Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 192, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LOGARITHMICALLY IMPROVED BLOW-UP CRITERIA FOR THE 3D NONHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VACUUM

QIANQIAN HOU, XIAOJING XU, ZHUAN YE

Abstract. This article is devoted to the study of the nonhomogeneous incompressible Navier-Stokes equations in space dimension three. By making use of the "weakly nonlinear" energy estimate approach introduced by Lei and Zhou in [16], we establish two logarithmically improved blow-up criteria of the strong or smooth solutions subject to vacuum for the 3D nonhomogeneous incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$. This results extend recent regularity criterion obtained by Kim (2006) [13].

1. Introduction

In this article we study a blow-up criterion of strong solutions to the 3D nonhomogeneous incompressible Navier-Stokes equation in the whole space $\mathbb{R}^3$,

$$\begin{gathered}\rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \Delta u + \nabla \pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, \rho u)|_{t=0} = (\rho_0, \rho_0 u_0),\end{gathered} \tag{1.1}$$

where $u = u(x,t) = (u_1(x,t),\, u_2(x,t),\, u_3(x,t))$, $\rho = \rho(x,t)$ and $\pi = \pi(x,t)$ denote the unknown velocity, density and pressure, respectively. The system (1.1) describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [17] for the detailed derivation.

In the past decades, there has been a lot of literature about the well-posedness theory of the incompressible Navier-Stokes equations (1.1). When the initial density is strictly positive, there has been proved that there is a unique strong solution to the problem (1.1) in dimension three, which is locally defined for large initial data, while globally defined for the case of small data (see for example [1, 2, 3, 8, 10, 12, 15]). On the other hand, for initial data which permits regions of vacuum, i.e. regions where the density ρ vanishes on some set, the problem becomes much more complicated. The global existence of weak solutions of the system (1.1) has been established (see [14, 17, 18]). However, the problem of uniqueness and regularity of such weak

2010 *Mathematics Subject Classification.* 35Q30, 35B40, 76D03, 76D05.
Key words and phrases. Nonhomogeneous Navier-Stokes equations; blow-up criterion; strong solution; vacuum.

Submitted April 29, 2015. Published July 14, 2016.

solutions is full of challenge and remains open. Very recently, Craig-Huang-Wang [7] proved the global existence of strong solution with vacuum of the system (1.1) under the assumption that the initial data $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough. We refer the interested readers to [4, 5, 6, 11, 9, 21] for many more results.

Recently, Choe and Kim [4] established an existence result on strong solutions with nonnegative densities for the system (1.1). More precisely, it was proved that if the data ρ_0 and u_0 satisfy the following regularity condition

$$0 \le \rho_0 \in L^{3/2} \cap H^2, \quad u_0 \in H_0^1 \cap H^2$$

and the compatibility condition

$$-\Delta u_0 + \nabla \pi_0 = \sqrt{\rho_0} g, \quad \operatorname{div} u_0 = 0,$$

with $(\pi_0,\, g) \in H^1 \times L^2$. Then there exist a time $T_\star \in (0,T)$ and a unique strong solution (ρ, u, π) to the system (1.1) such that

$$\begin{gathered}\rho \in L^\infty(0,T_\star; L^\infty \cap H^1), \quad \nabla u, \pi \in L^\infty(0,T_\star;H^1) \cap L^2(0,T_\star;W^{1,6}),\\ \rho_t \in L^\infty(0,T_\star;L^2), \quad \sqrt{\rho} u_t \in L^\infty(0,T_\star;L^2), \quad u_t \in L^2(0,T_\star;H_0^1),\end{gathered}$$

Here we would like to emphasize that Kim [13] established the so-called Serrin type regularity criterion to the system (1.1), which reads: If

$$u \in L^q(0,T;L^p_w(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \le 1,\ 3 < p \le \infty,$$

then the solution can be extended beyond time T. Here L^p_w denotes the weak L^p-space.

The aim of this article is to establish the logarithmic Serrin type regularity criterion, which improves the result of [13]. More precisely,

Theorem 1.1. *Suppose that (ρ, u, π) is the unique local strong solution (established by Choe and Kim [4]) in time interval $[0,T)$ to the system (1.1). If*

$$\int_0^T \frac{\|u(t)\|^r_{L^p_w(\mathbb{R}^3)}}{\ln\big(e + \|u(t)\|_{L^p_w(\mathbb{R}^3)}\big)}\, dt < \infty, \tag{1.2}$$

for $\frac{3}{p} + \frac{2}{r} = 1$ with $3 < p \le \infty$, then the solution (ρ, u, π) can be extended beyond time T. In other words, if the solution blows up at T^*, then

$$\int_0^{T^*} \frac{\|u(t)\|^r_{L^p_w(\mathbb{R}^3)}}{\ln\big(e + \|u(t)\|_{L^p_w(\mathbb{R}^3)}\big)}\, dt = \infty.$$

Our second result concerning the following regularity criterion in the Besov space with negative index reads as follows.

