

EIGENVALUE ESTIMATES FOR STATIONARY $p(x)$ -KIRCHHOFF PROBLEMS

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ABSTRACT. Using variational techniques we prove an eigenvalue theorem for a stationary $p(x)$ -Kirchhoff problem, and provide an estimate for the range of such eigenvalues. We employ a specific family of test functions in variable-exponent Sobolev spaces. Our approach permits to handle both non-degenerate and degenerate Kirchhoff coefficients.

1. INTRODUCTION AND FUNCTIONAL SET-UP

In this article we analyze the stationary Kirchhoff-type problem

$$\begin{aligned} -k\left(\int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx\right) \Delta_{p(x)} u &= \lambda f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, $k : [0, +\infty) \rightarrow \mathbb{R}$ is defined by

$$k(t) := a + bt^\gamma \quad \text{for all } t \in \mathbb{R}, \tag{1.2}$$

with $a \geq 0$, $b, \gamma > 0$, λ is a positive parameter, $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $p \in C^0(\overline{\Omega})$ is a regular function such that

$$1 < p^- := \inf_{x \in \Omega} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < +\infty. \tag{1.3}$$

The operator

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$$

is the $p(x)$ -Laplacian and reduces to the classical p -Laplacian when p is a constant. It has been recently used in several application contexts, the most notable being the image processing [7] and the motion of electrorheological fluids [15].

Historically equations like (1.1) made their first appearance in [11] as a model to describe free transverse vibrations of a clamped string in which the tension has a non-negligible dependence on the deformation. The nonlocal coefficient (1.2) generalizes the original one proposed by Kirchhoff (with $\gamma = 1$) to the sub/super-linear cases and has already been taken into consideration in literature (cf. [1, 2, 3, 13, 16] and references therein). In many issues, like the decay rate of solutions of wave equations, the growth of k plays, in particular, a central role (see [16] where γ is deeply related to the critical exponent $2n/(n-2)$, $n > 2$). In this paper

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we allow γ to range all over the positive reals and we take also account of the circumstance $a = 0$ (we recall that if k takes the value 0 then problem (1.1) is said to be degenerate; otherwise it is named non-degenerate).

Our aim is to provide a localization theorem for the eigenvalues of the problem under exam. Starting from some results obtained in [4] via variational techniques (see also [5, 6]), we first show that problem (1.1) indeed admits non-trivial solutions for some positive λ . Then, constructing a suitable family of test functions in the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$, depending in particular on the primitive of f and the geometry of Ω , we are able to provide a lower bound for such a λ . As previously mentioned, we manage to deal with both the non-degenerate and the degenerate case. In both situations we require that a certain relationship involving the growth rate of k , the dimension n and the “extreme” values p^-, p^+ of p be satisfied; when $a = 0$ this relationship is more restrictive.

The whole of these results is presented in Section 2. In the remainder of the current one we briefly introduce the functional framework in which (1.1) is set, i.e. the variable exponent Sobolev spaces. We refer to [8, 9, 12] for a more complete account on generalized Lebesgue and Sobolev spaces and for the proof of the properties just stated below.

Hereafter $p \in C^0(\overline{\Omega})$, satisfies (1.3) and is log-Hölder continuous, i.e.

$$|p(x) - p(y)| \leq -\frac{c}{\log|x - y|} \quad (1.4)$$

for some $c > 0$ and for all $x, y \in \Omega$ with $|x - y| \leq 1/2$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable: } \int_{\Omega} |u(x)|^{p(x)} dx < \infty\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} = \inf\{\sigma > 0 : \int_{\Omega} \left|\frac{u(x)}{\sigma}\right|^{p(x)} dx \leq 1\},$$

and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the standard norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$; in view of the assumptions on p , $|\nabla u|_{p(x)}$ turns out to be an equivalent norm on $W_0^{1,p(x)}(\Omega)$ and we will adopt the plain notation $\|u\| = |\nabla u|_{p(x)}$ for any $u \in W_0^{1,p(x)}(\Omega)$. $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces under the norms introduced above.

