lemma 3.16. *Assume that the Carathéodory function f satisfies* (A2), (A3). *Then there exist $k_1, k_2 > 0$, $\sigma_0 \in L^1(\Omega)$ and $\chi \in L^\infty(\Omega)$ with $\chi(x) > 0$ for every $x \in \Omega$ such that*

$$F(x,t) \ge \chi(x)|t|^\theta - k_1 - k_2\sigma_0(x), \text{ for } x \in \Omega,\ t \in \mathbb{R}.$$

Now, we are in a position to give the proofs of main theorems state above.

Proof of Theorem 3.7. Let us verify the conditions of the Fountain theorem. It is clear that the C^1-functional $H : X \to \mathbb{R}$ defined by

$$H(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega F(x,u)dx$$

is even and, by Lemma 3.11, it satisfies the (PS) condition. So, the functional H satisfies also the $(PS)_c$ condition for every $c > 0$, which gives the condition (3) of Fountain theorem.

Let us prove that for each $k \in \mathbb{N}^*$ there exists $r_k > 0$ such that

$$\inf_{u\in Z_k,\ \|u\|=r_k} H(u) \to \infty \text{ as } k \to \infty.$$

By Proposition 2.15, we deduce that

$$\Big|\int_\Omega F(x,u(x))dx\Big| \le \int_\Omega \big(c_1|u|^{\beta(x)} + \sigma(x)\big)dx \le c_1\varphi_\beta(u) + c_2,$$

where $c_2 = \int_\Omega \sigma(x)dx$. By Proposition 2.6 (1) and (2), $\varphi_\beta(u) \le 1$ if $|u|_{\beta(x)} \le 1$ and $\varphi_\beta(u) \le |u|^{\beta^+}_{\beta(x)}$ if $|u|_{\beta(x)} > 1$, respectively. Using Lemma 3.14, we also have $|u|_{\beta(x)} \le \beta_k\|u\|$, for all $u \in Z_k$. Then, for $u \in Z_k$ with $\|u\| \ge 1$, it follows that

$$\begin{aligned}
H(u) &\ge \frac{1}{p^+}\varphi_p(|\nabla u|) - \frac{1}{q^-}\int_\Omega |V(x)||u(x)|^{q(x)}dx - c_1\varphi_\beta(u) - c_2\\
&\ge \begin{cases} \frac{1}{p^+}\|u\|^{p^-} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}\|u\|^{q^+} - c_1 - c_2 & \text{if } |u|_{\beta(x)} \le 1\\ \frac{1}{p^+}\|u\|^{p^-} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}\|u\|^{q^+} - c_1\beta_k^{\beta^+}\|u\|^{\beta^+} - c_2 & \text{if } |u|_{\beta(x)} > 1 \end{cases}\\
&\ge \frac{1}{p^+}\|u\|^{p^-} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}\|u\|^{q^+} - c_1\beta_k^{\beta^+}\|u\|^{\beta^+} - c_3,
\end{aligned}$$

where $c_3 = c_1 + c_2$. For each $k \in \mathbb{N}^*$, define the real numbers r_k by

$$r_k = (c_1\beta^+\beta_k^{\beta^+})^{\frac{1}{p^- - \beta^+}}.$$

From hypothesis (A5), we know that $\beta^+ > p^-$, hence $\lim_{k\to\infty} r_k = +\infty$. Without any loss of generality, we can suppose that $r_k \geq 1$ for each $k \in \mathbb{N}^*$. Using the above inequality, for all $u \in Z_k$ with $\|u\| = r_k$, we infer that

