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GROWTH OF TRANSCENDENTAL SOLUTIONS TO HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

KARIMA HAMANI

Abstract. In this article, we study the growth of transcendental solutions of certain higher order linear differential equations with entire coefficients. Under some conditions, we prove that every transcendental solution is of infinite order. We also give an estimate of its hyper-order. We improve previous results by Peng and Chen [14].

1. Introduction and statement of results

In this article, we use fundamental results and the standard notation of the Nevanlinna's value distribution theory of meromorphic functions (see [8, 16]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f and $\sigma_2(f)$ to denote the hyper-order of f which is defined in [16] by

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

For the second order linear differential equation

$$f'' + e^{-z}f' + B(z)f = 0, \tag{1.1}$$

where $B(z)$ is an entire function of finite order, it is well known that every solution of (1.1) is an entire function and most solutions of (1.1) have an infinite order. Thus, a natural question is: what conditions on $B(z)$ will guarantee that every solution $f(\not\equiv 0)$ of (1.1) has an infinite order? Ozawa [13], Gundersen [6], Langley [11]. Amemiya and Ozawa [1] have studied the problem, where $B(z)$ is a nonconstant polynomial or a transcendental entire function with order $\sigma(B) \neq 1$. In 2002, Chen [3] investigated the growth of solutions of equation (1.1) in the case where $\sigma(B) = 1$.

In 1988, Gundersen [7] studied finite order solutions of second order linear differential equations, where coefficients satisfy certain conditions in some angle. This result was generalised to higher order linear differential equations by Laine and Yang [10]. Recently, the authors [9] have studied completely regular growth solutions of second order linear differential equations and discussed cases where coefficients and or solutions of these equations are exponential polynomials.

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Recently, Peng and Chen [14] have studied the order and the hyper-order of solutions of equation (1.1) and have proved the following result:

Theorem 1.1 ([14])**.** *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1a_2 \neq 0$ and $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + e^{-z}f' + (A_1(z)e^{a_1 z} + A_2(z)e^{a_2 z})f = 0 \tag{1.2}$$

is of infinite order and $\sigma_2(f) = 1$.

In this article, we continue the research in this type of problem. We consider the higher order linear differential equation

$$f^{(k)} + h_{k-1}(z)f^{(k-1)} + \cdots + h_1(z)f' + h_0(z)f = 0, \tag{1.3}$$

where $k > 2$ is an integer and $h_j(z)$ $(j = 0, \dots, k-1)$ are entire functions. We suppose that there exists only one coefficient of the form $h_s(z) = A_1(z)e^{P_1(z)} + A_2(z)e^{P_2(z)}$, where $P_l(z) = \sum_{i=0}^{n} a_{i,l}z^i$ $(l = 1, 2)$ are polynomials with degree $n \geq 1$ and $A_l(z)$ $(\not\equiv 0)$ $(l = 1, 2)$ are entire functions with $\sigma(A_l) < n$. The other coefficients have the form $h_j(z) = B_j(z)e^{Q_j(z)}$ $(j \neq s)$, where $Q_j(z) = \sum_{i=0}^{n} b_{i,j}z^i$ are polynomials with degree $n \geq 1$ and $B_j(z)$ $(\not\equiv 0)$ are entire functions with $\sigma(B_j) < n$. Under some conditions on the complex numbers $a_{n,l}$ $(l = 1, 2)$ and $b_{n,j}$ $(j \neq s)$, we will prove that every transcendental solution of equation (1.3) is of infinite order. We also give an estimation of its hyper-order. We will prove the following results:

Theorem 1.2. *Let $P_l(z) = \sum_{i=0}^{n} a_{i,l}z^i$ $(l = 1, 2)$ be polynomials with degree $n \geq 1$, where $a_{0,l}, \dots, a_{n,l}$ $(l = 1, 2)$ are complex numbers such that $a_{n,1} \neq a_{n,2}$, $A_l(z)$ $(\not\equiv 0)$ $(l = 1, 2)$ be entire functions with $\sigma(A_l) < n$ and $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $h_s(z) = A_1(z)e^{P_1(z)} + A_2(z)e^{P_2(z)}$ and for $j \neq s$, $h_j(z) = B_j(z)e^{Q_j(z)}$, where $B_j(z)$ $(\not\equiv 0)$ are entire functions with $\sigma(B_j) < n$, $Q_j(z) = \sum_{i=0}^{n} b_{i,j}z^i$ are polynomials with degree $n \geq 1$ and $b_{0,j}, \dots, b_{n,j}$ $(j \neq s)$ are complex numbers. Let I and J be two sets satisfying $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$ and $I \cup J = \{0, \dots, s-1, s+1, \dots, k-1\}$ such that for $j \in I$, $b_{n,j} = \alpha_j a_{n,1}$ $(0 < \alpha_j < 1)$ and for $j \in J$, $b_{n,j} = \beta_j a_{n,2}$ $(0 < \beta_j < 1)$. Set $a_{n,l} = |a_{n,l}|\ e^{i\theta_l}$, $\theta_l \in [0, 2\pi)$ $(l = 1, 2)$, $\alpha = \max\{\alpha_j : j \in I\}$ and $\beta = \max\{\beta_j : j \in J\}$.*

If $\theta_1 \neq \theta_2$ or $\theta_1 = \theta_2$ and (i) $|a_{n,1}| < (1-\beta)|a_{n,2}|$ or (ii) $|a_{n,2}| < (1-\alpha)|a_{n,1}|$, then every transcendental solution f of equation (1.3) is of infinite order and satisfies $\sigma_2(f) = n$.

Theorem 1.3. *Let $P_l(z) = \sum_{i=0}^{n} a_{i,l}z^i$ $(l = 1, 2)$ be polynomials with degree $n \geq 1$, where $a_{0,l}, \dots, a_{n,l}$ $(l = 1, 2)$ are complex numbers such that $a_{n,1} \neq a_{n,2}$ (suppose that $|a_{n,1}| \leq |a_{n,2}|$), $A_l(z)$ $(\not\equiv 0)$ $(l = 1, 2)$ be entire functions with $\sigma(A_l) < n$ and $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $h_s(z) = A_1(z)e^{P_1(z)} + A_2(z)e^{P_2(z)}$ and for $j \neq s$, $h_j(z) = B_j(z)e^{Q_j(z)}$, where $B_j(z)$ $(\not\equiv 0)$ are entire functions with $\sigma(B_j) < n$, $Q_j(z) = \sum_{i=0}^{n} b_{i,j}z^i$ are polynomials with degree $n \geq 1$ and $b_{0,j}, \dots, b_{n,j}$ $(j \neq s)$ are complex numbers. Let I and J be two sets satisfying $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$ and $I \cup J = \{0, \dots, s-1, s+1, \dots, k-1\}$ such that for $j \in I$, $b_{n,j} = \alpha_j a_{n,1}$ $(0 < \alpha_j < 1)$ and for $j \in J$, $b_{n,j}$ are real numbers satisfying $b_{n,j} < 0$.*