The function

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)} & \text{if } p(x) < n \\ +\infty & \text{if } p(x) \geq n. \end{cases}$$

represents the variable-exponent counterpart of the critical Sobolev exponent. If $q \in C^0(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for any $x \in \Omega$, then $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ continuously; if $\inf_{\Omega}(p^* - q) > 0$ the embedding is compact.

The functional

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega),$$

called the modular of $L^{p(x)}(\Omega)$, is closely related to the norm $|\cdot|_{p(x)}$, as clarified by the following proposition.

Proposition 1.1. *Let $u \in L^{p(x)}(\Omega)$; then*

- (i) $|u|_{p(x)} < 1$ ($= 1, > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ ($= 1, > 1$);
- (ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;
- (iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;
- (iv) $\lim_{m \rightarrow \infty} |u_m - u|_{p(x)} = 0 \Leftrightarrow \lim_{m \rightarrow \infty} \rho_{p(x)}(u_m - u) = 0$.

The functional

$$\Psi(u) = \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx \quad \text{for all } u \in W_0^{1,p(x)}(\Omega), \tag{1.5}$$

appearing explicitly in problem (1.1), is known to satisfy the following remarkable properties (cf. [10]).

Proposition 1.2. *Ψ is sequentially weakly lower semicontinuous, convex and C^1 in $W_0^{1,p(x)}(\Omega)$. Its derivative, given by*

$$\Psi'(u)(v) = \int_{\Omega} |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx$$

for any $u, v \in W_0^{1,p(x)}(\Omega)$, is a homeomorphism between the spaces $W_0^{1,p(x)}(\Omega)$ and $(W_0^{1,p(x)}(\Omega))^*$.

With the above premises, it is straightforward to realize that (1.1) represents the Euler-Lagrange equation associated with the functional

$$u \mapsto a\Psi(u) + \frac{b}{\gamma + 1} \Psi(u)^{\gamma+1} - \lambda \int_{\Omega} F(u) dx, \quad u \in W_0^{1,p(x)}(\Omega),$$

where

$$F(t) := \int_0^t f(\xi) d\xi \quad \text{for all } t \in \mathbb{R}.$$

Specifically, the weak solutions to (1.1) are those functions $u \in W_0^{1,p(x)}(\Omega)$ such that

$$k(\Psi(u)) \int_{\Omega} |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx - \lambda \int_{\Omega} f(u(x))v(x) dx = 0$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

For the sequel, we introduce the following subset of $W_0^{1,p(x)}(\Omega)$,

$$W := \{u \in W_0^{1,p(x)}(\Omega) : \int_{\Omega} F(u) dx > 0\},$$

and put

$$\Lambda := \inf_{u \in W} \frac{a(\gamma + 1)\Psi(u) + b\Psi(u)^{\gamma+1}}{(\gamma + 1) \int_{\Omega} F(u) dx}. \tag{1.6}$$

2. RESULTS AND APPLICATIONS

Before searching for possible bounds for the eigenvalues of (1.1), let us prove a theorem which guarantees the existence of such eigenvalues. We will treat separately the cases $a \neq 0$ and $a = 0$.

Theorem 2.1. *Let $k : [0, +\infty) \rightarrow \mathbb{R}$ be as in (1.2) with $a, b, \gamma > 0$, let*

$$\frac{\gamma + 1}{\gamma} p^- \leq n < \frac{p^- p^+}{p^+ - p^-}$$

and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

- (i) $\limsup_{|t| \rightarrow +\infty} |f(t)|/|t|^q < +\infty$ for some $q \in (0, \frac{(n+1)p^- - n}{n-p^-})$,
- (ii) $\limsup_{t \rightarrow 0} F(t)/|t|^{p^+} \leq 0$,
- (iii) $\sup_{t \in \mathbb{R}} F(t) > 0$.

Then, for each $\lambda \in (\Lambda, +\infty)$, problem (1.1) admits at least three weak solutions in $W_0^{1,p(x)}(\Omega)$.

The proof of Theorem 2.1 leans on the following multiplicity result (which, in turn, is based on an abstract variational tool of [14]).

Theorem 2.2 ([4, Theorem 3.1]). *Let $(p^+ - p^-)n < p^- p^+$, $k : [0, +\infty) \rightarrow \mathbb{R}$ a non-decreasing continuous function satisfying*

- (A1) $\inf_{[0, +\infty)} k > 0$,
- (A2) $\liminf_{t \rightarrow +\infty} \int_0^t k(\xi) d\xi / t^\sigma > 0$ for some $\sigma > 1$.

