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A WAVELESS FREE SURFACE FLOW PAST A SUBMERGED TRIANGULAR OBSTACLE IN PRESENCE OF SURFACE TENSION

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Abstract. We consider the Free surface flows passing a submerged triangular obstacle at the bottom of a channel. The problem is characterized by a nonlinear boundary condition on the surface of unknown configuration. The analytical exact solutions for these problems are not known. Following Dias and Vanden Broeck [6], we computed numerically the solutions via a series truncation method. These solutions depend on two parameters: the Weber number α characterizing the strength of the surface tension and the angle β at the base characterizing the shape of the apex. Although free surface flows with surface tension admit capillary waves, it is found that solution exist only for values of the Weber number greater than α_0 for different configurations of the triangular obstacle.

1. Introduction

We consider the steady two dimensional flow of an inviscid incompressible fluid passing a submerged triangular obstacle at the bottom of a channel (See 1), as we shall see the problem is characterized by the Weber number. Free surface flows around submerged bodies have been studied by many authors and researchers, for long years. They modeled their problems by considering bodies of regular shapes: flows around cylinder [10, 11], semi-circle [7, 8, 9, 12], triangles [6], and finite flat plates [13]. Choi [4, 5]) carried out an analytical asymptotic calculation over a small depression in a channel with a shallow water flow, taking into consideration gravity and neglecting surface tension. Dias and Vanden Broeck [6] considering the effect of gravity and neglected the surface tension, they computed the problem via a series truncation method solution, for different values of the Froude number. We used the same method to solve our problem considering the effect of surface tension and neglecting gravity. For very large values of the Weber number $\alpha \to \infty$, solutions are approximately the same and the free surface profiles coincide with the free streamline solution, in the absence of gravity and surface tension.

It is observed that there is a value α_0, $0 < \alpha_0 < 1$, of the Weber number for which there is no solution, if $\alpha < \alpha_0$, and a unique negative solitary-wave-like solution if $\alpha > \alpha_0$, Vanden Broeck [3] showed that, in presence of surface tension, capillary waves are exponentially small to all orders. This may explain the limiting value

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α_0 of the Weber number below which our procedure fails to describe a waveless solution of the problem.

2. Formulation of the problem

We consider the steady two-dimensional flow of a fluid over a triangular obstacle (See 1). The fluid is assumed to be inviscid, incompressible and the flow is irrotational. We neglect the effect of the gravity but we take into account the effect of surface tension. Far upstream and downstream "far from the triangular BCD", the flow is uniform with a constant velocity U and a constant depth L. As we shall see, the flow is characterized by two-parameters: the angle β at the base characterizing the shape of the apex and the Weber number α characterizing the strength of the surface tension and is defined by

$$\alpha = \frac{\rho U^2 L}{T} \tag{2.1}$$

where T is the surface tension and ρ is the density of the fluid.

When the effects of surface tension and gravity g are neglected, the classical exact solution can be found via the hodograph transformation Birkhoff[2]. If the effects of surface tension or gravity are considered, the boundary condition at the free surface is nonlinear and no exact analytical solution is known. Different combinations and some varieties of this problem have been considered. Considering the effect of the surface tension, our results confirm that there is a solution for different Weber number $\alpha > 0$, and for triangles of arbitrary size by varying the angle β. A system of cartesian coordinates is defined, with the x-axis along the horizontal bottom AB, DE and the y-axis going through the apex C of the triangle BCD.

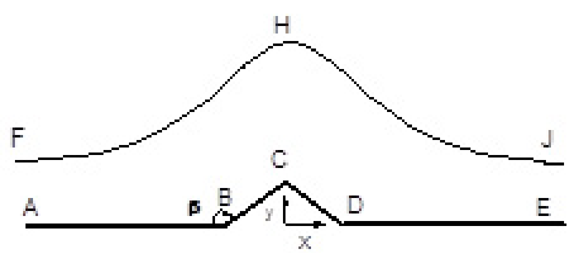

Figure 1. Sketch of the flow and of the system of coordinates.