If $a_{n,1}$ is a real number such that $(1-\alpha)a_{n,1} < b$, where $\alpha = \max\{\alpha_j : j \in I\}$ and $b = \min\{b_{n,j} : j \in J\}$, then every transcendental solution f of equation (1.3) *is of infinite order and satisfies $\sigma_2(f) = n$.*

Theorem 1.4. *Let $P_l(z) = \sum_{i=0}^{n} a_{i,l}z^i$ $(l = 1, 2)$ be polynomials with degree $n \geq 1$, where $a_{0,l}, \dots, a_{n,l}$ $(l = 1, 2)$ are complex numbers such that $a_{n,1} \neq a_{n,2}$ (suppose that $|a_{n,1}| \leq |a_{n,2}|$), $A_l(z)$ $(\not\equiv 0)$ $(l = 1, 2)$ be entire functions with $\sigma(A_l) < n$ and $h_j(z)$ $(j = 0, \dots, k-1)$ be entire functions. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $h_s(z) = A_1(z)e^{P_1(z)} + A_2(z)e^{P_2(z)}$ and for $j \neq s$, $h_j(z) = B_j(z)e^{Q_j(z)}$, where $B_j(z)$ $(\not\equiv 0)$ are entire functions with $\sigma(B_j) < n$, $Q_j(z) = \sum_{i=0}^{n} b_{i,j}z^i$ are polynomials with degree $n \geq 1$ and $b_{0,j}, \dots, b_{n,j}$ $(j \neq s)$ are complex numbers. Let I and J be two sets satisfying $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, s-1, s+1, \dots, k-1\}$ such that for $j \in I$, $b_{n,j} = \alpha_j a_{n,1} + \beta_j a_{n,2}$ $(0 < \alpha_j < 1)$, $(0 < \beta_j < 1)$ and for $j \in J$, $b_{n,j}$ are real numbers satisfying $b_{n,j} < 0$. Set $\alpha = \max\{\alpha_j : j \in I\}$, $\beta = \max\{\beta_j : j \in I\}$ and $b = \min\{b_{n,j} : j \in J\}$.*

If $a_{n,1}$ and $a_{n,2}$ are real numbers such that (i) $(1-\beta)a_{n,2} - b < a_{n,1} < 0$ or (ii) $(1-\alpha)a_{n,1} - b < a_{n,2} < 0$, then every transcendental solution f of equation (1.3) *is of infinite order and satisfies $\sigma_2(f) = n$.*

Remark 1.5. In Theorem 1.1, the authors have considered conditions only on one complex number a_1. But in Theorem 1.2 and Theorem 1.4, conditions are imposed to the two numbers $a_{n,l}$ $(l = 1, 2)$.

2. Preliminary Lemmas

Lemma 2.1 ([5])**.** *Let $f(z)$ be a transcendental meromorphic function of finite order σ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denotes a set of distinct pairs of integers satisfying $k_i > j_i \geqslant 0$ $(i = 1, 2, \dots, m)$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi)\backslash E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_1$, and for all $(k, j) \in \Gamma$, we have*

$$|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.2 ([2, 12])**.** *Let $P(z) = (\alpha+i\beta)z^n + \dots$ (α, β are real numbers, $|\alpha|+|\beta| \neq 0$) be a polynomial with degree $n \geq 1$, and $A(z)$ be an entire function with $\sigma(A) < n$. Set $f(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha\cos(n\theta) - \beta\sin(n\theta)$. Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero such that for any $\theta \in [0, 2\pi)\backslash E_2 \cup H$, where $H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set, there is a constant $R_2 > 1$ such that for $|z| = r \geqslant R_2$, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P, \theta)r^n\} \leq |f(re^{i\theta})| \leq \exp\{(1+\varepsilon)\delta(P, \theta)r^n\}, \tag{2.2}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P, \theta)r^n\} \leq |f(re^{i\theta})| \leq \exp\{(1-\varepsilon)\delta(P, \theta)r^n\}. \tag{2.3}$$

Lemma 2.3 ([7, 9])**.** *Let $d \geq 1$ be an integer, $f(z)$ be an entire function and suppose that $|f^{(d)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an* *infinite sequence of points* $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, *where* $r_m \to +\infty$, *such that* $f^{(d)}(z_m) \to \infty$ *and*

$$|\frac{f^{(j)}(z_m)}{f^{(d)}(z_m)}| \leqslant \frac{1}{(d-j)!}(1+o(1))|z_m|^{d-j} \quad (j = 0, \dots, d-1). \tag{2.4}$$

The following Lemma is a trivial consequence of theorems by Phragmèn-Lindelöf and Liouville (see [13, p. 214]):

Lemma 2.4 ([4]). *Let* $f(z)$ *be an entire function of finite order* ρ. *Suppose that there exists a set* $E_3 \subset [0, 2\pi)$ *that has linear measure zero such that for any ray* $\arg z = \theta \in [0, 2\pi)\backslash E_3$, $|f(re^{i\theta})| \le Mr^k$, *where* $M = M(\theta) > 0$ *is a constant and* k (> 0) *is a constant independent of* θ. *Then* $f(z)$ *is a polynomial with* $\deg f \le k$.

Lemma 2.5 ([5]). *Let* $f(z)$ *be a transcendental meromorphic function. Let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then there exist a set* $F_1 \subset (1, +\infty)$ *having finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* (i, j) $(i, j$ *are positive integers with* $i > j)$, *such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup F_1$, *we have*

$$|\frac{f^{(i)}(z)}{f^{(j)}(z)}| \le B\big[\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\big]^{i-j}. \tag{2.5}$$

Lemma 2.6 ([15]). *Let* $f(z)$ *be a transcendental entire function. For each sufficiently large* $|z| = r$, *let* $z_r = r\ e^{i\theta_r}$ *be a point satisfying* $|f(z_r)| = M(r, f)$. *Then there exist a constant* δ_r (> 0) *and a set* F_2 *of finite logarithmic measure, such that for all* z *satisfying* $|z| = r \notin F_2$ *and* $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, *we have*

$$|\frac{f(z)}{f^{(d)}(z)}| \le r^{2d} \quad (d \ge 1 \text{ is an integer}). \tag{2.6}$$

Lemma 2.7 ([7]). *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone nondecreasing functions such that* $\varphi(r) \leqslant \psi(r)$ *for all* $r \notin F_3 \cup [0, 1]$, *where* $F_3 \subset (1, +\infty)$ *is a set of finite logarithmic measure. Let* $\alpha > 1$ *be a given constant. Then there exists an* $r_0 = r_0(\alpha) > 0$ *such that* $\varphi(r) \leqslant \psi(\alpha r)$ *for all* $r > r_0$.