Furthermore, denoting by \mathcal{A} the class of all Carathéodory functions $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\sup_{(x,t) \in \Omega \times \mathbb{R}} \frac{|\varphi(x,t)|}{1 + |t|^{q(x)}} < +\infty \tag{2.1}$$

for some $q \in C^0(\bar{\Omega})$, $0 < q < p^* - 1$ in Ω , assume that $f \in \mathcal{A}$ and the following:

- (A3) $\sup_{u \in W_0^{1,p(x)}(\Omega)} \int_{\Omega} F(x,u) dx > 0$;
- (A4) $\limsup_{t \rightarrow 0} \sup_{x \in \Omega} F(x,t)/|t|^{p^+} \leq 0$;
- (A5) $\limsup_{|t| \rightarrow +\infty} \sup_{x \in \Omega} F(x,t)/|t|^{\sigma p^-} \leq 0$,

where, as usual, $F(x,t) = \int_0^t f(x,\xi) d\xi$ for each $(x,t) \in \Omega \times \mathbb{R}$.

Under such hypotheses, if we set

$$\tilde{\Lambda} := \inf \left\{ \frac{\int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx}{\int_{\Omega} F(x,u) dx} k(\xi) d\xi : u \in W_0^{1,p(x)}(\Omega), \int_{\Omega} F(x,u) dx > 0 \right\}, \tag{2.2}$$

for each compact interval $[\Lambda_1, \Lambda_2] \subset (\tilde{\Lambda}, +\infty)$ there exists $r > 0$ with the following property: for every $\lambda \in [\Lambda_1, \Lambda_2]$ and every $g \in \mathcal{A}$, there exists $\mu_1 > 0$ such that, for each $\mu \in [0, \mu_1]$, the problem

$$\begin{aligned} -k \left(\int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx \right) \Delta_{p(x)} u &= \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has at least three weak solutions whose norms in $W_0^{1,p(x)}(\Omega)$ are smaller than r .

Proof of Theorem 2.1. The function

$$k(t) = a + bt^\gamma, \quad t \geq 0,$$

with $a, b > 0$ clearly satisfies (A1) while (A2) is satisfied by $\sigma = \gamma + 1$. The membership of f in \mathcal{A} follows easily from (i) while (A3) and (A4) are consequences of (iii) and (ii), respectively.

It remains to show the validity of (A5). To this end, notice that due to (i) one has

$$\frac{|F(t)|}{|t|^{(\gamma+1)p^-}} \leq c_1|t|^{1-(\gamma+1)p^-} + c_2|t|^{q+1-(\gamma+1)p^-} \tag{2.3}$$

for some $c_1, c_2 > 0$ and for all $t \neq 0$. Since $(\gamma + 1)p^- \leq \gamma n$, we deduce that $(\gamma + 1)(p^-)^2 \leq (\gamma + 1)np^- - np^-$ and therefore that

$$\frac{np^-}{n - p^-} \leq (\gamma + 1)p^-.$$

Taking account of the above inequality and the range of q , passing to the lim sup as $|t| \rightarrow +\infty$ in (2.3) we get the verification of (A5). All the assumptions of Theorem 2.2 being fulfilled, the conclusion follows. \square

Remark 2.3. From the hypotheses of Theorem 2.1 it is immediately seen that when the $\Delta_{p(x)}$ degenerates into the p -Laplacian, no upper bound on the dimension is necessary. Moreover, as observed in [4] in the case $\gamma = 1$, when $n < 2p^-$ the multiplicity of solutions associated with a fixed $\lambda > \Lambda$ is lost and instead the uniqueness occurs.

The underlying idea to obtain a concrete estimate of Λ is to build an *ad-hoc* test function which lies in W and at which reasonably evaluating Ψ . To this end, denote by $B(x, r)$ the n -dimensional open ball centered at $x \in \mathbb{R}^n$ and of radius $r > 0$. As Ω is open we can certainly fix a point $x_0 \in \Omega$ and a number $\tau > 0$ so that $B(x_0, \tau) \subseteq \Omega$.