$$\begin{aligned}
H(u) &\geq \frac{1}{p^+}(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_k^{\beta^+})^{\frac{q^+}{p^- - \beta^+}} \\
&\quad - c_1\beta_k^{\beta^+}(c_1\beta^+\beta_k^{\beta^+})^{\frac{\beta^+}{p^- - \beta^+}} - c_3 \\
&= \frac{\beta^+ - p^+}{\beta^+ p^+}(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_k^{\beta^+})^{\frac{q^+}{p^- - \beta^+}} - c_3 \\
&= (c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}}\Big[\frac{\beta^+ - p^+}{\beta^+ p^+} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^- - q^+}{\beta^+ - p^-}} \\
&\quad - \frac{c_3}{(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}}}\Big].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\inf_{u\in Z_k,\, \|u\|=r_k} H(u) &\geq (c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}}\Big[\frac{\beta^+ - p^+}{\beta^+ p^+} \\
&\quad - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^- - q^+}{\beta^+ - p^-}} - \frac{c_3}{(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}}}\Big].
\end{aligned} \tag{3.13}$$

Using inequality (3.13) and hypothesis (A5), it is obvious that

$$\inf_{u\in Z_k,\ \|u\|=r_k} H(u) \to +\infty \text{ as } k \to \infty,$$

so condition (1) of Fountain theorem is satisfied.

It remain to prove that for each $k \in \mathbb{N}^*$ there exists $\rho_k > r_k > 0$ such that

$$\max_{u\in Y_k,\ \|u\|=\rho_k} H(u) \leq 0.$$

The functional $\|\cdot\|_\theta : X \to \mathbb{R}$ defined by

$$\|u\|_\theta = \Big(\int_\Omega \chi(x)|u(x)|^\theta dx\Big)^{1/\theta}$$

being a norm on the Banach space X, with χ as defined in Lemma 3.16. Then, on the finite dimensional subspace Y_k the norms $\|\cdot\|$ and $\|\cdot\|_\theta$ are equivalent, so there exists a constant $\delta > 0$ such that $\|u\|_\theta \geq \delta\|u\|$, for all $u \in Y_k$. Using Lemma 3.16, we also obtain $\int_\Omega F(x,u)dx \geq \|u\|_\theta^\theta - k_3$, where $k_3 = \int_\Omega (k_1 + k_2\sigma_0(x))dx$. Then, for all $u \in Y_k$ with $\|u\| \geq 1$, we have

$$\begin{aligned}
H(u) &\leq \frac{1}{p^-}\varphi_p(|\nabla u|) + \frac{1}{q^-}\int_\Omega |V(x)||u(x)|^{q(x)}dx - \|u\|_\theta^\theta + k_3 \\
&\leq \frac{1}{p^-}\|u\|^{p^+} + \frac{2C^{q^+}}{q^-}|V|_{s(x)}\|u\|^{q^+} - \delta^\theta\|u\|^\theta + k_3.
\end{aligned} \tag{3.14}$$

Hypothesis $\theta > p^+ > q^+$ implies that

$$\lim_{t\to\infty}\Big(\frac{1}{p^-}t^{p^+} + \frac{2C^{q^+}}{q^-}|V|_{s(x)}t^{q^+} - \delta^\theta t^\theta + k_3\Big) = -\infty.$$

Then, there exists $t_0 > 0$ such that for all $t \in [1, +\infty) \cap [t_0, +\infty)$

$$\frac{1}{p^-} t^{p^+} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} t^{q^+} - \delta^\theta t^\theta + k_3 \le -1. \tag{3.15}$$

By Choosing $\rho_k = \max\{r_k, t_0\} + 1$, inequality (3.15) is fulfilled for $t = \rho_k$. Then, for all $u \in Y_k$ with $\|u\| = \rho_k$, it follows that

$$\frac{1}{p^-} \|u\|^{p^+} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} \|u\|^{q^+} - \delta^\theta \|u\|^\theta + k_3 \le -1 < 0. \tag{3.16}$$

Combine (3.14) and (3.16), it is obvious that

$$\max_{u \in Y_k,\, \|u\| = \rho_k} H(u) \le 0,$$

which shows that the condition (2) of Fountain theorem is satisfied.

