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A PRIORI ESTIMATES AND EXISTENCE FOR QUASILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

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Abstract. This article concerns the existence of positive solutions for a nonlinear Neumann problem involving the m-Laplacian. The equation does not have a variational structure. We use a blow-up argument and a Liouville-type theorem to obtain a priori estimates and obtain the existence of positive solutions by the Krasnoselskii fixed point theorem.

1. Introduction and statement of main results

In this work we consider the problem

$$\begin{gathered}\Delta_m u + B(z,u,\nabla u) = 0 \quad \text{in } \Omega,\\ |\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = g(z,u) \quad \text{on } \partial\Omega,\end{gathered}\tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N \geq 2)$. $B(z,u,\mathbf{p}) : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a continuous function. $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative respect to $\partial\Omega$, $g(z,u) : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

A function $u \in W^{1,m}(\Omega) \cap C(\bar\Omega)$ is said to be a weak solution for (1.1) if

$$\int_\Omega |\nabla u|^{m-2}\nabla u \cdot \nabla\phi \, dz - \int_{\partial\Omega} g(z,u)\phi \, d\sigma = \int_\Omega B(z,u,\nabla u)\phi \, dz$$

for any $\phi \in C^\infty(\bar\Omega)$.

Similar problems have been studied in many articles, see e.g. [1]-[10]. When B depends on ∇u, variational methods are barely used to deal with equation (1.1). In this case, the question of the existence of solutions can be handled by a priori estimates and topological methods. Combining the blow-up (scaling) arguments with suitable Liouville-type theorems, we can derive a priori estimates. The method was introduced in [2], where Gidas and Spruck obtain a priori bounds for solutions of nonlinear elliptic boundary value problem with the nonlinearity depending on x and u. Later, the method was used to systems in [3]-[5] and more general cases concerning a single equation were studied in [6]-[12]. Ruiz[6] and Zou[7] consider nonlinear Dirichlet problem involving the m-Laplacian with the nonlinearity depending on x, u and ∇u under different conditions. In [8], the power of growth of

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u and ∇u maybe critical or supercritical. In [9], the authors obtain similar results of generalized mean curvature equations.

All articles mentioned before deal with the Dirichlet problems. We consider the m-Laplacian with nonlinear Neumann boundary conditions. Throughout this paper, we assume $m \in (1, N)$, $p \in (m-1, m^*)$, where $m^* = \frac{Nm}{N-m} - 1$. Let $\alpha = \frac{p-(m-1)}{m}$ and $0 < q < \frac{p+1}{m}(m-1)$. First, we list some conditions to the nonlinear terms B and g.

We say $B(z, u, \mathbf{p})$ satisfies a growth-limit condition (G-L) if there exist positive constants p and $K_i, i = 1, 2, 3$, such that the following:

(1) There exists a bounded function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$|B(z, u, \mathbf{p})| \le K_1[1 + u^p + F(|\mathbf{p}|)|\mathbf{p}|^{\frac{mp}{p+1}}]$$
for all $(z, u, \mathbf{p}) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^N$, and $F(|\mathbf{p}|) \to 0$ as $|\mathbf{p}| \to \infty$.

(2) There exists a continuous function $b : \bar{\Omega} \to \mathbb{R}_+$ such that for any sequences $\{(M_k, \mathbf{p}_k)\} \subset \mathbb{R}_+ \times \mathbb{R}^N$ satisfying $M_k \to \infty$ and $\mathbf{p}_k = O(M_k^{1+\alpha})$, it holds
$$\lim_{k\to\infty} \frac{B(z, M_k, \mathbf{p}_k)}{M_k^p} = b(z)$$
uniformly on Ω.

For the nonlinearity g on the boundary we assume the following conditions:

(A1) Assume that $g \in C(\partial\Omega \times \mathbb{R}, \mathbb{R})$. There exist constants $0 < \mu_1, \mu_2 < 1$ and a nondecreasing continuous function $\Gamma(t) : [0, \infty) \to (0, \infty)$ with $|\Gamma(t)| \le K_2(1 + t^q)$ such that
$$|g(z, u) - g(y, v)| \le \Gamma(\max\{|u|, |v|\})[|z - y|^{\mu_1} + |u - v|^{\mu_2}]$$
for all $(z, u), (y, v) \in \partial\Omega \times \mathbb{R}$.

(A2) $|g(z, u)| \le K_3(1 + |u|^q)$ for all $(z, u) \in \partial\Omega \times \mathbb{R}$.