Theorem 1.2. *Suppose that (ρ, u, π) is the unique local strong solution (established by Choe and Kim [4]) in time interval $[0,T)$ to the system (1.1). If*

$$\int_0^T \frac{\|u(t)\|^{\frac{2}{1-\delta}}_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)}}{\ln\big(e + \|u(t)\|_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)}\big)}\, dt < \infty, \tag{1.3}$$

for $0 < \delta < 1$, then the solution (ρ, u, π) can be extended beyond time T. In other words, if the solution blows up at T^*, then

$$\int_0^{T^*} \frac{\|u(t)\|_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)}^{\frac{2}{1-\delta}}}{\ln\big(e + \|u(t)\|_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)}\big)}\,dt = \infty.$$

Remark 1.3. At the moment we are not able to show above Theorem 1.2 still holds for the case $\delta = 0$, even we replace the logarithmic type assumption (1.3) by

$$\int_0^T \|u(t)\|^2_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)}\,dt < \infty. \tag{1.4}$$

Fortunately, we established a regularity criteria which are slightly weaker than (1.4),

$$\int_0^T \|u(t)\|^2_{\dot{B}^0_{\infty,2}(\mathbb{R}^3)}\,dt < \infty \quad \text{and} \quad \int_0^T \|u(t)\|^2_{\mathrm{BMO}(\mathbb{R}^3)}\,dt < \infty \quad \text{(see [19]).}$$

Let us state the following result corresponding to the case $\delta = 1$.

Theorem 1.4. *Suppose that (ρ, u, π) is the unique local strong solution (established by Choe and Kim [4]) in time interval $[0, T)$ to the system (1.1). If there exists a small constant η such that*

$$\sup_{0\le t\le T} \|u(t)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \le \eta, \tag{1.5}$$

then the solution (ρ, u, π) can be extended beyond time T.

2. Proof of Theorem 1.1

The proof is based on the “weakly nonlinear” energy estimate approach introduced firstly by Lei and Zhou in [16]. Since the local strong or smooth solutions to the system (1.1) was established by Choe and Kim [4], the key step in the proof of Theorem 1.1 is to prove a priori estimates.

If (1.2) holds, one can deduce that for any small $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) < T$ such that

$$\int_{T_0}^T \frac{\|u(t)\|^r_{L^p_w(\mathbb{R}^3)}}{\ln\big(e + \|u(t)\|_{L^p_w(\mathbb{R}^3)}\big)}\,dt \le \epsilon. \tag{2.1}$$

In what follows, we choose some suitable ϵ. In sequel, C stands for some real positive constant which may be different in each occurrence and depend on ρ_0, u_0, T_0, T and so on. It is easy to show the following the basic estimates

$$\begin{gathered} \|\sqrt{\rho}u(t)\|^2_{L^2} + \int_0^t \|\nabla u(s)\|^2_{L^2}\,ds \le C(\rho_0, u_0) < \infty, \\ \|\rho(t)\|_{L^q} \le \|\rho_0\|_{L^q} < \infty, \end{gathered} \tag{2.2}$$

for any $2 \le q \le \infty$.

Testing the second equation of (1.1) by u_t and integrating over $\mathbb{R}^3$, we see that by using the mass equation $(1.1)_1$ and divergence-free condition

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} + \|\sqrt{\rho}u_t\|^2_{L^2} \le \Big|\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u_t\,dx\Big|. \tag{2.3}$$

It follows from [13, 209)] that

$$\Big|\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u_t\,dx\Big| \le C(1 + \|u\|^r_{L^p_w})\|\nabla u\|^2_{L^2}.$$

For any $t \in (T_0, T)$, we denote

$$y(t) := \max_{\tau \in [T_0,\, t]} \|\Lambda^{\frac{3(p-2)}{2p}} u(\tau)\|_{L^2}, \quad 3 < p \le \infty.$$