By applying Theorem 3.9 ("Fountain theorem"), the C^1-functional H has a sequence of critical values tending to $+\infty$. Therefore, there is a sequence $(\pm u_n)_{n \in \mathbb{N}} \subseteq X$ of critical points for the functional H such that $H(\pm u_n) \to +\infty$ as $n \to \infty$. So, the proof is complete. □

Proof of Theorem 3.8. Let us verify the conditions of the Dual Fountain theorem. The C^1-functional H is even, because the function f is odd in its second argument (see hypothesis (A4)). By Lemma 3.13, the functional H satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$, in particular for every $c \in [d_{k_0}, 0)$, so condition (4) of Dual Fountain theorem is satisfied.

We first prove that for each $k \in \mathbb{N}^*$ there exists $r_k > 0$ such that

$$\max_{u \in Y_k,\ \|u\| = r_k} H(u) < 0.$$

The norm $\|\cdot\|_\theta$ defined previously being equivalent with the norm $\|\cdot\|$ on the finite dimensional subspace Y_k, there exists a constant $\delta > 0$ such that $\|u\|_\theta \ge \delta \|u\|$, for all $u \in Y_k$. As in the proof of Theorem 3.7, for all $u \in Y_k$ with $\|u\| \ge 1$, the following inequality holds

$$H(u) \le \frac{1}{p^-} \|u\|^{p^+} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} \|u\|^{q^+} - \delta^\theta \|u\|^\theta + k_3. \tag{3.17}$$

Hypothesis $\theta > p^+ > q^+$ implies that

$$\lim_{t \to \infty} \Big(\frac{1}{p^-} t^{p^+} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} t^{q^+} - \delta^\theta t^\theta + k_3 \Big) = -\infty.$$

So, there exists a constant $t_1 \in (1, +\infty)$ such that for all $t \in [t_1, +\infty)$

$$\frac{1}{p^-} t^{p^+} + \frac{2C^{q^+}}{q^-} |V|_{s(x)} t^{q^+} - \delta^\theta t^\theta + k_3 \le -1. \tag{3.18}$$

Inequalities (3.17) and (3.18) show that, for any $u \in Y_k$ with $\|u\| = t_1,\ H(u) \le -1$. Choosing $r_k = t_1$ for each $k \in \mathbb{N}^*$, we deduce that

$$\max_{u \in Y_k,\ \|u\| = r_k} H(u) \le -1 < 0,$$

so condition (2) of Dual Fountain theorem is satisfied.

Second, we prove that there is $k_0 \in \mathbb{N}^*$ such that for each $k \in \mathbb{N}^*,\ k \geq k_0$, there exists $\rho_k > r_k > 0$ for which

$$\inf_{u\in Z_k,\ \|u\|=\rho_k} H(u) \geq 0.$$

In a similar way to the proof of Theorem 3.7, for all $u \in Z_k$ with $\|u\| \geq 1$, the following inequality holds

$$H(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}\|u\|^{q^+} - c_1\beta_k^{\beta^+}\|u\|^{\beta^+} - c_3. \tag{3.19}$$

We also have

$$\begin{aligned}
&\lim_{k\to\infty}(c_1\beta^+\beta_k^{\beta^+})^{\frac{1}{p^- - \beta^+}} \\
&= \lim_{k\to\infty}\Big[\frac{\beta^+ - p^+}{\beta^+ p^+}(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_k^{\beta^+})^{\frac{q^+}{p^- - \beta^+}} - c_3\Big] = +\infty.
\end{aligned}$$

Then, there is $k_0 \in \mathbb{N}^*$ such that for all $k \geq k_0$, $(c_1\beta^+\beta_k^{\beta^+})^{\frac{1}{p^- - \beta^+}} > t_1$ and

$$\frac{\beta^+ - p^+}{\beta^+ p^+}(c_1\beta^+\beta_k^{\beta^+})^{\frac{p^-}{p^- - \beta^+}} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_k^{\beta^+})^{\frac{q^+}{p^- - \beta^+}} - c_3 \geq 0.$$