(A3) $g(z, u) \ge 0$ for all $(z, u) \in \partial\Omega \times \mathbb{R}_+$ and $g(z, 0) = 0$ for all $z \in \partial\Omega$.

The main ingredients of our arguments are a priori estimates on the pairs (u, λ) solving the problem
$$\begin{gathered} \Delta_m u + B(z, u, \nabla u) + \lambda = 0 \quad \text{in } \Omega, \\ |\nabla u|^{m-2} \frac{\partial u}{\partial \nu} = g(z, u) \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$
By the blow-up method, we first suppose by contradiction that there exists a sequence of unbounded solutions. Then by suitable scaling argument and taking advantage of the regularity results in [13] (see also [14]) we obtain a subsequence which converges to a nonnegative solution. That contradicts Liouville-type theorem on the entire space $\mathbb{R}^N$ or on the half-space $\mathbb{R}^N_+$. Our main results can be stated as follows.

Theorem 1.1. *Let Ω be a bounded smooth domain and assume that conditions* (G-L), (A1) *and* (A2) *hold. Then there exists a positive constant C such that $\sup_{z\in\Omega} u(z) + \lambda \le C$ for all non-negative C^1solutions u of* (1.2).

By this a priori estimates we can derive the existence of solutions for (1.1). For this purpose, we need some further hypotheses.

We say B satisfies a positivity condition:

(A4) There exists $L > 0$ such that $B(z, u, \mathbf{p}) + L|u|^{m-1} \ge 0$for all $(z, u, \mathbf{p}) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^N$.

We call B and g "super-linear" at the origin if

(A5) There exists $L > 0$ such that $B(z,u,\mathbf{p}) + L|u|^{m-1} = o(|u|^{m-1} + |\mathbf{p}|^{m-1})$, $(z,u,\mathbf{p}) \in \Omega\times\mathbb{R}_+\times\mathbb{R}^N$, $g(z,u) = o(|u|^{m-1})$, $(z,u) \in \partial\Omega\times\mathbb{R}_+$ as $(u,\mathbf{p}) \to 0$ uniformly on $\overline{\Omega}$.

Theorem 1.2. *Let Ω be a bounded smooth domain and assume that B and g satisfy conditions* (G-L), (A1)–(A5). *Then* (1.1) *has a positive solution.*

This paper is structured as follows. In section 2, we obtain the a priori estimates for solutions of (1.2). In section 3, we obtain the existence result of (1.1) by the Krasnoselskii fixed-point theorem.

2. A priori estimates

In this section, we prove Theorem 1.1, the main part of this article. The regularity for solutions and Liouville theorem play an important role in the proof. We first list two lemmas which will be used later.

Lemma 2.1 ($C^{1,\beta}$ Regularity [13])**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\beta,\mu_1,\mu_2 \in (0,1)$. Suppose $B : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy the condition*

$$|B(x,u,\mathbf{p})| \le \Lambda(|u|)(1+|\mathbf{p}|^m), \quad \forall (x,u,\mathbf{p}) \in \Omega\times\mathbb{R}\times\mathbb{R}^N \tag{2.1}$$

Suppose $g \in C(\partial\Omega\times\mathbb{R},\mathbb{R})$ satisfy the condition

$$|g(x,\vartheta) - g(y,\omega)| \le \Lambda(\max\{|\vartheta|,|\omega|\})[|x-y|^{\mu_1} + |\vartheta-\omega|^{\mu_2}], \quad \forall x,y \in \partial\Omega, \forall \vartheta,\omega \in \mathbb{R},$$

where $\Lambda : [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function.
If $u \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ is a bounded generalized solution of the boundary value problem

$$\begin{gathered} \Delta_m u + B(z,u,\nabla u) = 0, \quad z \in \Omega, \\ |\nabla u|^{m-2}\frac{\partial u}{\partial\nu} = g(z,u), \quad z \in \partial\Omega, \end{gathered} \tag{2.2}$$

and satisfy $\sup_\Omega |u| \le M_0$, then there is a positive constant

$$\beta = \beta(m,N,\Lambda(M_0),M_0,\mu_1,\mu_2,\sup|g(\partial\Omega\times[-M_0,M_0])|,\Omega)$$

such that u is in $C^{1,\beta}(\partial\Omega)$; moreover

$$|u|_{C^{1,\beta}(\overline{\Omega})} \le C(m,N,\Lambda(M_0),M_0,\mu_1,\mu_2,\sup|g(\partial\Omega\times[-M_0,M_0])|,\Omega) \tag{2.3}$$

Lemma 2.2. *Let $b > 0$ be a constant. Then the problem*

$$\begin{gathered} \Delta_m u + bu^p = 0 \quad \text{in } \mathbb{R}^N_+ \\ |\nabla u|^{m-2}\frac{\partial u}{\partial\nu} = 0 \quad \text{on } \partial\mathbb{R}^N_+, \end{gathered}$$

does not admit any non-negative non-trivial solutions when $p \in (m-1, m^*)$.