It should be noted that the function $y(t)$ is nondecreasing. As a consequence of Gronwall inequality, we can conclude that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \int_{T_0}^t \|\sqrt{\rho} u_t(s)\|_{L^2}^2\, ds \\
&\le \|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A \int_{T_0}^t (1 + \|u(s)\|_{L^p_w}^r)\, ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A \int_{T_0}^t \frac{\|u(s)\|_{L^p_w}^r}{\ln\big(e + \|u(s)\|_{L^p_w}\big)} \ln\big(e + \|u(s)\|_{L^p_w}\big)\, ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A \int_{T_0}^t \frac{\|u(s)\|_{L^p_w}^r}{\ln\big(e + \|u(s)\|_{L^p_w}\big)} \ln\big(e + \|\Lambda^{\frac{3(p-2)}{2p}} u(s)\|_{L^2}\big)\, ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A \int_{T_0}^t \frac{\|u(s)\|_{L^p_w}^r}{\ln\big(e + \|u(s)\|_{L^p_w}\big)} \ln\big(e + y(s)\big)\, ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A \int_{T_0}^t \frac{\|u(s)\|_{L^p_w}^r}{\ln\big(e + \|u(s)\|_{L^p_w}\big)}\, ds \cdot \ln\big(e + y(t)\big)\Big] \\
&\le C\big(e + y(t)\big)^{A\epsilon},
\end{aligned} \tag{2.4}$$

where A is an absolute constant and we have used the following facts

$$\|u\|_{L^p_w(\mathbb{R}^3)} \le C\|u\|_{L^p(\mathbb{R}^3)} \le C\|\Lambda^{\frac{3(p-2)}{2p}} u\|_{L^2(\mathbb{R}^3)}.$$

Thus, we infer that

$$\|\nabla u(t)\|_{L^2}^2 + \int_{T_0}^t \|\sqrt{\rho} u_t(s)\|_{L^2}^2\, ds \le C\big(e + y(t)\big)^{A\epsilon}. \tag{2.5}$$

In view of the mass equation $(1.1)_1$, we can rewrite the second equation of (1.1) as

$$-\Delta u + \nabla \pi = -\rho u_t - \rho u \cdot \nabla u.$$

Applying the Helmholtz-Weyl operator to above equation, then using the boundedness of Calderón-Zygmund (or the Stokes theorem), it is not hard to deduce that

$$\begin{aligned}
\|\Delta u\|_{L^2} &\le C(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2}) \\
&\le C(\|\rho u_t\|_{L^2} + \|\rho\|_{L^\infty} \|u\|_{L^\infty} \|\nabla u\|_{L^2}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|\rho\|_{L^\infty} \|u\|_{L^6}^{\frac12} \|\Delta u\|_{L^2}^{\frac12} \|\nabla u\|_{L^2}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|\rho_0\|_{L^\infty} \|\nabla u\|_{L^2}^{\frac12} \|\Delta u\|_{L^2}^{\frac12} \|\nabla u\|_{L^2}) \\
&\le \frac12 \|\Delta u\|_{L^2} + C(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}^3),
\end{aligned} \tag{2.6}$$

and for any $p > 3$,

$$\begin{aligned} \|\Delta u\|_{L^{\frac{3p}{2p+3}}} &\le C(\|\rho u_t\|_{L^{\frac{3p}{2p+3}}} + \|\rho u\cdot\nabla u\|_{L^{\frac{3p}{2p+3}}}) \\ &\le C(\|\sqrt{\rho}\|_{L^{\frac{6p}{p+6}}}\|\sqrt{\rho}u_t\|_{L^2} + \|\rho\|_{L^p}\|u\|_{L^6}\|\nabla u\|_{L^2}) \\ &\le C(\|\sqrt{\rho_0}\|_{L^{\frac{6p}{p+6}}}\|\sqrt{\rho}u_t\|_{L^2} + \|\rho_0\|_{L^p}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}) \\ &\le C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^2). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) leads to

$$\|\Delta u\|_{L^2} + \|\Delta u\|_{L^{\frac{3p}{2p+3}}} \le C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^3). \tag{2.8}$$

Note that by (2.5), we obtain

$$\begin{aligned} \int_{T_0}^t \|\Delta u(s)\|_{L^2}^2\,ds &\le C\int_{T_0}^t (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^6)(s)\,ds \\ &\le C\big(e+y(t)\big)^{3A\epsilon}. \end{aligned} \tag{2.9}$$