We define dimensionless variables by taking U as the unit velocity and L as the unit length. We denote by u and v the components of the velocity in the x and y directions, respectively. Since the flow is potential, it can be described by two functions: a potential function ϕ and a stream function ψ. Without loss of generality, we choose $\phi = 0$ at C and $\psi = 0$ on the stream line ABCDE. It follows from the choice of the dimensionless variables that $\psi = 1$ on the free stream line FGHJ.

(1) In the far field (as $|x| \to \infty$), the flow is supposed to be uniform, hence

$$\phi(x, y) = Ux \quad \text{as } |x| \to \infty \tag{2.2}$$

(2) On the rigid boundary $(ABCDE)$, the normal velocity has to vanish, that is:
$$\frac{\partial \phi}{\partial \overrightarrow{\eta}} = 0, \tag{2.3}$$
where $\overrightarrow{\eta}$ is the unit normal vector on the boundary $(ABCDE)$.

(3) On the free surface, the atmospheric pressure P_0 is constant, hence the Bernoulli equation yields:
$$\overline{p} + \frac{1}{2}\rho \overline{q}^2 = \overline{p}_0 + \frac{1}{2}\rho U^2 \quad \text{on FHJ } \psi = 1 \tag{2.4}$$

Here $\overline{p}$ and $\overline{q}$ are the fluid pressure and the speed just inside the free surface, respectively. The right-hand side of (2.4) is evaluated from the condition in the far field.

A relationship between $\overline{p}$ and $\overline{p}_0$ is given by Laplace's capillarity formula
$$\overline{p} - \overline{p}_0 = TK \tag{2.5}$$
Here K is the curvature of the free surface and T is the surface tension.
If we substitute (2.5) into (2.4), and in dimensionless variables, (2.4) becomes
$$\frac{1}{2}q^2 - \frac{1}{\alpha}K = \frac{1}{2} \quad \text{on } FHJ, \tag{2.6}$$
where α is the Weber number defined by (2.1).

The physical flow problem as described above can be formulated as a boundary value problem in the potential function $\phi(x, y)$:
$$\begin{gathered} \Delta\phi = 0 \quad \text{in the flow domain}, \quad \phi(x,y) = x, \quad |x| \to \infty \\ \frac{\partial \phi}{\partial \overrightarrow{\eta}} = 0 \quad \text{on the rigid boundary ABCDE} \\ |\nabla \phi|^2 - \frac{2}{\alpha}K = 1 \quad \text{on the free surface.} \end{gathered} \tag{2.7}$$
Solving the problem in this form is very difficult especially that the nonlinear boundary condition is specified on an unknown boundary (the free surface). Instead of solving the problem in its partial differential equation form in ϕ, we take advantage of the property that for the bidimensional potential flow (as is in our problem) and if the plane in which the flow is embedded is identified to the complex plane, the complex velocity $\xi = u - iv$ and the complex potential function $f = \phi + i\psi$ are analytic functions of the complex variable $z = x + iy$. Hence, we use all the necessary properties of analytic functions of a complex variable: integral formulation, series formulation, conformal mapping, etc.. Therefore, in the f-plane, the flow is the strip $0 < \psi < 1$ (See 2).

The free surface, the bottom channel and the triangle are parts of a streamline, are mapped onto the straight lines $\psi = 1$ and $\psi = 0$, respectively.

In order that the curvature be well defined, we introduce the function $\tau - i\theta$ as
$$\xi = \frac{df}{dz} = u - iv = e^{\tau - i\theta}, \tag{2.8}$$
where e^τ represents the strength of the velocity, $e^\tau = \sqrt{u^2 + v^2}$ and θ is the angle between the x-axis and the vector velocity. In these new variables, the Bernoulli equation (2.6) becomes
$$e^{2\tau} - \frac{2}{\alpha}\Big|\frac{\partial \theta}{\partial \phi}\Big| e^\tau = 1 \quad \text{on FHJ } (\psi = 1) \tag{2.9}$$