By Choosing $\rho_k = (c_1\beta^+\beta_{k_0}^{\beta^+})^{\frac{1}{p^- - \beta^+}}$ for $k \geq k_0$, it follows that $\rho_k > r_k = t_1 > 0$ for each $k \in \mathbb{N}^*$. Using (3.19), it is obvious that

$$H(u) \geq \frac{\beta^+ - p^+}{\beta^+ p^+}(c_1\beta^+\beta_{k_0}^{\beta^+})^{\frac{p^-}{p^- - \beta^+}} - \frac{2C^{q^+}}{q^-}|V|_{s(x)}(c_1\beta^+\beta_{k_0}^{\beta^+})^{\frac{q^+}{p^- - \beta^+}} - c_3 \geq 0,$$

for all $u \in Z_k,\ \|u\| = \rho_k$. Finally, this last inequality gives

$$\inf_{u\in Z_k,\ \|u\|=\rho_k} H(u) \geq 0,$$

which shows that the condition (1) of Dual Fountain theorem is satisfied. Next, we prove that

$$\inf_{u\in Z_k,\ \|u\|\leq\rho_k} H(u) \to 0 \text{ as } k \to \infty.$$

Let us denote

$$b_k = \max_{u\in Y_k,\, \|u\|=r_k} H(u), \quad d_k = \inf_{u\in Z_k,\, \|u\|\leq\rho_k} H(u).$$

It is easy to remark that $Y_k \cap Z_k \neq 0$ for each $k \in \mathbb{N}^*$. For $k \geq k_0$, let $u_0 \in Y_k \cap Z_k$, with $u_0 \neq 0$, and $u_k = \frac{r_k}{\|u_0\|}u_0$, then $\|u_k\| = r_k$. Since $0 < r_k < \rho_k$ for each $k \geq k_0$, it follows that

$$d_k \leq H(u_k) \leq b_k < 0, \text{ for each } k \geq k_0.$$

From hypothesis (A2), a straightforward computation gives

$$|F(x,t)| \leq \frac{c}{\beta(x)}|t|^{\beta(x)} + h(x)|t|, \quad \text{for } x \in \Omega,\ t \in \mathbb{R}. \tag{3.20}$$

Let us consider the functionals $\Psi_1,\ \Psi_2,\ \Psi_3 : X \to \mathbb{R}$ defined by

$$\Psi_1(u) = \int_\Omega \frac{|V(x)|}{q(x)}|u(x)|^{q(x)}dx, \quad \Psi_2(u) = \int_\Omega \frac{c}{\beta(x)}|u(x)|^{\beta(x)}dx,$$
$$\Psi_3(u) = \int_\Omega h(x)|u(x)|dx.$$

Obviously, $\Psi_1(0) = \Psi_2(0) = \Psi_3(0) = 0$.

Proposition 3.17. *The functionals Ψ_1, Ψ_2 and Ψ_3 are strongly continuous.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence and $u \in X$ such that $u_n \to u$ weakly in X. We have to show that $\Psi_1(u_n) \to \Psi_1(u)$ in $\mathbb{R}$. Using the inequality

$$|\Psi_1(u_n) - \Psi_1(u)| \leq \frac{1}{q^-}\int_\Omega |V(x)||u_n|^{q(x)} - |u|^{q(x)}|dx,$$

and the well known inequality

$$||a|^p - |b|^p| \leq \gamma|a-b|\left(|a|+|b|\right)^{p-1}, \quad \text{for all } a,b\in\mathbb{R}, \tag{3.21}$$

where γ is a positive constant, we obtain

$$\begin{aligned}
|\Psi_1(u_n) - \Psi_1(u)| &\leq \frac{\gamma}{q^-}\int_\Omega |V(x)||u_n - u|\left(|u_n|+|u|\right)^{q(x)-1}dx \\
&\leq \frac{\gamma c_0}{q^-}\|V\|_{s(x)}|u_n - u|_{\alpha(x)}|\left(|u_n|+|u|\right)^{q(x)-1}|_{\frac{q(x)}{q(x)-1}} \\
&\leq \frac{\gamma c_0}{p^-}\|V\|_{s(x)}|u_n - u|_{q(x)}\left(|u_n|_{q(x)}+|u|_{q(x)}\right)^r,
\end{aligned}$$