Combining (2.5) and (2.9), we obtain

$$\|\nabla u(t)\|_{L^2}^6 + \int_{T_0}^t (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)(s)\,ds \le C\big(e+y(t)\big)^{3A\epsilon}. \tag{2.10}$$

Differentiating the momentum equation with respect to t, multiplying by u_t, and then integrating over whole space, one can obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 &= -\int_{\mathbb{R}^2}\rho_t u_t\cdot u_t\,dx - \int_{\mathbb{R}^2}(\rho u)_t\cdot\nabla u\cdot u_t\,dx \\ &:= J_1 + J_2. \end{aligned} \tag{2.11}$$

By the mass equation, we derive

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^2}\operatorname{div}(\rho u)u_t\cdot u_t\,dx \\ &\le 2\Big|\int_{\mathbb{R}^2}\rho u\nabla u_t\cdot u_t\,dx\Big| \\ &\le C\|u\|_{L^6}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^3} \\ &\le C\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12} \\ &\le C\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|u_t\|_{L^6}^{\frac12} \\ &\le C\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac12} \\ &\le \frac{1}{8}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\sqrt{\rho}u_t\|_{L^2}^2. \end{aligned} \tag{2.12}$$

Again we resort to the mass equation to obtain

$$\begin{aligned} J_2 &= -\int_{\mathbb{R}^2}\rho u_t\cdot\nabla u\cdot u_t\,dx - \int_{\mathbb{R}^2}\rho_t u\cdot\nabla u\cdot u_t\,dx \\ &= -\int_{\mathbb{R}^2}\rho u_t\cdot\nabla u\cdot u_t\,dx + \int_{\mathbb{R}^2}\operatorname{div}(\rho u)u\cdot\nabla u\cdot u_t\,dx \\ &= -\int_{\mathbb{R}^2}\rho u_t\cdot\nabla u\cdot u_t\,dx - \int_{\mathbb{R}^2}(\rho u)\nabla(u\cdot\nabla u\cdot u_t)\,dx \\ &= J_{21} + J_{22}. \end{aligned}$$

The Young inequality and Sobolev embedding theorem entail us to obtain

$$\begin{aligned}
J_{21} &\le C\|\sqrt{\rho}u_t\|_{L^4}^2\|\nabla u\|_{L^2}\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}^{\frac14}\|\sqrt{\rho}u_t\|_{L^6}^{\frac34})^2\|\nabla u\|_{L^2}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|u_t\|_{L^6}^{3/2}\|\nabla u\|_{L^2}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{3/2}\|\nabla u\|_{L^2}\\
&\le \frac18\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\sqrt{\rho}u_t\|_{L^2}^2.
\end{aligned}$$

Similarly, we obtain by using Young inequality and Sobolev embedding theorem

$$\begin{aligned}
J_{22} &\le \Big|\int_{\mathbb{R}^2}(\rho u)\nabla u\cdot\nabla u\cdot u_t\,dx\Big| + \Big|\int_{\mathbb{R}^2}(\rho u)u\cdot\nabla^2 u\cdot u_t\,dx\Big|\\
&\quad + \Big|\int_{\mathbb{R}^2}(\rho u)u\cdot\nabla u\cdot\nabla u_t\,dx\Big|\\
&\le C\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6} + C\|u\|_{L^6}^2\|\Delta u\|_{L^2}\|u_t\|_{L^6} + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le \frac18\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2.
\end{aligned}$$

Plugging the above estimates into inequality (2.11) we arrive at

$$\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^4(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2). \tag{2.13}$$

Integrating above differential inequality and using the estimate (2.10), it gives

$$\begin{aligned}
&\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \int_{T_0}^t\|\nabla u_t(s)\|_{L^2}^2\,ds\\
&\le C\int_{T_0}^t\|\nabla u\|_{L^2}^4(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\,ds\\
&\le C\int_{T_0}^t\big(e+y(s)\big)^{2A\epsilon}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\,ds\\
&\le C\big(e+y(t)\big)^{2A\epsilon}\int_{T_0}^t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2)\,ds\\
&\le C\big(e+y(t)\big)^{5A\epsilon}.
\end{aligned} \tag{2.14}$$

Next, we split the range $3 < p \le \infty$ into two cases, namely $6 \le p \le \infty$ and $3 < p < 6$.