Now, for any $t \in \mathbb{R}$ and $\eta \in (0, 1)$, define $u_{t,\eta}$ to be

$$u_{t,\eta}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0, \tau) \\ \frac{t}{(1-\eta)\tau} (\tau - |x - x_0|) & \text{if } x \in B(x_0, \tau) \setminus B(x_0, \eta\tau) \\ t & \text{if } x \in B(x_0, \eta\tau), \end{cases} \tag{2.4}$$

where $|\cdot|$ stands for the usual Euclidean norm in \mathbb{R}^n . Moreover, indicate by $\omega_n = 2\pi^{n/2}/n\Gamma(\frac{n}{2})$ the Lebesgue measure of the unit ball in \mathbb{R}^n , where Γ is the Euler Gamma function, and for any $c > 0$ put

$$c_{p^\pm} := \max\{c^{p^-}, c^{p^+}\}.$$

We are now in a position to prove the following result.

Theorem 2.4. *Assume that all the hypotheses of Theorem 2.1 hold and let Λ be defined by (1.6). Then there exists $t_0 \in \mathbb{R}$ and $\eta_0 = \eta_0(t_0) \in (0, 1)$ such that $\Lambda \leq \Lambda^*$, where*

$$\Lambda^* = \frac{a(\gamma + 1)(p^-(1 - \eta_0)^{p^+})^\gamma (1 - \eta_0^n) \left(\frac{|t_0|}{\tau}\right)_{p^\pm} + b\omega_n^\gamma (1 - \eta_0^n)^{\gamma+1} \left(\frac{|t_0|}{\tau}\right)_{p^\pm}^{\gamma+1} \tau^{n\gamma}}{(\gamma + 1)(p^-(1 - \eta_0)^{p^+})^{\gamma+1} (F(t_0)\eta_0^n - (1 - \eta_0^n) \max_{|t| \leq |t_0|} |F(t)|)}. \tag{2.5}$$

Proof. Assumption (iii) of Theorem 2.1 of course provides the existence of a number $t_0 \in \mathbb{R}$ such that $F(t_0) > 0$. Now let $\eta_0 \in (0, 1)$ be such that

$$\frac{\max_{|t| \leq |t_0|} |F(t)|}{F(t_0) + \max_{|t| \leq |t_0|} |F(t)|} < \eta_0^n < 1$$

and, with the same notation as (2.4), consider the function u_{t_0, η_0} .

Since $|u_{t_0, \eta_0}(x)| \leq |t_0|$ for every $x \in B(x_0, \tau) \setminus B(x_0, \eta_0 \tau)$, it follows that

$$\int_{B(x_0, \tau) \setminus B(x_0, \eta_0 \tau)} F(u_{t_0, \eta_0}(x)) dx \geq -(1 - \eta_0^n) \omega_n \tau^n \max_{|t| \leq |t_0|} |F(t)|,$$

and therefore

$$\begin{aligned} & \int_{\Omega} F(u_{t_0, \eta_0}(x)) dx \\ &= \int_{B(x_0, \eta_0 \tau)} F(u_{t_0, \eta_0}(x)) dx + \int_{B(x_0, \tau) \setminus B(x_0, \eta_0 \tau)} F(u_{t_0, \eta_0}(x)) dx \\ &\geq F(t_0) \eta_0^n \omega_n \tau^n + \int_{B(x_0, \tau) \setminus B(x_0, \eta_0 \tau)} F(u_{t_0, \eta_0}(x)) dx \\ &\geq \left(F(t_0) \eta_0^n - (1 - \eta_0^n) \max_{|t| \leq |t_0|} |F(t)| \right) \omega_n \tau^n > 0. \end{aligned} \quad (2.6)$$

On the other hand we can deduce the following upper estimate for Ψ :

$$\begin{aligned} \Psi(u_{t_0, \eta_0}) &= \int_{\Omega} \frac{|\nabla u_{t_0, \eta_0}(x)|^{p(x)}}{p(x)} dx \\ &\leq \frac{1}{p^-} \int_{B(x_0, \tau) \setminus B(x_0, \eta_0 \tau)} \frac{|t_0|^{p(x)}}{((1 - \eta_0) \tau)^{p(x)}} dx \\ &\leq \frac{(1 - \eta_0^n) \omega_n \left(\frac{|t_0|}{\tau} \right)_{p^\pm} \tau^n}{p^- (1 - \eta_0)^{p^+}}. \end{aligned} \quad (2.7)$$