Lemma 2.8 ([4]). *Let* $k \ge 2$ *be an integer and* $A_j(z)$ $(j = 0, 1, \dots, k-1)$ *be entire functions of finite order. Set* $\rho = \max\{\sigma(A_j) : j = 0, 1, \dots, k-1\}$. *If* f *is a solution of equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0, \tag{2.7}$$

then $\sigma_2(f) \le \rho$.

3. Proof of Theorem 1.2

Assume f is a transcendental solution of (1.3).

First step. We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \rho < +\infty$. By Lemma 2.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_1$, we have

$$|\frac{f^{(j)}(z)}{f^{(i)}(z)}| \le |z|^{k\rho} \ (0 \le i < j \le k). \tag{3.1}$$

By Lemma 2.2, for any given $\varepsilon > 0$, there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi)/E_2 \cup H_1$ and r is sufficiently large, then $A_l(z)e^{P_l(z)}$ $(l=1,2)$ and $B_j(z)e^{Q_j(z)}$ $(j\neq s)$ satisfy (2.2) or (2.3), where $H_1=\{\theta\in[0,2\pi):\delta(P_1,\theta)$ or $\delta(P_2,\theta)=0\}$.

Case 1. Suppose that $\theta_1\neq\theta_2$. Set $H_2=\{\theta\in[0,2\pi):\delta(P_1,\theta)=\delta(P_2,\theta)\}$. Since $\theta_1\neq\theta_2$, it follows that H_2 has linear measure zero. For any given $\theta\in[0,2\pi)\backslash E_1\cup E_2\cup H_1\cup H_2$, we have

$$\begin{gathered}\delta(P_1,\theta)\neq 0,\quad \delta(P_2,\theta)\neq 0\quad\text{and}\\ \delta(P_1,\theta)>\delta(P_2,\theta)\quad\text{or}\quad \delta(P_1,\theta)<\delta(P_2,\theta).\end{gathered}\tag{3.2}$$

Set $\delta_1=\delta(P_1,\theta)$ and $\delta_2=\delta(P_2,\theta)$.

Subcase 1.1: $\delta_1>\delta_2$**.** Here we also divide our proof in three subcases:

(a): $\delta_1>\delta_2>0$**.** Set $\delta_3=\max\{\delta_2,\delta(Q_j,\theta):j\in I\}$. Then $0<\delta_3<\delta_1$. Thus for any given ε $(0<\varepsilon<\frac{\delta_1-\delta_3}{2(\delta_1+\delta_3)})$ and all z satisfying $\arg z=\theta$ and $|z|=r$ sufficiently large, we have

$$|A_1(z)e^{P_1(z)}|\geq\exp\{(1-\varepsilon)\delta_1r^n\},\tag{3.3}$$

$$|A_2(z)e^{P_2(z)}|\leq\exp\{(1+\varepsilon)\delta_3r^n\}\tag{3.4}$$

$$|B_j(z)e^{Q_j(z)}|\leq\exp\{(1+\varepsilon)\delta_3r^n\}\ (j\neq s).\tag{3.5}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z=\theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\dots)$, where $r_m\to+\infty$ such that $f^{(s)}(z_m)\to\infty$ and

$$\Big|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\Big|\leqslant\frac{1}{(s-j)!}(1+o(1))|z_m|^{s-j}\quad(j=0,\dots,s-1).\tag{3.6}$$

By (1.3), (3.1) and (3.3)–(3.6), for the above z_m, we obtain

$$\exp\{(1-\varepsilon)\delta_1r_m^n\}\leq M_1r_m^{d_1}\exp\{(1+\varepsilon)\delta_3r_m^n\},\tag{3.7}$$

where M_1, $d_1(>0)$ are constants. This is a contradiction. Hence $|f^{(s)}(z)|\leq M$ on $\arg z=\theta$, where M (>0) is a constant. We can easily obtain

$$|f(z)|\leq M|z|^s\tag{3.8}$$

on $\arg z=\theta$. By Lemma 2.4, (3.8) and the fact that $E_1\cup E_2\cup H_1\cup H_2$ has linear measure zero, we obtain that $f(z)$ is a polynomial with $\deg f\leq s$, which contradicts our assumption. Therefore $\sigma(f)=+\infty$.

(b): $\delta_1>0>\delta_2$**.** Thus for any given ε $(0<\varepsilon<\frac{1-\alpha}{2(1+\alpha)})$ and all z satisfying $\arg z=\theta$ and $|z|=r$ sufficiently large, we have (3.3),

$$|A_2(z)e^{P_2(z)}|\leq\exp\{(1-\varepsilon)\delta_2r^n\}<1,\tag{3.9}$$

$$|B_j(z)e^{Q_j(z)}|\leq\exp\{(1+\varepsilon)\alpha\delta_1r^n\}\ (j\in I),\tag{3.10}$$

$$|B_j(z)e^{Q_j(z)}|\leq\exp\{(1-\varepsilon)\delta(Q_j,\theta)r^n\}<1\ (j\in J).\tag{3.11}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z=\theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\dots)$, where $r_m\to+\infty$ such that $f^{(s)}(z_m)\to\infty$ and (3.6) holds.

By (1.3), (3.1), (3.3), (3.6) and (3.9)-(3.11), for the above z_m, we obtain

$$\exp\{(1-\varepsilon)\delta_1r_m^n\}\leq M_2r_m^{d_2}\exp\{(1+\varepsilon)\alpha\delta_1r_m^n\},\tag{3.12}$$

where M_2, d_2 (>0) are constants. This is a contradiction. Hence $|f^{(s)}(z)| \leq M$ on $\arg z = \theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z = \theta$. Using similar arguments as above, we deduce that $\sigma(f) = +\infty$.