Case: $6 \le p \le \infty$. We can show that

$$\|\Lambda^{\frac{3(p-2)}{2p}}u\|_{L^2(\mathbb{R}^3)} \le C\|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{p+6}{2p}}\|\Delta u\|_{L^2(\mathbb{R}^3)}^{\frac{p-6}{2p}}, \quad 6 \le p \le \infty.$$

Recalling estimate (2.8)

$$\|\Delta u\|_{L^2} \le C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^3), \tag{2.15}$$

we can conclude that

$$\begin{aligned}
&\|\Lambda^{\frac{3(p-2)}{2p}}u\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{\frac{p+6}{2p}}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla u\|_{L^2}^6)^{\frac{p-6}{4p}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\big(e+y(t)\big)^{\frac{p+6}{4p}A\epsilon}\Big(\big(e+y(t)\big)^{5A\epsilon}+\big(e+y(t)\big)^{2A\epsilon}+\big(e+y(t)\big)^{3A\epsilon}\Big)^{\frac{p-6}{4p}}\\
&\le C\big(e+y(t)\big)^{\frac{p+6}{4p}A\epsilon}\big(e+y(t)\big)^{\frac{p-6}{4p}5A\epsilon}\\
&\le C\big(e+y(t)\big)^{\frac{3}{2}A\epsilon}.
\end{aligned}
$$

Finally, we infer from above inequality that

$$y(t) \le C\big(e+y(t)\big)^{\frac{3}{2}A\epsilon}.$$

Selecting $\epsilon < \frac{2}{3A}$ such that $\frac{3}{2}A\epsilon < 1$, it is easy to get

$$y(t) \le C(T_0, T, \|\nabla u(T_0)\|_{L^2}), \quad \forall T_0 \le t < T.$$

Noticing that the righthand of above estimate is independent of t for all $T_0 \le t \le T$, it is easy to observe that

$$\max_{\tau\in[T_0,T]} y(t) \le C(T_0, T, \|\nabla u(T_0)\|_{L^2}) < \infty.$$

By (2.10), we obtain

$$\max_{\tau\in[T_0,T]} \|\nabla u(\tau)\|_{L^2} \le C(T_0, T, \|\nabla u(T_0)\|_{L^2}) < \infty.$$

Consequently, it also holds that

$$\max_{\tau\in[0,T]} \|\nabla u(\tau)\|_{L^2} \le C(T_0, T, \rho_0, u_0, \|\nabla u(T_0)\|_{L^2}) < \infty.$$

By the embedding inequality

$$\|u\|_{L^6(\mathbb{R}^3)} \le C\|\nabla u\|_{L^2(\mathbb{R}^3)},$$

one can obtain that

$$u \in L^4(0,T; L^6(\mathbb{R}^3)), \quad \frac{3}{6}+\frac{2}{4}=1.$$

Now the regularity criterion established in [13] allows us to extend the solution (ρ, u, π) beyond time T.

Case: $3 < p < 6$. The embedding inequality

$$\|\Lambda^{\frac{3(p-2)}{2p}} u\|_{L^2(\mathbb{R}^3)} \le C\|\Delta u\|_{L^{\frac{3p}{2p+3}}(\mathbb{R}^3)}$$

direct yields

$$
\begin{aligned}
\|\Lambda^{\frac{3(p-2)}{2p}} u\|_{L^2} &\le C\|\Delta u\|_{L^{\frac{3p}{2p+3}}}\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^2)\\
&\le C\Big(\big(e+y(t)\big)^{5A/2} + \big(e+y(t)\big)^{A\epsilon}\Big)\\
&\le C\big(e+y(t)\big)^{5A\epsilon/2}.
\end{aligned}
$$

It is worth noting that the case $6 \le p \le \infty$ can also be handled by the argument used for the case $3 < p < 6$. Again, we arrive at

$$y(t) \le C\big(e+y(t)\big)^{5A/2}.$$

The remainder proof is the same as the previous case. Thus, this completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

As above, under the condition (1.2), we can infer that for any small $\epsilon > 0$, there exists $T_0 = T_0(\epsilon) < T$ such that

$$\int_{T_0}^{T} \frac{\|u(t)\|_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)}^{\frac{2}{1-\delta}}}{\ln\big(e + \|u(t)\|_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)}\big)}\,dt \le \epsilon. \tag{3.1}$$