So, thanks to Proposition 1.1 and the inequalities (2.6), (2.7), $u_{t_0, \eta_0} \in W$ and, recalling the definition of Λ , one has

$$\begin{aligned} \Lambda &\leq \frac{a(\gamma + 1) \Psi(u_{t_0, \eta_0}) + b \Psi(u_{t_0, \eta_0})^{\gamma+1}}{(\gamma + 1) \int_{\Omega} F(u_{t_0, \eta_0}) dx} \\ &\leq \frac{\frac{a(\gamma+1)(1-\eta_0^n) \omega_n \left(\frac{|t_0|}{\tau} \right)_{p^\pm} \tau^n}{p^- (1-\eta_0)^{p^+}} + \frac{b(1-\eta_0^n)^{\gamma+1} \omega_n^{\gamma+1} \left(\frac{|t_0|}{\tau} \right)_{p^\pm}^{\gamma+1} \tau^{n(\gamma+1)}}{(p^- (1-\eta_0)^{p^+})^{\gamma+1}}}{(\gamma + 1) \left(F(t_0) \eta_0^n - (1 - \eta_0^n) \max_{|t| \leq |t_0|} |F(t)| \right) \omega_n \tau^n} \\ &= \Lambda^*, \end{aligned}$$

as claimed. \square

Remark 2.5. Let us notice that Λ^* is a function of the nonlinearity f through t_0 and η_0 , of the elliptic operator through the constants a, b, γ, p^-, p^+ , while it is related to the geometry of the domain by τ .

Moreover, for fixed τ and t_0 , the sharp value of Λ^* can be computed. Indeed, thought of as a function of η , clearly $\Lambda^*(\eta)$ is continuous and, set

$$\eta^* := \left(\frac{\max_{|t| \leq |t_0|} |F(t)|}{F(t_0) + \max_{|t| \leq |t_0|} |F(t)|} \right)^{1/n},$$

one has

$$\lim_{\eta \rightarrow \eta^*} \Lambda^*(\eta) = +\infty.$$

On the other hand, a simple application of de l'Hôpital's rule shows that

$$\lim_{\eta \rightarrow 1^-} \Lambda^*(\eta) = +\infty$$

and hence $\eta \mapsto \Lambda^*(\eta)$ admits a minimum in $(\eta^*, 1)$.

Example 2.6. Let $a, b > 0$, $\gamma = 1$, $\Omega = B(0, 1) \subset \mathbb{R}^3$,

$$p(x) = |x|^2 + \frac{5}{2}$$

and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(t) = \begin{cases} t - \sqrt{t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Clearly one has

$$\begin{aligned} \lim_{|t| \rightarrow +\infty} \frac{|f(t)|}{|t|^q} &= 0, \quad \text{for all } q \in (1, 14) \\ \limsup_{t \rightarrow 0} \frac{F(t)}{|t|^{7/2}} &= 0, \\ \sup_{t \in \mathbb{R}} F(t) &= +\infty. \end{aligned}$$

Therefore all the assumptions of Theorem 2.1 are met. Setting $t_0 = 2$ and $\eta_0 \in (\sqrt[3]{(13 + 8\sqrt{2})/41}, 1)$, and considering the problem

$$\begin{aligned} -\left(a + b \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx\right) \Delta_{p(x)} u &= \lambda f(u) \quad \text{in } B(0, 1) \\ u &= 0 \quad \text{on } \partial B(0, 1), \end{aligned} \tag{2.8}$$

for any λ varying inside the interval

$$\left(\frac{2^{\frac{11}{2}} \cdot 15a(1 - \eta_0)^{\frac{7}{2}}(1 - \eta_0^3) + 2^{11}\pi b(1 - \eta_0^3)^2}{25(1 - \eta_0)^7((13 - 8\sqrt{2})\eta_0^3 - 1)}, +\infty\right)$$

the above problem admits at least three weak solutions in $W_0^{1,p(x)}(B(0, 1))$.