(c): $0 > \delta_1 > \delta_2$**.** Thus for any given ε $(0 < 2\varepsilon < 1)$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$|A_l(z)e^{P_l(z)}| \leq \exp\{(1-\varepsilon)\delta(P_l,\theta)r^n\} \quad (l = 1,2), \tag{3.13}$$

$$|B_j(z)e^{Q_j(z)}| \leq \exp\{(1-\varepsilon)\delta(Q_j,\theta)r^n\}(j \neq s). \tag{3.14}$$

By (1.3), we obtain

$$-1 = h_{k-1}(z)\frac{f^{(k-1)}(z)}{f^{(k)}(z)} + \cdots + h_s(z)\frac{f^{(s)}(z)}{f^{(k)}(z)} + \cdots + h_0(z)\frac{f(z)}{f^{(k)}(z)}. \tag{3.15}$$

Now we prove that $|f^{(k)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(k)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, where $r_m \to +\infty$ such that $f^{(k)}(z_m) \to \infty$ and

$$\Big|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\Big| \leqslant \frac{1}{(k-j)!}(1+o(1))|z_m|^{k-j} \quad (j = 0, \dots, k-1). \tag{3.16}$$

Substituting (3.1), (3.13), (3.14) and (3.16) into (3.15), as $r_m \to +\infty$, we obtain $1 \leq 0$. This contradiction implies $|f^{(k)}(z)| \leq M'$ on $\arg z = \theta$, where M' (>0) is a constant. We can easily obtain that $|f(z)| \leq M'|z|^k$ on $\arg z = \theta$. From this and the fact $E_1 \cup E_2 \cup H_1 \cup H_2$ has linear measure zero, we obtain by Lemma 2.4 that $f(z)$ is a polynomial with $\deg f \leq k$, which contradicts our assumption. Therefore $\sigma(f) = +\infty$.

Subcase 1.2: $\delta_1 < \delta_2$**.** Using the same reasoning as in subcase 1.1 , we can also obtain that $f(z)$ is a polynomial, which contradicts our assumption. Therefore $\sigma(f) = +\infty$.

Case 2. Suppose that $\theta_1 = \theta_2$. For any given $\theta \in [0, 2\pi)/E_1 \cup E_2 \cup H_1$, where E_1, E_2 and H_1 are defined above, we have

$$\delta(P_1,\theta) > 0 \quad \text{or} \quad \delta(P_1,\theta) < 0. \tag{3.17}$$

Subcase 2.1: $\delta(P_1,\theta) > 0$**.**

(i) If $|a_{n,1}| < (1-\beta)|a_{n,2}|$, for any given ε $(0 < \varepsilon < \frac{(1-\beta)|a_{n,2}|-|a_{n,1}|}{2[(1+\beta)|a_{n,2}|+|a_{n,1}|]})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$|A_1(z)e^{P_1(z)}| \leq \exp\{(1+\varepsilon)\delta(P_1,\theta)r^n\}, \tag{3.18}$$

$$|A_2(z)e^{P_2(z)}| \geq \exp\{(1-\varepsilon)\delta(P_2,\theta)r^n\}, \tag{3.19}$$

$$|B_j(z)e^{Q_j(z)}| \leq \exp\{(1+\varepsilon)\alpha\delta(P_1,\theta)r^n\} \ (j \in I), \tag{3.20}$$

$$|B_j(z)e^{Q_j(z)}| \leq \exp\{(1+\varepsilon)\beta\delta(P_2,\theta)r^n\} \ (j \in J). \tag{3.21}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, where $r_m \to +\infty$ such that $f^{(s)}(z_m) \to \infty$ and (3.6) holds.

By (1.3), (3.1), (3.6) and (3.18)-(3.21), for the above z_m, we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(P_2,\theta)r_m^n\} \\ &\le M_3 r_m^{d_3}\exp\{(1+\varepsilon)\delta(P_1,\theta)r_m^n\}\exp\{(1+\varepsilon)\beta\delta(P_2,\theta)r_m^n\}, \end{aligned} \tag{3.22}$$

where M_3, d_3 (>0) are constants. By (3.22), we have

$$\exp\{\gamma_1 r_m^n\} \le M_3 r_m^{d_3}, \tag{3.23}$$

where

$$\gamma_1 = (1-\varepsilon)\delta(P_2,\theta) - (1+\varepsilon)\delta(P_1,\theta) - (1+\varepsilon)\beta\delta(P_2,\theta).$$

Since

$$0 < \varepsilon < \frac{(1-\beta)|a_{n,2}| - |a_{n,1}|}{2[(1+\beta)|a_{n,2}| + |a_{n,1}|]},$$

$\theta_1 = \theta_2$ and $\cos(\theta_1 + n\theta) > 0$, we have

$$\begin{aligned} \gamma_1 &= \{(1-\beta)|a_{n,2}| - |a_{n,1}| - \varepsilon[(1+\beta)|a_{n,2}| + |a_{n,1}|]\}\cos(\theta_1+n\theta) \\ &> \frac{(1-\beta)|a_{n,2}| - |a_{n,1}|}{2}\cos(\theta_1+n\theta) > 0. \end{aligned}$$

Hence (3.23) is a contradiction. Hence $|f^{(s)}(z)| \le M$ on $\arg z = \theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z = \theta$. By Lemma 2.4, (3.8) and the fact that $E_1 \cup E_2 \cup H_1$ has linear measure zero, we obtain that $f(z)$ is a polynomial with $\deg f \le s$, which contradicts our assumption. Therefore $\sigma(f) = +\infty$.

(ii) If $|a_{n,2}| < (1-\alpha)|a_{n,1}|$, for any given ε ($0 < \varepsilon < \frac{(1-\alpha)|a_{n,1}| - |a_{n,2}|}{2[(1+\alpha)|a_{n,1}| + |a_{n,2}|]}$) and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$|A_1(z)e^{P_1(z)}| \ge \exp\{(1-\varepsilon)\delta(P_1,\theta)r^n\}, \tag{3.24}$$

$$|A_2(z)e^{P_2(z)}| \le \exp\{(1+\varepsilon)\delta(P_2,\theta)r^n\}. \tag{3.25}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ ($m = 1, 2, \dots$), where $r_m \to +\infty$ such that $f^{(s)}(z_m) \to \infty$ and (3.6) holds.

By (1.3), (3.1), (3.6), (3.20), (3.21), (3.24) and (3.25), for the above z_m, we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(P_1,\theta)r_m^n\} \\ &\le M_4 r_m^{d_4}\exp\{(1+\varepsilon)\alpha\delta(P_1,\theta)r_m^n\}\exp\{(1+\varepsilon)\delta(P_2,\theta)r_m^n\}, \end{aligned} \tag{3.26}$$

where M_4, d_4 (>0) are constants. By (3.26), we have

$$\exp\{\gamma_2 r_m^n\} \le M_4 r_m^{d_4}, \tag{3.27}$$

where

$$\gamma_2 = (1-\varepsilon)\delta(P_1,\theta) - (1+\varepsilon)\delta(P_2,\theta) - (1+\varepsilon)\alpha\delta(P_1,\theta) > 0.$$

Since (3.27) is a contradiction, $|f^{(s)}(z)| \le M$ on $\arg z = \theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z = \theta$. Using similar arguments as above, we conclude that $\sigma(f) = +\infty$.

Subcase 2.2: $\delta(P_1,\theta) < 0$**.** Using the same reasoning as in subcase 1.1(c), we can also conclude that $\sigma(f) = +\infty$.