The well-known Stokes theorem ensures that

$$\begin{aligned}
\|\Delta u\|_{L^2} &\le C(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2}) \\
&\le C(\|\rho u_t\|_{L^2} + \|\rho\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla\cdot(u\otimes u)\|_{L^2}) \quad \big(\operatorname{div} u = 0\big) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|uu\|_{\dot{H}^1}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|uu\|_{\dot{B}^1_{2,2}}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}\|u\|_{\dot{B}^{1+\delta}_{2,2}}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\delta}\|\Delta u\|_{L^2}^{\delta}) \quad \big(0<\delta<1\big) \\
&\le \frac{1}{2}\|\Delta u\|_{L^2} + C(\|\sqrt{\rho} u_t\|_{L^2} + \|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{1}{1-\delta}}\|\nabla u\|_{L^2}),
\end{aligned} \tag{3.2}$$

where we have used the following facts

$$\begin{gathered}
\|ff\|_{\dot{B}^1_{2,2}} \le C\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|f\|_{\dot{B}^{1+\alpha}_{2,2}}, \quad \text{for any } \alpha > 0, \quad \text{(see, e.g., [20])} \\
\|f\|_{\dot{H}^1} \approx \|f\|_{\dot{B}^1_{2,2}} \quad \text{and} \quad \|u\|_{\dot{B}^{1+\delta}_{2,2}} \le C\|\nabla u\|_{L^2}^{1-\delta}\|\Delta u\|_{L^2}^{\delta}, \quad 0<\delta<1.
\end{gathered}$$

Applying Stokes theorem once again gives

$$\begin{aligned}
\|\Delta u\|_{L^{\frac{3}{2+\delta}}} &\le C(\|\rho u_t\|_{L^{\frac{3}{2+\delta}}} + \|\rho u\cdot\nabla u\|_{L^{\frac{3}{2+\delta}}}) \\
&\le C(\|\sqrt{\rho}\|_{L^{\frac{6}{1+2\delta}}}\|\sqrt{\rho} u_t\|_{L^2} + \|\rho\|_{L^{\frac{3}{\delta}}}\|u\|_{L^6}\|\nabla u\|_{L^2}) \\
&\le C(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}^2).
\end{aligned} \tag{3.3}$$

Thus, one deduces from (2.6) and (3.3) that

$$\|\Delta u\|_{L^2} + \|\Delta u\|_{L^{\frac{3}{2+\delta}}} \le C(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^3). \tag{3.4}$$

Multiplying the second equation of (1.1) by u_t and integrating over whole space, one can obtain that for any $0 < \delta < 1$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\le \Big|\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u_t\,dx\Big| \\
&\le C\|\sqrt{\rho}\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|\nabla\cdot(u\otimes u)\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \quad \big(\operatorname{div} u = 0\big) \\
&\le C\|uu\|_{\dot{B}^1_{2,2}}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}\|u\|_{\dot{B}^{1+\delta}_{2,2}}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\delta}\|\Delta u\|_{L^2}^{\delta}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\delta}\big(\|\sqrt{\rho}u_t\|_{L^2} + \|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{1}{1-\delta}}\|\nabla u\|_{L^2}\big)^{\delta}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\delta}\|\sqrt{\rho}u_t\|_{L^2}^{1+\delta} + C\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{1}{1-\delta}}\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2}{1-\delta}}\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{3.5}
$$

For any $t \in (T_0, T)$, we denote

$$
y(t) := \max_{\tau\in[T_0,\,t]}\|\Lambda^{\frac{3}{2}-\delta}u(\tau)\|_{L^2}.
$$

Applying Gronwall inequality to (3.5), we conclude

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \int_{T_0}^t \|\sqrt{\rho}u_t(s)\|_{L^2}^2\,ds \\
&\le \|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A\int_{T_0}^t \|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2}{1-\delta}}\,ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A\int_{T_0}^t \frac{\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2}{1-\delta}}}{\ln\big(e+\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}\big)}\ln\big(e+\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}\big)\,ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A\int_{T_0}^t \frac{\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2}{1-\delta}}}{\ln\big(e+\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}\big)}\ln\big(e+\|\Lambda^{\frac{3}{2}-\delta}(s)\|_{L^2}\big)\,ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A\int_{T_0}^t \frac{\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2}{1-\delta}}}{\ln\big(e+\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}\big)}\ln\big(e+y(s)\big)\,ds\Big] \\
&\le C\|\nabla u(T_0)\|_{L^2}^2 \exp\Big[A\int_{T_0}^t \frac{\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}^{\frac{2}{1-\delta}}}{\ln\big(e+\|u(s)\|_{\dot{B}^{-\delta}_{\infty,\infty}}\big)}\,ds\cdot\ln\big(e+y(t)\big)\Big] \\
&\le C\big(e+y(t)\big)^{A\epsilon},
\end{aligned}
\tag{3.6}
$$