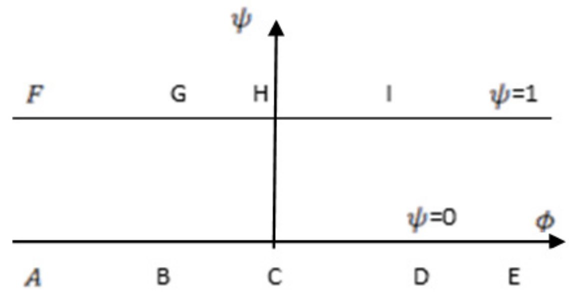

FIGURE 2. The flow configuration in the complex potential plane.

The kinematic condition is expressed as

$$\beta = 0 \quad \text{on AB and DE,} \tag{2.10}$$

$$\begin{aligned} \theta &= \beta \quad \text{on BC,} \\ \theta &= \beta_2 \quad \text{on CD,} \end{aligned} \tag{2.11}$$

We shall seek $\tau - i\theta$ as an analytic function of $f = \phi + i\psi$ in the strip $0 < \psi < 1$, satisfying the conditions (2.9), (2.10) and (2.11).

3. Numerical procedure

We define a new variable t by the relation

$$f = \frac{2}{\pi}\log(\frac{1+t}{1-t}) \tag{3.1}$$

This transformation maps the flow domain into the upper half of the unit disc in the complex t plane (See 3).

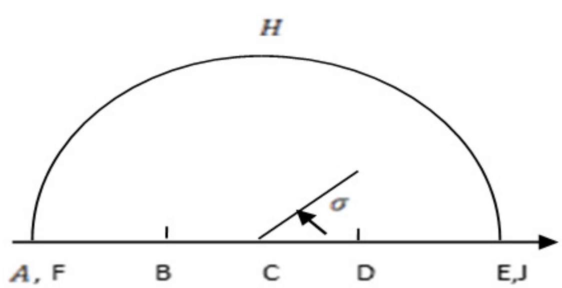

FIGURE 3. The flow domain in the t-plane.

The free surface is mapped onto the upper half unit circle and the rigid bottom is mapped onto the diameter. The apex C of the triangle is mapped into the origin, the apex B into a point t_B, $-1 < t_B < 0$ and the apex D is mapped into a point t_D, $0 < t_D < 1$. The y-axis is the median of the segment BD. Because of the symmetry of the flow, we have $t_B = -t_D$. Since there is no singularities in the flow domain, except the flow around the corners B, C and D, and since the transformation (3.1) is conformal except at the points B, C and D, the flow function $\xi = u - iv$ should be analytical, in the upper half unit disc in the t-plane except at the points t_B, $t_C = 0$ and t_D.

Local behavior of ξ at B, D and C. At the points B, D and C, the flow is around or into an angle. Hence, $\xi = u - iv$ is regular except at those points and a local analysis is required.

Asymptotic behavior $t = t_B$, $t = t_D$. In the z-plane and the vicinities of B, D and C, the flow is around angle of measure β, β_2 and β_1, hence the appropriate singularities are

$$\begin{gathered}
\xi = O\Big(\Big(\frac{t-t_B}{1-t_B}\Big)^{1-\frac{\beta}{\pi}}\Big) \quad \text{as } t \to t_B \\
\xi = O\Big(\Big(\frac{t-t_D}{1-t_D}\Big)^{1-\frac{\beta_2}{\pi}}\Big) \quad \text{as } t \to t_D \\
\xi = O(t^{1-\frac{\beta_1}{\pi}}) \quad \text{as } t \to 0\,.
\end{gathered}$$

The angles β, β_1 and β_2 satisfy the relation $\beta + \beta_1 + \beta_2 = 3\pi$. Now, that we have the local behavior of the flow near the singularities, we seek $\xi(t)$ in the form

$$\xi = \Big(\frac{t-t_B}{1-t_B}\Big)^{1-\frac{\beta}{\pi}}\Big(\frac{t-t_D}{1-t_D}\Big)^{1-\frac{\beta_2}{\pi}}(t^{1-\frac{\beta_1}{\pi}})\Omega(t)\,. \tag{3.2}$$