Second step. Now we prove that $\sigma_2(f) = n$. By Lemma 2.5, there exists a constant $B > 0$ and a set $F_1 \subset (1, +\infty)$ having finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup F_1$, we have

$$|\frac{f^{(j)}(z)}{f^{(i)}(z)}| \leq Br[T(2r, f)]^{j+1}(0 \leq i < j \leq k). \tag{3.28}$$

For each sufficiently large $|z| = r$, we take a point $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r, f)$. By Lemma 2.6, there exists a constant δ_r (> 0) and a set F_2 of finite logarithmic measure such that for all all z satisfying $|z| = r \notin F_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$|\frac{f(z)}{f^{(d)}(z)}| \leq r^{2d} \quad (d = s, k). \tag{3.29}$$

Case 1. Suppose that $\theta_1 \neq \theta_2$. For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1 \cup H_2$, we have (3.2), where E_2, H_1 and H_2 are defined above. Set $\delta_1 = \delta(P_1, \theta)$ and $\delta_2 = \delta(P_2, \theta)$.

Subcase 1.1: $\delta_1 > \delta_2$**.** Here we also divide our proof in three subcases:

(a): $\delta_1 > \delta_2 > 0$**.** Thus for any given ε $(0 < \varepsilon < \frac{\delta_1 - \delta_3}{2(\delta_1 + \delta_3)})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.3)–(3.5), where δ_3 is defined above. By (1.3), (3.3)–(3.5), (3.28) and (3.29), for all z satisfying $|z| = r \notin [0, 1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1 \cup H_2$, we obtain

$$\exp\{(1 - \varepsilon)\delta_1 r^n\} \leq M_5 r^{2s+1} \exp\{(1 + \varepsilon)\delta_3 r^n\}[T(2r, f)]^{k+1}, \tag{3.30}$$

where $M_5(> 0)$ is a constant. Hence by using Lemma 2.7 and (3.30), we obtain $\sigma_2(f) \geq n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

(b): $\delta_1 > 0 > \delta_2$**.** Thus for any given ε $(0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.3) and (3.9)–(3.11). By (1.3), (3.3), (3.9)—(3.11), (3.28) and (3.29), for all z satisfying $|z| = r \notin [0, 1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1 \cup H_2$, we obtain

$$\exp\{(1 - \varepsilon)\delta_1 r^n\} \leq M_6 r^{2s+1} \exp\{(1 + \varepsilon)\alpha\delta_1 r^n\}[T(2r, f)]^{k+1}, \tag{3.31}$$

where $M_6(> 0)$ is a constant. Hence by using Lemma 2.7 and (3.31), we obtain $\sigma_2(f) \geq n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

(c): $0 > \delta_1 > \delta_2$**.** Set $\gamma = \min\{\alpha_j, \beta_j : j \neq s\}$. Thus for any given ε $(0 < 2\varepsilon < 1)$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$|A_l(z)e^{P_l(z)}| \leq \exp\{(1 - \varepsilon)\gamma\delta_1 r^n\} \quad (l = 1, 2), \tag{3.32}$$

$$|B_j(z)e^{Q_j(z)}| \leq \exp\{(1 - \varepsilon)\gamma\delta_1 r^n\} \ (j \neq s). \tag{3.33}$$

By (1.3), (3.28), (3.29), (3.32) and (3.33), for all z satisfying $|z| = r \notin [0, 1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1 \cup H_2$, we obtain

$$1 \leq M_7 r^{2k+1} \exp\{(1 - \varepsilon)\gamma\delta_1 r^n\}[T(2r, f)]^{k+1}, \tag{3.34}$$

where M_7 (> 0) is a constant. Hence by using Lemma 2.7 and (3.34), we obtain $\sigma_2(f) \geq n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

Subcase 1.2: $\delta_1 < \delta_2$**.** Using the same reasoning as in subcase 1.1 of the second step, we can also obtain that $\sigma_2(f) = n$.

Case 2. Suppose that $\theta_1 = \theta_2$. For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1$, where E_2 and H_1 are defined above, we have (3.17).

Subcase 2.1: $\delta(P_1, \theta) > 0$**.**

(i) If $|a_{n,1}| < (1-\beta)|a_{n,2}|$, for any given ε,

$$0 < \varepsilon < \frac{(1-\beta)|a_{n,2}| - |a_{n,1}|}{2[(1+\beta)|a_{n,2}| + |a_{n,1}|]},$$

and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.18) -(3.21). By (1.3), (3.18)–(3.21), (3.28) and (3.29), for all z satisfying $|z| = r \notin [0,1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1$, we obtain

$$\exp\{\gamma_1 r^n\} \le M_8 r^{2s+1}[T(2r, f)]^{k+1}, \tag{3.35}$$

where M_8 (> 0) is a constant and γ_1 is defined above. Hence by using Lemma 2.7 and (3.35), we obtain $\sigma_2(f) \ge n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

(ii) If $|a_{n,2}| < (1-\alpha)|a_{n,1}|$, for any given ε $(0 < \varepsilon < \frac{(1-\alpha)|a_{n,1}|-|a_{n,2}|}{2[(1+\alpha)|a_{n,1}|+|a_{n,2}|]})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.20) , (3.21), (3.24) and (3.25). By (1.3), (3.20), (3.21), (3.24), (3.25), (3.28) and (3.29) for all z satisfying $|z| = r \notin [0,1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r]\backslash E_2 \cup H_1$, we obtain

$$\exp\{\gamma_2 r^n\} \le M_9 r^{2s+1}[T(2r, f)]^{k+1}, \tag{3.36}$$

where M_9 (> 0) is a constant and γ_2 is defined above. Hence by using Lemma 2.7 and (3.36), we obtain $\sigma_2(f) \ge n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

Subcase 2.2: $\delta(P_1, \theta) < 0$**.** Using the same reasoning as in subcase 1.1(c) of the second step, we can also obtain that $\sigma_2(f) = n$.

4. Proof of Theorem 1.3

Assume f is a transcendental solution of (1.3).

First step. We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \rho < +\infty$. By Lemma 2.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi)\backslash E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_1$, we have (3.1). Set $a_{n,l} = |a_{n,l}|e^{i\theta_l}$, $\theta_l \in [0, 2\pi)$ $(l = 1, 2)$.

Assume that $a_{n,1}$ is a real number such that $(1-\alpha)a_{n,1} < b$, which is $\theta_1 = \pi$. By Lemmas 2.2, for any given $\varepsilon > 0$, there exist a set $E_2 \subset [0, 2\pi)$ that has linear measure zero, such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi)/E_2 \cup H_1$ and r is sufficiently large, then $A_l(z)e^{P_l(z)}$ $(l = 1, 2)$ and $B_j(z)e^{Q_j(z)}$ $(j \ne s)$ satisfy (2.2) or (2.3), where H_1 is defined as in the proof of Theorem 1.2.