where c_0 is a positive constant of Hölder inequality and $r \in \{q^- - 1, q^+ - 1\}$. Since the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, then $\lim_{n\to\infty}|u_n - u|_{q(x)} = 0$ and $\lim_{n\to\infty}\left(|u_n|_{q(x)}+|u|_{q(x)}\right)^r = 2^r|u|^r$. Hence,

$$\lim_{n\to\infty}\Psi_1(u_n) = \Psi_1(u).$$

The embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$ being compact and continuous, similar computations show that

$$\lim_{n\to\infty}\Psi_2(u_n) = \Psi_2(u) \quad \text{and} \quad \lim_{n\to\infty}\Psi_3(u_n) = \Psi_3(u).$$

□

Now, denote

$$\begin{aligned}
\lambda_k &= \sup\{|\Psi_1(u)| : u\in Z_k,\ \|u\|\leq 1\},\\
\gamma_k &= \sup\{|\Psi_2(u)| : u\in Z_k,\ \|u\|\leq 1\},\\
\varepsilon_k &= \sup\{|\Psi_3(u)| : u\in Z_k,\ \|u\|\leq 1\}.
\end{aligned}$$

By Lemma 3.15 and Proposition 3.17, we obtain

$$\lim_{k\to\infty}\lambda_k = \lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}\varepsilon_k = 0. \tag{3.22}$$

Choose $v \in Z_k$ with $\|v\| \leq 1$ and $1 < t \leq \rho_k$, using inequality (3.20), we get

$$\begin{aligned}
H(tv) &= \int_\Omega \frac{1}{p(x)}|\nabla tv(x)|^{p(x)} + \int_\Omega \frac{V(x)}{q(x)}|tv(x)|^{q(x)}dx - \int_\Omega F(x,tv(x))dx \\
&\geq -\int_\Omega \frac{|V(x)|}{q(x)}|tv(x)|^{q(x)}dx - \int_\Omega \frac{c}{\beta(x)}|tv(x)|^{\beta(x)}dx - \int_\Omega h(x)|tv(x)|dx \\
&= -\Psi_1(tv) - \Psi_2(tv) - \Psi_3(tv).
\end{aligned}$$

Since $\Psi_1(tv) \leq \rho_k^{q^+}\lambda_k$, $\Psi_2(tv) \leq \rho_k^{\beta^+}\gamma_k$ and $\Psi_3(tv) \leq \rho_k\varepsilon_k$, it follows that

$$H(tv) \geq -\rho_k^{q^+}\lambda_k - \rho_k^{\beta^+}\gamma_k - \rho_k\varepsilon_k, \text{ for all } t\in]1,\rho_k] \text{ and } v\in Z_k \text{ with } \|v\|\leq 1.$$

Hence, from the last inequality, we deduce that

$$-\rho_k^{q^+}\lambda_k - \rho_k^{\beta^+}\gamma_k - \rho_k\varepsilon_k \leq d_k < 0, \quad \text{for all } k\geq k_0. \tag{3.23}$$

The real number $\rho_k = (c_1\beta^+\beta_{k_0}^{\beta^+})^{\frac{1}{p^- - \beta^+}}$ being a positive constant, expressions (3.22) and (3.23) yield $\lim_{k\to\infty} d_k = 0$, so the condition (3) of Dual Fountain theorem is satisfied.

By applying Theorem 3.10 ("Dual Fountain theorem"), the C^1-functional H has a sequence of negative critical values converging to 0. Therefore, there is a sequence $(\pm u_n)_{n\in\mathbb{N}} \subseteq X$ of critical points for the functional H such that $H(\pm u_n) \leq 0$ for each $n \in \mathbb{N}$ and $H(\pm u_n) \to 0$ as $n \to \infty$. The proof is complete. □

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