Now let us pass to consider the degenerate case. As already mentioned, a more restrictive relationship among γ , p^- , p^+ and n needs to occur.

Theorem 2.7. Let $k : [0, +\infty) \rightarrow \mathbb{R}$ be as in (1.2) with $a = 0$, $b, \gamma > 0$, let

$$\frac{(2(\gamma + 1)p^- - 1)p^-}{(2\gamma + 1)p^- - 1} \leq n < \frac{(\gamma + 1)p^- p^+}{(\gamma + 1)p^+ - p^-}$$

and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

- (i) $\limsup_{|t| \rightarrow +\infty} |f(t)|/|t|^q < +\infty$ for some $q \in (0, \frac{(n+1)p^- - n}{2(n-p^-)})$,
- (ii) $\limsup_{t \rightarrow 0} F(t)/|t|^{(\gamma+1)p^+} \leq 0$,
- (iii) $\sup_{t \in \mathbb{R}} F(t) > 0$.

Then, the conclusion of Theorem 2.1 holds.

The proof of the above theorem is a consequence of the following result, which can be deduced with slight modifications from [4, Theorem 3.2].

Theorem 2.8. *Let $\gamma > 0$, $((\gamma + 1)p^+ - p^-)n < (\gamma + 1)p^-p^+$, $k : [0, +\infty) \rightarrow \mathbb{R}$ a non-decreasing continuous function satisfying*

$$(A1') \quad k(t) \geq k_1 t^\gamma \text{ for some } k_1 > 0 \text{ and for any } t \geq 0$$

and (A2). Moreover let $f \in \mathcal{A}$ satisfy (A3), (A5) and

$$(A4') \quad \limsup_{t \rightarrow 0} \sup_{x \in \Omega} F(x, t)/|t|^{(\gamma+1)p^+} \leq 0.$$

Then, set $\tilde{\Lambda}$ as in (2.2), the same conclusion as Theorem 2.2 holds.

Proof of Theorem 2.7. It is obvious that the function

$$k(t) = bt^\gamma, \quad t \geq 0$$

satisfies (A1') and (A2) for $\sigma = \gamma + 1$. Moreover, on account of (i)-(jjj), $f \in \mathcal{A}$ and promptly verifies (A3) and (A4'). Finally, notice that thanks to (i) one has

$$\frac{|F(t)|}{|t|^{(\gamma+1)p^-}} \leq c_1 |t|^{1-(\gamma+1)p^-} + c_2 |t|^{q+1-(\gamma+1)p^-} \quad (2.9)$$

for some $c_1, c_2 > 0$ and for all $t \neq 0$. Since

$$(2(\gamma + 1)p^- - 1)p^- \leq n((2\gamma + 1)p^- - 1),$$

it follows that

$$2(\gamma + 1)(p^-)^2 - p^- \leq 2(\gamma + 1)np^- - np^- - n$$

and hence

$$\frac{(n + 1)p^- - p^-}{2(n - p^-)} \leq (\gamma + 1)p^-. \quad (2.10)$$

Inequality (2.10) and the definition of q imply that, if we take the lim sup as $|t| \rightarrow +\infty$ in (2.9), (A5) is fulfilled as well. So the conclusion follows from Theorem 2.8. \square

By the same line of reasoning as Theorem 2.4, we can obtain an estimate of Λ also in the degenerate case.

Theorem 2.9. *Assume that all the hypotheses of Theorem 2.7 hold. Then there exists $t_0 \in \mathbb{R}$ and $\eta_0 = \eta_0(t_0) \in (0, 1)$ such that*

$$\Lambda \leq \frac{b\omega_n^\gamma (1 - \eta_0^n)^{\gamma+1} \left(\frac{|t_0|}{\tau}\right)_{p^\pm}^{\gamma+1} \tau^{n\gamma}}{(\gamma + 1)(p^-(1 - \eta_0)^{p^+})^{\gamma+1} (F(t_0)\eta_0^n - (1 - \eta_0^n) \max_{|t| \leq |t_0|} |F(t)|)}. \quad (2.11)$$