Case 1. Suppose that $\theta_1 \ne \theta_2$. For any given $\theta \in [0, 2\pi]/E_1 \cup E_2 \cup H_1 \cup H_2$, we have (3.2), where H_2 is defined as in the proof of Theorem 1.2. Since $(1-\alpha)a_{n,s} < b$, it follows that $|b_{n,j}| < |a_{n,1}|$ $(j \in J)$. Set $\delta_1 = \delta(P_1, \theta)$ and $\delta_2 = \delta(P_2, \theta)$.

Subcase 1.1: $\delta_1 > \delta_2$**.** If (a): $\delta_1 > \delta_2 > 0$ or (b): $\delta_1 > 0 > \delta_2$, it follows that $0 < \delta(Q_j, \theta) < \delta_1$ $(j \in J)$. Hence by using the same reasoning as in subcase 1.1(a) of the first step in the proof of Theorem 1.2, we can also obtain that $\sigma(f) = +\infty$. If (c): $0 > \delta_1 > \delta_2$, by using similar reasoning as in subcase 1.1(c) of the first step in the proof of Theorem 1.2, we can also obtain $\sigma(f) = +\infty$.

Subcase 1.2: $\delta_2 > \delta_1$**.** Here we also divide our proof in three subcases:

(a): $\delta_2 > \delta_1 > 0$. Thus for any given ε $(0 < \varepsilon < \frac{\delta_2-\delta_1}{2(\delta_2+\delta_1)})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have

$$|A_2(z)e^{P_2(z)}| \geq \exp\{(1-\varepsilon)\delta_2 r^n\}, \tag{4.1}$$

$$|A_1(z)e^{P_1(z)}| \leq \exp\{(1+\varepsilon)\delta_1 r^n\}, \tag{4.2}$$

$$|B_j(z)e^{Q_j(z)}| \leq \exp\{(1+\varepsilon)\delta_1 r^n\} \ (j \neq s). \tag{4.3}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, where $r_m \to +\infty$ such that $f^{(s)}(z_m) \to \infty$ and (3.6) holds.

By (1.3), (3.1), (3.6) and (4.1)–(4.3), for the above z_m, we obtain

$$\exp\{(1-\varepsilon)\delta_2 r_m^n\} \leq M_1 r_m^{d_1} \exp\{(1+\varepsilon)\delta_1 r_m^n\}, \tag{4.4}$$

where M_1, $d_1(>0)$ are constants. This is a contradiction. Hence $|f^{(s)}(z)| \leq M$ on $\arg z = \theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z = \theta$. By Lemma 2.4, (3.8) and the fact that $E_1 \cup E_2 \cup H_1 \cup H_2$ has linear measure zero, we obtain that $f(z)$ is a polynomial with $\deg f \leq s$ which contradicts our assumption. Therefore $\sigma(f) = +\infty$.

(b): $\delta_2 > 0 > \delta_1$. Thus for any given ε $(0 < 2\varepsilon < 1)$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (4.1),

$$|A_1(z)e^{P_1(z)}| \leq \exp\{(1-\varepsilon)\delta_1 r^n\} < 1, \tag{4.5}$$

$$|B_j(z)e^{Q_j(z)}| \leq \exp\{(1-\varepsilon)\delta(Q_j,\theta) r^n\} < 1 \ (j \neq s). \tag{4.6}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, where $r_m \to +\infty$ such that $f^{(s)}(z_m) \to \infty$ and (3.6) holds.

By (1.3), (3.1), (3.6), (4.1), (4.5) and (4.6), for the above z_m, we obtain

$$\exp\{(1-\varepsilon)\delta_2 r_m^n\} \leq M_2 r_m^{d_2}, \tag{4.7}$$

where M_2, $d_2(>0)$ are constants. This is a contradiction. Hence $|f^{(s)}(z)| \leq M$ on $\arg z = \theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z = \theta$. Using similar arguments as above, we conclude that $\sigma(f) = +\infty$.

(c): $0 > \delta_2 > \delta_1$. Using similar reasoning as in subcase 1.1.(c) of the first step in the proof of Theorem 1.2, we can also obtain that $\sigma(f) = +\infty$.

Case 2. Assume that $\theta_1 = \theta_2$. Then $\theta_1 = \theta_2 = \pi$. For any given $\theta \in [0, 2\pi)/E_1 \cup E_2 \cup H_1$, we have (3.17).

Subcase 2.1: $\delta(P_1, \theta) > 0$. Since $|a_{n,1}| \leq |a_{n,2}|, a_{n,1} \neq a_{n,2}$ and $\theta_1 = \theta_2$, it follows that $|a_{n,1}| < |a_{n,2}|$. Thus for any given ε $(0 < \varepsilon < \frac{|a_{n,2}|-|a_{n,1}|}{2(|a_{n,2}|+|a_{n,1}|)})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.18)–(3.20) and

$$\begin{aligned} |B_j(z)e^{Q_j(z)}| &\leq \exp\{(1+\varepsilon)\delta(Q_j,\theta) r^n\} \\ &\leq \exp\{(1+\varepsilon) b r^n \cos(n\theta)\} (j \in J). \end{aligned} \tag{4.8}$$

By (3.18) and (3.19), we obtain

$$|A_1(z)e^{P_1(z)} + A_2(z)e^{P_2(z)}| \geq \exp\{(1+\varepsilon)\delta(P_1,\theta) r^n\}[\exp\{\gamma_1 r^n\} - 1], \tag{4.9}$$

where
$$\gamma_1 = (1-\varepsilon)\delta(P_2,\theta) - (1+\varepsilon)\delta(P_1,\theta) > 0.$$
Hence from (4.9), we obtain
$$|A_1(z)e^{P_1(z)} + A_2(z)e^{P_2(z)}| \geq (1-o(1))\exp\{(1+\varepsilon)\delta(P_1,\theta)r^n\}\exp\{\gamma_1 r^n\}. \quad (4.10)$$
Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, where $r_m \to +\infty$ such that $f^{(s)}(z_m) \to \infty$ and (3.6) holds.