We sketch a proof of Lemma 2.2, our approach is similar to the one used in [15]. Assume that the equation has a non-negative non-trivial solution ω. By reflection with respect to the hyperplane $z_N = 0$, we obtain $\tilde\omega$ which is a non-negative non-trivial solution of corresponding equation on entire space, as the reader can see in [7]. That is a contradiction and we prove Lemma 2.2.

Proof of Theorem 1.1. We argue by contradiction and suppose that the conclusion is not true. Then there exists a sequence of positive solutions $\{u_k, \lambda_k\}$ of (1.2) such that

$$\lim_{k\to\infty}(\|u_k\|_{L^\infty(\Omega)} + \lambda_k) = \infty. \tag{2.4}$$

For $u_k \in C(\bar\Omega)$, there exists $\xi^k \in \Omega$, such that $M_k = \max_{z\in\Omega} u_k(z) = u_k(\xi^k)$, $k = 1, 2, \dots$ We introduce the transform

$$w_k(y) = N_k^{-1}u_k(z), \quad y = (z - \zeta^k)N_k^\alpha \tag{2.5}$$

where N_k, ζ^k will be determined later. Denote $\Omega_k = \{y \in \mathbb{R}^N | z = N_k^{-\alpha}y + \zeta^k \in \Omega\}$ being the image of Ω after the transform (2.5). By direct calculations, w_k satisfies

$$\begin{gathered}\Delta_m w_k + N_k^{-(1+\alpha)(m-1)-\alpha}[B(N_k^{-\alpha}y + \zeta^k, N_k w_k, N_k^{1+\alpha}\nabla w_k) + \lambda_k] = 0 \quad \text{in}\Omega_k,\\ |\nabla w_k|^{m-2}\frac{\partial w_k}{\partial\nu} = N_k^{-(1+\alpha)(m-1)}g(N_k^{-\alpha}y + \zeta^k, N_k w_k) \quad \text{on}\partial\Omega_k.\end{gathered} \tag{2.6}$$

For convenience, we denote

$$\begin{gathered}\theta_k(y, w_k, \nabla w_k) = N_k^{-(1+\alpha)(m-1)-\alpha}[B(N_k^{-\alpha}y + \zeta^k, N_k w_k, N_k^{1+\alpha}\nabla w_k) + \lambda_k] \quad \text{in } \Omega_k,\\ \sigma_k(y, w_k) = N_k^{-(1+\alpha)(m-1)}g(N_k^{-\alpha}y + \zeta^k, N_k w_k) \quad \text{on } \partial\Omega_k.\end{gathered}$$

We divide the proof into two cases.

Case 1. For a subsequence, but still indexed by k, it holds

$$\lim_{k\to\infty}\frac{\lambda_k}{M_k^p} = 0,$$

which implies that $M_k \to \infty$ as $k \to \infty$. In the transform (2.5), take $N_k = M_k, \zeta^k = \xi^k$, then

$$\lim_{k\to\infty}\frac{\lambda_k}{N_k^p} = \lim_{k\to\infty}\frac{\lambda_k}{M_k^p} = 0$$

and

$$0 < w_k(y) \le \frac{M_k}{N_k} = 1, \quad y \in \Omega_k; \quad w_k(0) = 1. \tag{2.7}$$