Example 2.10. Let $\gamma = 1/2$, $\Omega = B(0, 1/2) \subset \mathbb{R}^3$, $p(x) = |x|^2 + \frac{3}{2}$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(t) = \begin{cases} t^{1/2} - t^{1/3} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Being

$$\lim_{|t| \rightarrow +\infty} \frac{|f(t)|}{|t|^q} = 0 \quad \text{for all } q \in (1/2, 1)$$

$$\limsup_{t \rightarrow 0} \frac{F(t)}{|t|^{21/8}} = 0,$$

$$\sup_{t \in \mathbb{R}} F(t) = +\infty,$$

all the requirements of Theorem 2.7 are satisfied. Setting $t_0 = 4$ and $\eta_0 \in (\sqrt[3]{4}/2, 1)$, and considering the problem

$$\begin{aligned} - \left(\int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx \right)^{1/2} \Delta_{p(x)} u &= \lambda f(u) \quad \text{in } B(0, 1/2) \\ u &= 0 \quad \text{on } \partial B(0, 1/2), \end{aligned} \quad (2.12)$$

we deduce that for each λ running in the interval

$$\left(\frac{2^{\frac{39}{8}} \pi^{1/2} b (1 - \eta_0^3)^{\frac{3}{2}}}{9(16 - 9\sqrt[3]{4})(1 - \eta_0)^{\frac{7}{4}}(2\eta_0^3 - 1)}, +\infty \right)$$

there exist at least three weak solutions in $W_0^{1,p(x)}(B(0, 1/2))$.

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REFERENCES

- [1] G. Autuori, P. Pucci. *Kirchhoff systems with nonlinear source and boundary damping terms*. Comm. Pure Appl. Anal., **9**(5): 1161–1188, 2010.
- [2] G. Autuori, P. Pucci, M. C. Salvatori. *Asymptotic stability for anisotropic Kirchhoff systems*. J. Math. Anal. Appl., **352**: 149–165, 2009.
- [3] G. Autuori, P. Pucci, M.C. Salvatori. *Global nonexistence for nonlinear Kirchhoff systems*. Arch. Rat. Mech. Anal., **196**: 489–516, 2010.
- [4] F. Cammaroto L. Vilasi. *Multiple solutions for a Kirchhoff-type problem involving the $p(x)$ -Laplacian operator*. Nonlinear Analysis, **74**: 1841–1852, 2011.
- [5] F. Cammaroto L. Vilasi. *Existence of three solutions for a degenerate Kirchhoff-type transmission problem*. Num. Func. Anal. Opt., **35**(7-9): 911–931, 2014.
- [6] F. Cammaroto, L. Vilasi. *Existence of three solutions for a nonlocal transmission problem*. J. Nonl. Conv. Anal., **17**(3): 421–439, 2016.
- [7] Y. Chen, S. Levine, M. Rao. *Functionals with $p(x)$ -growth in image processing*. Duquesne University, Department of Mathematics and Computer Science, Technical Report 2004-01.
- [8] L. Diening, P. Harjulehto, P. Hästö, M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, Lect. Notes Math., volume **2017**. Springer-Verlag, Berlin, 2011.
- [9] X. L. Fan, D. Zhao. *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$* . J. Math. Anal. Appl., **263**: 424–446, 2001.
- [10] X.L. Fan, Q. H. Zhang. *Existence of solutions for $p(x)$ -Laplacian Dirichlet problem*. Nonlinear Analysis, **52**: 1843–1852, 2003.
- [11] G. R. Kirchhoff. *Vorlesungen über mathematische Physik: Mechanik*. Teubner, Leipzig, 1883.
- [12] O. Kováčik, J. Rákosník. *On spaces $L^{p(x)}$ and $W^{k,p(x)}$* . Czechoslovak Math. J., **41**(4): 592–618, 1991.
- [13] K. Ono. *Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings*. J. Differential Equations, **37**: 273–301, 1997.
- [14] B. Ricceri. *A further three critical points theorem*. Nonlinear Analysis, **71**: 4151–4157, 2009.
- [15] M. Růžička. *Electrorheological Fluids: Modeling and Mathematical Theory*. Springer, Berlin, 2000.
- [16] X. Zhu. *Decay rate of solutions of a wave equation with damping and external force*. Nonlinear Analysis, **46**: 335–345, 2001.

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