By (1.3), (3.1), (3.6), (3.20), (4.8) and (4.10), we obtain
$$\begin{aligned}&(1-o(1))\exp\{(1+\varepsilon)\delta(P_1,\theta)r_m^n\}\exp\{\gamma_1 r_m^n\}\\ &\leq M_3 r_m^{d_3}\exp\{(1+\varepsilon)\alpha\delta(P_1,\theta)r_m^n\}\exp\{(1+\varepsilon)br_m^n\cos(n\theta)\},\end{aligned} \quad (4.11)$$
where M_3, d_3 (>0) are constants. Hence
$$(1-o(1))\exp\{\gamma_2 r_m^n\} \leq M_3 r_m^{d_3}, \quad (4.12)$$
where
$$\gamma_2 = (1+\varepsilon)[(1-\alpha)\delta(P_1,\theta) - b\cos(n\theta)] + \gamma_1.$$
Since $\gamma_1 > 0$, $\cos(n\theta) < 0$, $\delta(P_1,\theta) = -|a_{n,1}|\cos(n\theta)$ and $(1-\alpha)a_{n,1} < b < 0$, we have
$$\gamma_2 = -(1+\varepsilon)[(1-\alpha)|a_{n,1}| + b]\cos(n\theta) + \gamma_1 > \gamma_1 > 0.$$
Hence (4.11) is a contradiction. Hence $|f^{(s)}(z)| \leq M$ on $\arg z = \theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z = \theta$. By Lemma 2.4, (3.8) and the fact that $E_1 \cup E_2 \cup H_1$ has linear measure zero, we obtain that $f(z)$ is a polynomial with $\deg f \leq s$ which contradicts our assumption. Therefore $\sigma(f) = +\infty$.

Subcase 2.2: $\delta(P_1,\theta) < 0$**.** Using similar reasoning as in subcase 1.1(c) of the first step in the proof of Theorem 1.2, we can also obtain that $\sigma(f) = +\infty$.

Second step. Now we prove that $\sigma_2(f) = n$. By Lemma 2.5, there exist a constant $B > 0$ and a set $F_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup F_1$, we have (3.28). For each sufficiently large $|z| = r$, we take a point $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r,f)$. By Lemma 2.6, there exists a constant δ_r (>0) and a set F_2 of finite logarithmic measure such that for all all z satisfying $|z| = r \notin F_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.29).

Case 1 Suppose that $\theta_1 \neq \theta_2$. For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_1 \cup H_2$, we have (3.2), where E_2, H_1 and H_2 are defined above. Set $\delta_1 = \delta(P_1,\theta)$ and $\delta_2 = \delta(P_2,\theta)$.

Subcase 1.1: $\delta_1 > \delta_2$**.** Using the same reasoning as in subcase 1.1(a) of the second step in the proof of Theorem 1.2, we can also obtain that $\sigma_2(f) = n$.

Subcase 1.2: $\delta_2 > \delta_1$**.** Here we also divide our proof in three subcases:

(a): $\delta_2 > \delta_1 > 0$**.** Thus for any given ε $(0 < \varepsilon < \frac{\delta_2 - \delta_1}{2(\delta_2 + \delta_1)})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (4.1)–(4.3). By (1.3), (3.28), (3.29) and (4.1)–(4.3), for all z satisfying $|z| = r \notin [0,1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_1 \cup H_2$, we obtain
$$\exp\{(1-\varepsilon)\delta_2 r^n\} \leq M_4 r^{2s+1}\exp\{(1+\varepsilon)\delta_1 r^n\}[T(2r,f)]^{k+1}, \quad (4.13)$$
where M_4 (>0) is a constant. Hence by using Lemma 2.7 and (4.13), we obtain $\sigma_2(f) \geq n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

(b): $\delta_2 > 0 > \delta_1$**.** Thus for any given ε $(0 < 2\varepsilon < 1)$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (4.1), (4.5) and (4.6). By (1.3), (3.28), (3.29), (4.1), (4.5) and (4.6), for all z satisfying $|z| = r \notin [0,1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_1 \cup H_2$, we obtain

$$\exp\{(1-\varepsilon)\delta_2 r^n\} \le M_5 r^{2s+1}[T(2r,f)]^{k+1}, \tag{4.14}$$

where $M_5(> 0)$ is a constant. Hence by using Lemma 2.7 and (4.14), we obtain $\sigma_2(f) \ge n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

(c): $0 > \delta_2 > \delta_1$**.** Using similar reasoning as in subcase 1.1(c) of the second step in the proof of Theorem 1.2, we can also obtain that $\sigma_2(f) = n$.

Case 2. Assume that $\theta_1 = \theta_2$. For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_1$, where E_2 and H_1 are defined above, we have (3.17).

Subcase 2.1: $\delta(P_1, \theta) > 0$**.** For any given ε,

$$0 < \varepsilon < \frac{|a_{n,2}| - |a_{n,1}|}{2(|a_{n,2}| + |a_{n,1}|)},$$

and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.20), (4.8) and (4.10). By (1.3), (3.20), (3.28), (3.29), (4.8) and (4.10), for all z satisfying $|z| = r \notin [0,1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_1$, we obtain

$$(1 - o(1)) \exp\{\gamma_2 r^n\} \le M_6 r^{2s+1}[T(2r,f)]^{k+1}, \tag{4.15}$$

where M_6 (> 0) is a constant. Hence by using Lemma 2.7 and (4.15), we obtain $\sigma_2(f) \ge n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

Subcase 2.2: $\delta(P_1, \theta) < 0$**.** Using similar reasoning as in subcase 1.1(c) of the second step in the proof of Theorem 1.2, we can also obtain that $\sigma_2(f) = n$.

5. Proof of Theorem 1.4

Assume f is a transcendental solution of equation (1.3).

First step. We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \rho < +\infty$. By Lemma 2.1, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\theta \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_1$, we have (3.1). Set $a_{n,l} = |a_{n,l}| e^{i\theta_l}$, $\theta_l \in [0, 2\pi)$ $(l = 1, 2)$.

Assume that $a_{n,1}$ and $a_{n,2}$ are real numbers such that $(1-\beta)a_{n,2} - b < a_{n,1} < 0$ or $(1-\alpha)a_{n,1} - b < a_{n,2} < 0$, which is $\theta_1 = \theta_2 = \pi$. By Lemma 2.2, for any given $\varepsilon > 0$, there exists a set $E_2 \subset [0, 2\pi)$ that has linear measure zero, such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi)/E_2 \cup H_3$ and r is sufficiently large, then $A_l(z)e^{P_l(z)}$ $(l = 1, 2)$ and $B_j(z)e^{Q_j(z)}$ $(j \neq s)$ satisfy (2.2) or (2.3), where $H_3 = \{\theta \in [0, 2\pi) : \cos(n\theta) = 0\}$.

For any given $\theta \in [0, 2\pi) \backslash E_1 \cup E_2 \cup H_3$, we have (3.17).