The function $\Omega(t)$ is bounded and continuous on the unit circle and analytic in the interior of the unit disk. Hence $\Omega(t)$ can be expressed as an exponential of analytical function. Therefore, we can write $\xi(t)$ as

$$\xi = \Big(\frac{t-t_B}{1-t_B}\Big)^{1-\frac{\beta}{\pi}}\Big(\frac{t-t_D}{1-t_D}\Big)^{1-\frac{\beta_2}{\pi}}(t^{1-\frac{\beta_1}{\pi}})\exp\Big(\sum_{n=0}^{\infty} a_n t^{2n}\Big) \tag{3.3}$$

By choosing all the coefficients a_n to be real, the function (3.3) satisfies (2.10) and (2.11). The coefficients a_n have to be determined to satisfy (2.9). We use the notation $t = |t|e^{i\sigma}$, so that points, on the free surface FHJ, are given by $t = e^{i\sigma}$, $0 < \sigma < \pi$. Using (3.3), we rewrite (2.9) in the form

$$e^{2\overline{\tau}} - \frac{\pi}{\alpha}\sin(\sigma)\Big|\frac{\partial\overline{\theta}}{\partial\sigma}\Big|e^{2\overline{\tau}} = 1 \tag{3.4}$$

Here $\overline{\tau}(\sigma)$ and $\overline{\theta}(\sigma)$ denote the values of τ and θ, on the free surface FHJ. We solve the problem numerically by truncating the infinite series in (3.4), after N terms. We find the N coefficients a_n by collocation. Thus, we introduce N mesh points

$$\sigma_j = \frac{\pi}{N}\Big(j - \frac{1}{2}\Big) \quad j = 0, \dots, N-1 \tag{3.5}$$

Using (3.5), we obtain $[\overline{\tau}(\sigma)]_{\sigma=\sigma_j}$, $[\overline{\theta}(\sigma)]_{\sigma=\sigma_j}$ and $[\frac{\partial\overline{\theta}}{\partial\sigma}]_{\sigma=\sigma_j}$ in terms of coefficients a_n.Thus, we obtain N nonlinear algebraic equations of N unknowns $(a_n, n = 0, \dots, N-1)$. The Weber number α and the measure β of the angle at the base are two parameters. The resulting system is solved using Newton's method. The shape of the free surface is obtained by integrating numerically the relation

$$\begin{aligned}
\frac{\partial x}{\partial\sigma} &= \exp(-\tau(\sigma))\cos(\theta(\sigma))\frac{\partial\phi}{\partial\sigma} \\
\frac{\partial y}{\partial\sigma} &= \exp(-\tau(\sigma))\sin(\theta(\sigma))\frac{\partial\phi}{\partial\sigma}
\end{aligned} \tag{3.6}$$

4. Discussion of results

The numerical scheme, described in section 3, is used to compute solutions for different values of the Weber number α and the angle β.

Flow without surface tension. For $\alpha \to \infty$ and for all inclination angle β, exact analytical solutions can be computed via free stream line theory due to Birkhoff (See [1]). We computed these solutions numerically using the procedure described above and our results agree with the theoretical and experimental results (See 4). For $\beta = \frac{3\pi}{4}$, all the coefficients a_n vanish, and the procedure gives the exact solution.

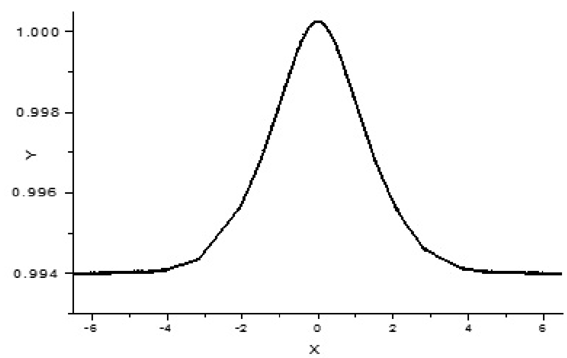

FIGURE 4. Free surface configuration without surface tension (-) Via analytical computation by free streamline theory (•) Via numerical integration using the present scheme