Using the Part 1 of growth-limit condition (G-L) we obtain that (for k large enough)

$$|\theta_k(y, w_k, \nabla w_k)| \le K_1(3 + |\nabla w_k|^m) + 1, \tag{2.8}$$

In condition (A1), constant $\mu_2 \in (0, 1)$ can be replaced by $\mu_3 \in (0, min\{\mu_2, \frac{p+1}{m}(m-1) - q\}]$ such that

$$|g(z, u) - g(y, v)| \le \tilde K_2[1 + (\max\{|u|, |v|\})^q](|z - y|^{\mu_1} + |u - v|^{\mu_3}) \tag{2.9}$$

for all $(z, u), (y, v) \in \partial\Omega \times \mathbb{R}$. by assumptions (A1) and (A2). Then we have

$$\begin{aligned}&|\sigma_k(x, \omega) - \sigma_k(y, \vartheta)|\\ &\le \tilde K_2 M_k^{-\frac{p+1}{m}(m-1)}[1 + M_k^q(\max\{|\omega|, |\vartheta|\})^q](M_k^{-\alpha\mu_1}|x - y|^{\mu_1} + M_k^{\mu_3}|\omega - \vartheta|^{\mu_3})\\ &\le \tilde K_2[1 + (\max\{|\omega|, |\vartheta|\})^q](|x - y|^{\mu_1} + |\omega - \vartheta|^{\mu_3}),\end{aligned} \tag{2.10}$$

By condition (A2), we have

$$M_k^{-\frac{p+1}{m}(m-1)}g(M_k^{-\frac{p-(m-1)}{m}}y + \xi^k, M_k w_k) \le K_3 M_k^{-\frac{p+1}{m}(m-1)}(1 + |M_k w_k|^q). \tag{2.11}$$

The transform (2.5) flatten the boundary $\partial\Omega$, then for k large enough, $\|\partial\Omega_k\|_{1,\beta_0} \le \|\partial\Omega\|_{1,\beta_0}$. Now we use a $C^{1,\beta}$ regularity result Lemma 2.1. From [13] (see also [14]) and (2.8), (2.10) to conclude that there exist positive constants $\beta = \beta(K_1, N, m) \in (0, \beta_0)$ and $C = C(K_1, K_2, K_3, N, m, \Gamma(1), \sup_{\partial\Omega\times[-1,1]} g(z,t), \Omega) > 0$ such that

$$\|w_k\|_{C^{1,\beta}(\overline{\Omega_k})} \le C, \tag{2.12}$$

where C is a constant independent of k.

Set $d_k = \operatorname{dist}(\xi^k, \partial\Omega)$, then $\operatorname{dist}(O, \partial\Omega_k) = M_k^\alpha d_k$. Next we consider two subcases:

Unbounded $\{M_k^\alpha d_k\}$. The sequence $\{M_k^\alpha d_k\}$ is unbounded, we assume there exists a subsequence $\{M_k^\alpha d_k\} \to \infty$ as $k \to \infty$. With the aid of (2.7) and (2.12), we can apply the Arzela-Ascoli theorem and the diagonal line argument to infer that there exists $w \in C^1(\mathbb{R}^N)$, such that

$$\lim_{k\to\infty} w_k(y) = w(y) \ge 0, w(0) = 1, \tag{2.13}$$

uniformly on any compact subset of $\mathbb{R}^N$ in C^1-topology.

Multiplying (2.6) by a test function $\phi \in C^\infty(\mathbb{R}^N)$ and integrating by parts on Ω_k, we obtain

$$\begin{aligned}
&\int_{\Omega_k} |\nabla w_k|^{m-2}\nabla w_k \cdot \nabla\phi dy - \int_{\partial\Omega_k} M_k^{-\frac{p+1}{m}(m-1)} g(M_k^{-\frac{p-(m-1)}{m}} y + \xi^k, M_k w_k)\phi ds \\
&= M_k^{-p} \int_{\partial\Omega_k} [B(M_k^{-\frac{p-(m-1)}{m}} y + \xi^k, M_k w_k, M_k^{\frac{p+1}{m}} \nabla w_k) + \lambda_k]\phi dy.
\end{aligned}$$

On account of (2.11), we have $\lim_{k\to\infty} M_k^{-\frac{p+1}{m}(m-1)} g(M_k^{-\frac{p-(m-1)}{m}} y+\xi^k, M_k w_k) = 0$. Combining the condition (G-L) part 2 with the above equality, we obtain

$$\Delta_m w + b(\xi^0) w^p = 0 \text{in} \mathbb{R}^N,$$

where $\xi^0 = \lim_{k\to\infty} \xi^k \in \bar{\Omega}$, but $w(0) = 1$. This contradicts the Liouville-type theorem on entire space $\mathbb{R}^N$ [7, Therorem 1.1].