Case 1: $\delta(P_1, \theta) > 0$**.**

(i) If $(1-\beta)a_{n,2} - b < a_{n,1} < 0$, for any given ε $(0 < \varepsilon < \frac{(1-\beta)|a_{n,2}| - |a_{n,1}| + b}{2[(1+\beta)|a_{n,2}| + |a_{n,1}| - b]})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.18), (3.19), (4.8), and

$$|B_j(z)e^{Q_j(z)}| \le \exp\{(1+\varepsilon)\alpha\delta(P_1, \theta)r^n\} \exp\{(1+\varepsilon)\beta\delta(P_2, \theta)r^n\} \quad (j \in I). \tag{5.1}$$

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z = \theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \dots)$, where $r_m \to +\infty$ such that $f^{(s)}(z_m) \to \infty$

and (3.6) holds. By (1.3), (3.1), (3.6), (3.18), (3.19), (4.8) and (5.1), for the above z_m, we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(P_2,\theta)r_m^n\} \\ &\le M_1 r_m^{d_1}\exp\{(1+\varepsilon)[\delta(P_1,\theta)+\beta\delta(P_2,\theta)+b\cos(n\theta)]r_m^n\}, \end{aligned} \tag{5.2}$$

where M_1, d_1 (>0) are constants. By (5.2), we have

$$\exp\{\gamma_1 r_m^n\} \le M_1 r_m^{d_1}, \tag{5.3}$$

where

$$\gamma_1 = (1-\varepsilon)\delta(P_2,\theta)-(1+\varepsilon)[\delta(P_1,\theta)+\beta\delta(P_2,\theta)+b\cos(n\theta)].$$

From

$$0<\varepsilon<\frac{(1-\beta)|a_{n,2}|-|a_{n,1}|+b}{2[(1+\beta)|a_{n,2}|+|a_{n,1}|-b]}$$

and $\cos(n\theta)<0$, we obtain

$$\begin{aligned} \gamma_1 &= -\{(1-\beta)|a_{n,2}|-|a_{n,1}|+b-\varepsilon[(1+\beta)|a_{n,2}|+|a_{n,1}|-b]\}\cos(n\theta) \\ &> -\frac{[(1-\beta)|a_{n,2}|-|a_{n,1}|+b]}{2}\cos(n\theta)>0. \end{aligned}$$

Thus (5.3) is a contradiction. Hence $|f^{(s)}(z)|\le M$ on $\arg z=\theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z=\theta$. By Lemma 2.4, (3.8) and the fact that $E_1\cup E_2\cup H_1$ has linear measure zero, we obtain that $f(z)$ is a polynomial with $\deg f\le s$, which contradicts our assumption. Therefore $\sigma(f)=+\infty$.

(ii) If $(1-\alpha)a_{n,1}-b<a_{n,2}<0$, for any given ε,

$$0<\varepsilon<\frac{(1-\alpha)|a_{n,1}|-|a_{n,2}|+b}{2[(1+\alpha)|a_{n,1}|+|a_{n,2}|-b)]}),$$

and all z satisfying $\arg z=\theta$ and $|z|=r$ sufficiently large, we have (3.24) and (3.25).

Now we prove that $|f^{(s)}(z)|$ is bounded on the ray $\arg z=\theta$. If $|f^{(s)}(z)|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.3, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ ($m=1,2,\dots$), where $r_m\to+\infty$ such that $f^{(s)}(z_m)\to\infty$ and (3.6) holds.

By (1.3), (3.1), (3.6), (3.24), (3.25), (4.8) and (5.1), for the above z_m, we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(P_1,\theta)r_m^n\} \\ &\le M_2 r_m^{d_2}\exp\{(1+\varepsilon)[\delta(P_2,\theta)+\alpha\delta(P_1,\theta)+b\cos(n\theta)]r_m^n\}, \end{aligned} \tag{5.4}$$

where M_2, d_2 (>0) are constants. By (5.4), we have

$$\exp\{\gamma_2 r_m^n\} \le M_2 r_m^{d_2}, \tag{5.5}$$

where

$$\gamma_2 = (1-\varepsilon)\delta(P_1,\theta)-(1+\varepsilon)[\delta(P_2,\theta)+\alpha\delta(P_1,\theta)+b\cos(n\theta)]>0.$$

Thus (5.5) is a contradiction. Hence $|f^{(s)}(z)|\le M$ on $\arg z=\theta$, where M (>0) is a constant. We can easily obtain (3.8) on $\arg z=\theta$. Using similar arguments as above, we deduce that $\sigma(f)=+\infty$.

Case 2: $\delta(P_1,\theta)<0$. Using similar reasoning as in subcase 1.1(c) of the first step in the proof of Theorem 1.2, we can also obtain that $\sigma(f)=+\infty$.

Second step. We prove that $\sigma_2(f) = n$. By Lemma 2.5, there exist a constant $B > 0$ and a set $F_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have (3.28). For each sufficiently large $|z| = r$, we take a point $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r, f)$. By Lemma 2.6, there exists a constant δ_r (> 0) and a set F_2 of finite logarithmic measure such that for all all z satisfying $|z| = r \notin F_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have (3.29). For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_3$, we have (3.17), where E_2 and H_3 are defined above.

Case 1: $\delta(P_1, \theta) > 0$.

(i) If $(1-\beta)a_{n,2} - b < a_{n,1} < 0$, for any given ε $(0 < \varepsilon < \frac{(1-\beta)|a_{n,2}|-|a_{n,1}|+b}{2[(1+\beta)|a_{n,2}|+|a_{n,1}|-b]})$ and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.18), (3.19), (4.8) and (5.1). By (1.3), (3.18), (3.19), (3.28), (3.29), (4.8) and (5.1), for all z satisfying $|z| = r \notin [0, 1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_3$, we obtain

$$\exp\{\gamma_1 r^n\} \leq M_3 r^{2s+1} [T(2r, f)]^{k+1}, \tag{5.6}$$

where M_3 (> 0) is a constant and γ_1 is defined above. Hence by using Lemma 2.7 and (5.6), we obtain $\sigma_2(f) \geq n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

(ii) If $(1-\alpha)a_{n,1} - b < a_{n,2} < 0$, for any given ε,

$$(0 < \varepsilon < \frac{(1-\alpha)|a_{n,1}| - |a_{n,2}| + b}{2[(1+\alpha)|a_{n,1}| + |a_{n,2}| - b)]}),$$

and all z satisfying $\arg z = \theta$ and $|z| = r$ sufficiently large, we have (3.23) and (3.24). By (1.3), (3.24), (3.25), (3.28), (3.29) and (4.8), for all z satisfying $|z| = r \notin [0, 1] \cup F_1 \cup F_2$ and $\arg z \in [\theta_r - \delta_r, \theta_r + \delta_r] \backslash E_2 \cup H_3$, we obtain

$$\exp\{\gamma_2 r^n\} \leq M_4 r^{2s+1} [T(2r, f)]^{k+1}, \tag{5.7}$$

where M_4 (> 0) is a constant and γ_2 is defined above. Hence by using Lemma 2.7 and (5.7), we obtain $\sigma_2(f) \geq n$. From this and Lemma 2.8, we have $\sigma_2(f) = n$.

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Karima Hamani

Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem, Algeria

E-mail address: hamanikarima@yahoo.fr, karima.hamani@univ-mosta.dz