where we have used

$$
\|u\|_{\dot{B}^{-\delta}_{\infty,\infty}(\mathbb{R}^3)} \le C\|\Lambda^{\frac{3}{2}-\delta}u\|_{L^2(\mathbb{R}^3)},
$$

which can be easily derived by the Littlewood-Paley technique with the Berstein inequality. By (3.4), it is easy to see that

$$\int_{T_0}^{t}\|\Delta u\|_{L^2}^2(s)\,ds \le C\int_{T_0}^{t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^6)(s)\,ds \le C\big(e+y(t)\big)^{3A\epsilon},$$

which together with (3.6) imply

$$\|\nabla u(t)\|_{L^2}^6+\int_{T_0}^{t}(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\Delta u\|_{L^2}^2)(s)\,ds \le C\big(e+y(t)\big)^{3A\epsilon}. \tag{3.7}$$

With the same argument as in Section 2, one can infer that

$$\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^4(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\Delta u\|_{L^2}^2).$$

Thus, integrating the above inequality over $[T_0,t]$ results in (see also (2.14))

$$\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\int_{T_0}^{t}\|\nabla u_t(s)\|_{L^2}^2\,ds \le C\big(e+y(t)\big)^{5A\epsilon}, \tag{3.8}$$

which along with (3.4) give

$$\|\Delta u\|_{L^2}^2+\|\Delta u\|_{L^{\frac{3}{2+\delta}}}^2 \le C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^4+\|\nabla u\|_{L^2}^6) \le C\big(e+y(t)\big)^{5A\epsilon}. \tag{3.9}$$

Note the interpolation inequality

$$\|\Lambda^{\frac{3}{2}-\delta}u\|_{L^2(\mathbb{R}^3)} \le C\|\Delta u\|_{L^{\frac{3}{2+\delta}}(\mathbb{R}^3)}, \quad 0<\delta<1. \tag{3.10}$$

Thus, we conclude the following by combining the inequalities (3.9) and (3.10)

$$y(t) \le C\big(e+y(t)\big)^{5A\epsilon}.$$

The remainder proof is the same as the previous section. Thus, this completes the proof of Theorem 1.2.

4. roof of Theorem 1.4

As above, we only establish several a priori estimates for the strong solutions. Now we recall the following bilinear estimate which is an easy consequence of [20, Lemma 1],

$$\|ff\|_{\dot B^1_{2,2}} \le C\|f\|_{\dot B^{-1}_{\infty,\infty}}\|f\|_{\dot B^2_{2,2}}. \tag{4.1}$$

Applying the Stokes theorem (or (2.6)) yields

$$\begin{aligned}
\|\Delta u\|_{L^2} &\le (\|\rho u_t\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2})\\
&\le (\|\rho u_t\|_{L^2}+\|\rho\|_{L^\infty}\|u\cdot\nabla u\|_{L^2})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla\cdot(u\otimes u)\|_{L^2}) \quad \big(\operatorname{div} u=0\big)\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|uu\|_{\dot H^1})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|uu\|_{\dot B^1_{2,2}})\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}+\|u\|_{\dot B^{-1}_{\infty,\infty}}\|u\|_{\dot B^2_{2,2}}) \quad \big(\text{see (4.1)}\big)\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}+C\|u\|_{\dot B^{-1}_{\infty,\infty}}\|\Delta u\|_{L^2}.
\end{aligned} \tag{4.2}$$

Thanks to condition (1.5), one has

$$C\|u\|_{\dot B^{-1}_{\infty,\infty}} \le \frac{1}{2},$$

which leads to

$$\|\Delta u\|_{L^2} \le C\|\sqrt{\rho}u_t\|_{L^2}. \tag{4.3}$$

As a consequence, this gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 &\le \Big|\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u_t\,dx\Big| \\
&\le \|\sqrt{\rho}\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le \|\nabla\cdot(u\otimes u)\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \quad \big(\operatorname{div} u = 0\big) \\
&\le C\|uu\|_{\dot{B}^1_{2,2}}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-1}_{\infty,\infty}}\|u\|_{\dot{B}^2_{2,2}}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\Delta u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} \\
&\le C\|u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\le \frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2,
\end{aligned} \tag{4.4}$$

which implies

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 \le 0.$$

Thus

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}u_t(s)\|_{L^2}^2\,ds \le \|\nabla u_0\|_{L^2}^2 \le C < \infty$$

for any $0 \le t < T$. As in proving Theorem 1.1, we get the desired result. The proof of Theorem 1.4 is complete.