Bounded $\{M_k^\alpha d_k\}$. The sequence $\{M_k^\alpha d_k\}$ is bounded as $k \to \infty$. So there exists a subsequence such that $\{M_k^\alpha d_k\} \to \varepsilon \ge 0$. Denote $z = (z', z_N) = (z_1, \dots, z_{N-1}, z_N)$ for any $z \in \mathbb{R}^N$. With proper translation and rotation, one may assume $\xi^k = (0', |\xi^k|), d_k = \operatorname{dist}(O, \xi^k) = |\xi^k|$, where $O = (0', 0) \in \partial\Omega$ is the origin in $\mathbb{R}^N$ and ξ^k is the positive z_N-direction. By the transform (2.5) for any $y \in \Omega_k$, we have $y_N > -\varepsilon$ and the sequence of the domains Ω_k converges to the half-space, namely $\lim_{k\to\infty} \Omega_k = \mathbb{R}^N_\varepsilon := \{y \in \mathbb{R}^n | y_N > -\varepsilon\}$.

By similar arguments as in 2.1.1, we deduce from (2.5)-(2.12) that there exists $w \in C^1(\overline{\mathbb{R}^N_\varepsilon})$ such that

$$\lim_{k\to\infty} w_k(y) = w(y) \ge 0, \quad w(0) = 1, \tag{2.14}$$

uniformly on any compact subset of $\overline{\mathbb{R}^N_\varepsilon}$ in C^1-topology. By the same approach in *2.1.1*, we have

$$\begin{gathered}
\Delta_m w + b(\xi^0) w^p = 0 \quad \text{in } \mathbb{R}^N_\varepsilon, \\
|\nabla w|^{m-2} \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\mathbb{R}^N_\varepsilon,
\end{gathered}$$

where $\xi^0 = \lim_{k\to\infty} \xi^k \in \overline{\Omega}$. On account of the Liouville-type theorem on half space $\mathbb{R}^N_\varepsilon$ in Lemma 2.2, this is a contradiction.

Case 2. There exists $c_0 > 0$ such that

$$\liminf_{k\to\infty} \frac{\lambda_k^{1/p}}{M_k} = c_0,$$

which implies that $\lambda_k \to \infty$ as $k \to \infty$. Fix any $x_0 \in \Omega$ and take $N_k = \lambda_k^{1/p}$, $\zeta^k = x_0$ in (2.5), then we have

$$\lim_{k\to\infty} \frac{\lambda_k}{N_k^p} = 1,$$

$$0 < w_k(y) \le \frac{M_k}{N_k} = \frac{1}{c_0}, y \in \Omega_k.$$

Now since $\operatorname{dist}(O, \partial\Omega_k) = N_k^\alpha \operatorname{dist}(x_0, \partial\Omega) \to \infty$ as $k \to \infty$, then Ω_k converges to the entire space $\mathbb{R}^N$. By similar procedure in *2.1*, we obtain that there exists $w \in C^1(\mathbb{R}^N)$ such that

$$\lim_{k\to\infty} w_k(y) = w(y) \ge 0$$

uniformly on any compact subset of $\mathbb{R}^N$ in C^1-topology and w satisfies

$$\Delta_m w + b(x_0) w^p + 1 = 0 \quad \text{in } \mathbb{R}^N.$$

This contradicts the Liouville-type theorem on entire space $\mathbb{R}^N$ [7, Lemma 2.8 Part 1].

In conclusion, the hypothesis (2.4) is invalid. We completed the proof. $\square$

3. Existence

In this section, we prove the existence of a positive solution for (1.1). We use a version of a fixed point theorem of Krasnoselskii [6]. In this procedure, the a priori estimates Theorem 1.1 are crucial.

Lemma 3.1. *Let $\mathcal{C}$ be a cone in a Banach X space and $\Lambda : \mathcal{C} \to \mathcal{C}$ a compact operator such that $\Lambda(0) = 0$. Assume that there exists $r > 0$, satisfying:*

(1) *$u \neq t\Lambda(u)$ for all $\|u\| = r, t \in [0, 1]$.*

Assume also that there exists a compact homotopy $H : [0, 1] \times \mathcal{C} \to \mathcal{C}$, and $R > r$ such that

(2) *$\Lambda(u) = H(0, u)$ for all $u \in \mathcal{C}$;*
(3) *$H(t, u) \neq u$for any $\|u\| = R, t \in [0, 1]$;*
(4) *$H(1, u) \neq u$for any $\|u\| \le R$.*

Let $D = \{u \in \mathcal{C} : r < \|u\| < R\}$, then Λ has a fixed point in D.