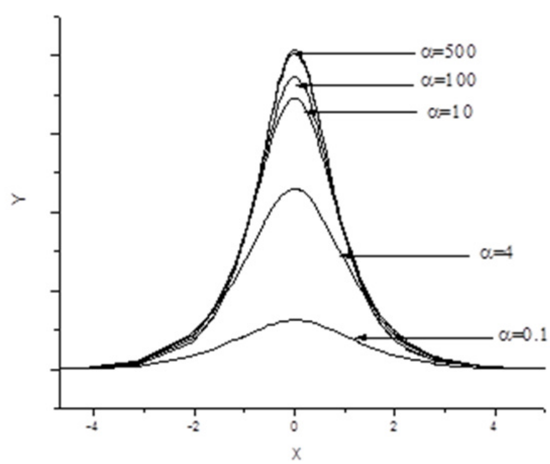

FIGURE 5. Free surface shapes for different values of the Weber number α with $\beta = \frac{3\pi}{4}$

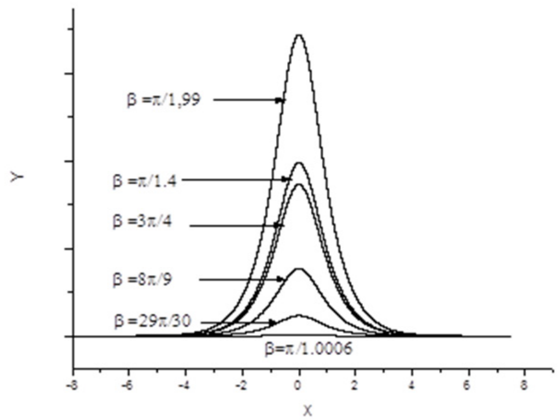

FIGURE 6. Free surface shapes for the Weber number $\alpha = 5$ and different values of β

Flow with surface tension effect. In presence of the effect of surface tension or force of gravity, there are no exact solutions known. The numerical procedure described above was used to compute solutions for various of α and β. The coefficients in equation (3.3) were found to decrease very rapidly and the algorithm converges for few iterations when Weber number $\alpha > 1$. For example with error less than a 10^{-8}. When $\alpha \to 0$, the algorithm converges less rapidly and ceases to converge, when $\alpha < \alpha_0$, for some critical values $0 < \alpha_0 < 1$. The critical value α_0 depends on the angle β. The existence of this critical value of the Weber number can be explained from the procedure used in this article itself. The procedure used relies on the series expansion (3.3) of the analytic complex velocity $\xi = u - iv$, which does not take into account capillary waves. In this article, Chapman (See [3]) showed that capillary waves are exponentially small to all order. Hence, the capillary waves are not dominant and the expansion (3.3) describes the flow very well unless the Weber number is sufficiently small. For all the values of the Weber number $\alpha > \alpha_0$ and for the angle $\frac{\pi}{2} < \beta < \pi$, the free surface profile looks like a symmetric negative solitary wave with the maximum crest is just above the apex C of the triangular. In (See 5), we showed different free surface profiles for $\beta = \frac{3\pi}{4}$ and different values of the Weber number. It is observed that the maximum crest is obtained for $\alpha \to \infty$ and decreases as $\alpha \to 0$. For $\alpha \geq 300$, all free surface profiles for different values of α are the same within graphical accuracy and coincide with the graph of the exact solution without surface tension. This suggests that the surface tension can be neglected if $\alpha \geq 300$. To obtain different configuration of the triangular, we varied the angle β, $\frac{\pi}{2} < \beta < \pi$ and fixed the Weber number α. Profiles of the free surface for different values of the angle β and $\alpha = 5$ is shown in (See 6).

We remark that when the angle β increases $\beta \to \pi$, the profiles of the free surface take the form of a uniform flow over a horizontal plan.

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