Acknowledgements. The authors would like to express their hearty thanks to the anonymous referees for their insightful comments and many valuable suggestions, which greatly improved the exposition of the manuscript.

References

[1] H. Abidi; *Équation de Navier-Stokes avec densité et viscosité variables dans l'espace critique*, Rev. Mat. Iberoam 23 (2007) 537-586.

[2] H. Abidi, G. Gui, P. Zhang; *On the wellposedness of three-dimensional inhomogeneous Navier-Stokes equations in the critical spaces*, Arch. Rational Mech. Anal. 204 (2012) 189-230.

[3] S. Antontesv, A. Kazhikov, V. Monakhov; *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, North-Holland, Amsterdam (1990).

[4] H. Choe, H. Kim; *Strong solutions of the Navier-Stokes equations for nonhomogeneous incompressible fluids*, Comm. Partial Differential Equations, 28 (2003) 1183-1201.

[5] Y. Cho, H. Kim; *Unique solvability for the density-dependent Navier-Stokes equations*, Nonlinear Anal., 59 (2004) 465-489.

[6] Y. Cho, H. Kim; *On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities*, Manuscripta Math. 120 (2006) 91-129.

[7] W. Craig, X. Huang, Y. Wang; *Global wellposedness for the 3D inhomogeneous incompressible Navier-Stokes equations*, J. Math. Fluid Mech. 15 (2013) 747-758.

[8] R. Danchin; *Density-dependent incompressible viscous fluids in critical spaces*, Proc. R. Soc. Edinburgh Sect. A 133 (2003) 1311-1334.

[9] B. Desjardins; *Regularity results for two-dimensional flows of multiphase viscous fluids*, Arch. Rational Mech. Anal., 137 (1997) 135-158.

[10] G. Gui, J. Huang, P. Zhang; *Large global solutions to 3-D inhomogeneous Navier-Stokes equations slowly varying in one variable*, J. Funct. Anal. 261 (2011) 3181-3210.

[11] X. Huang, Y. Wang; *Global strong solution to the 2D nonhomogeneous incompressible MHD system*, Journal of Differential Equations, 254 (2013) 511-527.

[12] A. Kazhikov; *Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid*, (Russian). Dokl. Akad. Nauk SSSR 216 (1974) 1008-1010.

[13] H. Kim; *A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations*, SIAM J. Math. Anal. 37 (2006) 1417-1434.

[14] J. Kim; *Weak solutions of an initial boundary value problem for an incompressible viscous fluid with nonnegative density*, SIAM J. Math. Anal. 18 (1987) 8-96.

[15] O. Ladyzhenskaya, V. Solonnikov; *Unique solvability of an initial and boundary value problem for viscous incompressible non-homogeneous fluids*, J. Soviet Math. 9 (1978) 697-749.

[16] Z. Lei, Y. Zhou; *BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity*, Discrete Contin. Dyn. Syst. 25 (2009), 575-583.

[17] P. Lions; *Mathematical topics in fluid mechanics. Incompressible models.* Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications, vol. 1. Clarendon Press/Oxford University Press, New York (1996).

[18] J. Simon; *Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure*, SIAM J. Math. Anal. 21 (1990) 1093-1117.

[19] Z. Ye, X. Xu; *A note on blow-up criterion of strong solutions for the 3D inhomogeneous incompressible Navier-Stokes equations with vacuum*, Math. Phys. Anal. Geom. 18 (2015), no. 1, Art. 14, 10 pp.

[20] B. Yuan, B. Zhang; *Blow-up criterion of strong solutions to the Navier-Stokes equations in Besov spaces with negative indices*, J. Differential Equations 242 (2007) 1-10.

[21] P. Zhang, C. Zhao, J. Zhang; *Global regularity of the three-dimensional equations for non-homogeneous incompressible fluids*, Nonlinear Anal., 110 (2014) 61-76.

Qianqian Hou
Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
E-mail address: qianqian.hou@connect.polyu.hk

Xiaojing Xu
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
E-mail address: xjxu@bnu.edu.cn

Zhuan Ye (corresponding author)
Department of Mathematics and Statistics, Jiangsu Normal University, 101 Shanghai Road, Xuzhou 221116, Jiangsu, China
E-mail address: yezhuan815@126.com