Proof of Theorem 1.2. We use Lemma 3.1. For each $f \in C(\overline{\Omega}), h \in C^\gamma(\partial\Omega)$, we denote by $K(f, h) \in C^{1,\beta}(\overline{\Omega})$ the unique weak solution of the problem

$$-\Delta_m u + L|u|^{m-2}u = f \quad \text{in } \Omega,$$

$$|\nabla u|^{m-2} \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial\Omega.$$

The operator $K : C(\overline{\Omega}) \times C^\gamma(\partial\Omega) \to C^{1,\beta}(\overline{\Omega})$ is bounded, continuous and positive, that is $K(f, h) \ge 0$ provided $f, h \ge 0$ [16, Proposition 2.7(3),(4)]. Define T :

$C^1(\overline{\Omega}) \to C(\overline{\Omega}) \times C^\gamma(\partial\Omega)$, $T(u) = (B(z,u,\nabla u) + L|u|^{m-2}u, g(x,u))$. T is bounded and continuous. Define $\Lambda = K \circ T : C^1(\overline{\Omega}) \to C^{1,\beta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$.

It is clear that the fixed-point of operator Λ is a solution of (1.1). The operator Λ is continuous and compact since $K \circ T$ is continuous and bounded and the embedding $C^{1,\beta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ is compact.

Let $X := C^1(\overline{\Omega})$, $\mathcal{C} = \{u \in X | u \geq 0\}$ is a cone in X. In the sequel, $\|\cdot\|$ denotes the supremum C^1-norm on $\overline{\Omega}$. $\Lambda(0) = 0$ since $K(0,0) = 0$. By the weak comparison principle for the m-Laplace operator with Neumann boundary condition (by the Maximum principle in [17]) and conditions (A4) and (A3), we have $\Lambda : \mathcal{C} \to \mathcal{C}$.

First we verify condition (1) of Lemma 3.1. Consider $u = \lambda\Lambda(u)$ in $\mathcal{C}\backslash\{0\}$ for certain $\lambda \in [0,1]$, that is, u satisfies the following equation

$$
\begin{gathered}
-\Delta_m u + L|u|^{m-2}u = \lambda^{m-1}[B(z,u,\nabla u) + L|u|^{m-2}u] \quad \text{in}\Omega, \\
|\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = \lambda^{m-1} g(x,u) \quad \text{on}\partial\Omega,
\end{gathered} \tag{3.1}
$$

By taking u as a test function and using the condition (A5), we have

$$
\begin{aligned}
&\int_\Omega |\nabla u|^m dz + \int_\Omega L|u|^m dz \\
&= \lambda^{m-1}\int_\Omega (B(z,u,\nabla u)u + L|u|^m)dz + \lambda^{m-1}\int_{\partial\Omega} g(z,u)u ds \\
&= \int_\Omega o(|u|^m + |\nabla u|^m)dz + \int_{\partial\Omega} o(|u|^m) ds
\end{aligned}
$$

as $\|u\| \to 0$. Hence we can choose $r > 0$ small enough such that equation $u = \lambda\Lambda(u)$ has no positive solutions in $B_r(0)\backslash\{0\}$ for all $\lambda \in [0,1]$.

Now we verify (2)–(4) of Lemma 3.1. By Theorem 1.1, there exists a positive constant λ_0, such that problem (1.2) has no solution. Define $H : [0,1] \times \mathcal{C} \to \mathcal{C}$ as $H(t,u) = K \circ (T(u) + t(\lambda_0, 0))$. Clearly, $u = H(t,u)$ is equivalent to

$$
\begin{gathered}
\Delta_m u + B(z,u,\nabla u) + t\lambda_0 = 0 \quad \text{in } \Omega, \\
|\nabla u|^{m-2}\frac{\partial u}{\partial \nu} = g(z,u) \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.2}
$$

Obviously $H(0,u) = \Lambda(u)$ for any $u \in \mathcal{C}$, namely, (2) holds. By Theorem 1.1 solutions of (3.2) are a priori bounded in the uniform norm. There exists a constant $R > r$, such that each solution of (3.2) satisfies $\|u\|_{C^1(\overline{\Omega})} < R$, and then (3) holds. When $t = 1$, (3.2) has no solution in view of the choice of the number λ_0, this implies (4) holds. Therefore the mapping Λ has a fixed point $u \in \mathcal{C}$ and $r < \|u\| < R$, which is a non-negative solution of (1.1). The proof is complete. $\square$

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