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LINEAR AND LOGISTIC MODELS WITH TIME DEPENDENT COEFFICIENTS

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Abstract. We sutdy the effects of some properties of the carrying capacity on the solution of the linear and logistic differential equations. We present results concerning the behaviour and the asymptotic behaviour of their solutions. Special attention is paid when the carrying capacity is an increasing or a decreasing positive function. For more general carrying capacity, we obtain bounds for the corresponding solution by constructing appropriate subsolution and supersolution. We also present a decomposition of the solution of the linear, and logistic, differential equation as a product of the carrying capacity and the solution to the corresponding differential equation with a constant carrying capacity.

1. Introduction

In this article we shall study the solutions of the linear and logistic differential equations defined respectively by

$$\dot{x}(t) = \alpha(t) - \beta(t)x(t), \quad t \geq t_0, \tag{1.1}$$

and

$$\dot{x}(t) = \beta(t)x(t) - \alpha(t)x^2(t), \quad t \geq t_0, \tag{1.2}$$

where $\alpha(t)$ and $\beta(t)$ are strictly positive and continuous functions on $[t_0, +\infty)$, with $t_0 \in \mathbb{R}$. More precisely, we will consider the effect of $\alpha(t)$, $\beta(t)$, and the ratio $k(t) = \alpha(t)/\beta(t)$ on the behavior of the solutions of these models for any positive initial value $x(t_0) > 0$.

Considering that the mapping $y : x \mapsto y = x^{-1}$ transforms (1.2) into the linear differential equation

$$\dot{y}(t) = \alpha(t) - \beta(t)y(t), \quad t \geq t_0,$$

we will focus our study on (1.1). Indeed, all results obtained for (1.1) could then be directly applied to (1.2).

Considering the preceding mapping used to rely solutions of (1.2) to solutions of (1.1), we will call the ratio $k(t) = \alpha(t)/\beta(t)$ the carrying capacity for the linear model (1.1) despite the fact that this expression, for the logistic equation (1.2), refer to $\beta(t)/\alpha(t) = 1/k(t)$.

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With a constant carrying capacity $k(t) = k$, these models, and many extensions of them, have extensively been used to describe and improve the possible relationship between independent and dependent variables in terms of mathematical equations. It happended in many fields of applied sciences, like ecology, sociology, medicine, and other domains [7, 8, 10, 14, 16, 17, 23]. However, according to Coleman [2, 3, 4], and later to Meyer [11, 12], changes in the environment affect the carrying capacity. Hence, modeling phenomena with unchanging carrying capacity is often unrealistic. Several authors have reformulated these standards models with constant carrying capacity to accommodate phenomena with varying [19, 20, 21], logistically varying [11, 12], increasing [5, 6, 13] or sinusoidally varying [4, 3, 18] carrying capacity. In [1], the author has given some examples on real situations in which the carrying capacity k changes with time continuously. Many other researchers have been interested in the problem of the existence and uniqueness solution of the solution of (1.2) with bounded time dependent carrying capacity [9, 15]. In [22], the author argued that it is difficult to make precise statement about the asymptotic behaviour of the solution to a non-autonomous differential equation when the coefficients α and β are time dependent functions. In [9] the authors have proved that a monotone bounded carrying capacity is an attractor forward in time of all positive solution of (1.2) in the sense that the limit at infinity of the difference between any solution to the differential equation and the carrying capacity vanishes.

Despite the intensive examples of real situations involving growth phenomena with unbounded time dependent carrying capacity (see [1] for example), these situations has surprisingly received little attention in the literature compared to the massive literature devoted to the problems with bounded coefficients.

The main purpose of this paper is to address this knowledge gap through a qualitative study. We study in a thorough way the effect of an unbounded carrying capacity on the behaviour and the asymptotic behaviour of the solution of the linear and logistic differential equations. We shall pay particular attention to the cases when the carrying capacity $k(t)$ is an increasing or decreasing positive function. Moreover, the asymptotic behaviour of the solutions to these differential equations is not well described when the carrying capacity $k(t)$ is time dependent and unbounded. On this basis comes the second aim of this paper which consists of reformulating the solutions $x(t)$ of (1.1) and (1.2) as a product of a simple function $\tilde{z}(t)$ and a carrying capacity $\tilde{k}(t)$ such that $\lim_{t\to+\infty} \tilde{z}(t) = 1$, and $\lim_{t\to+\infty}(x(t) - \tilde{k}(t)) = 0$.

The present paper is organized as follows. In Section 2, we start by giving some properties about the solution of the linear differential equation. Then, we present some results on the behaviour and the asymptotic behaviour of its solution when the time dependent coefficients are not necessary bounded. We also show that when the carrying capacity is an increasing and unbounded function, the limit at infinity of the difference between the solution of the linear differential equation and the carrying capacity is not always equal to zero. In Section 3, we provide monotonic bounds for the solution of (1.1) when $k(t)$ is neither an increasing nor a decreasing function. Section 4 addresses the problem of decomposing the solution into a product of a carrying capacity and a simple analytic function. Finally, we present a conclusion.

2. The linear differential equation

Let $I = [t_0, +\infty)$ be an interval such that $t_0 \in \mathbb{R}$ and consider the linear differential equation

$$\dot{x}(t) = \alpha(t) - \beta(t)x(t) = \beta(t)(k(t) - x(t)) \quad t \geq t_0, \tag{2.1}$$

where $\alpha(t), \beta(t) : I \to (0, +\infty)$ are two continuous functions, and $k(t) = \alpha(t)/\beta(t)$. Subject to the initial condition $x(t_0) = x_0$, (2.1) has the unique solution given by

$$x(t) = e^{-\int_{t_0}^{t} \beta(u)du}\Big(x_0 + \int_{t_0}^{t} \alpha(\tau) e^{\int_{t_0}^{\tau} \beta(u)du} d\tau\Big). \tag{2.2}$$

Under the assumptions $k(t_0) = k_0$, and

$$\begin{aligned} k \in AC^1_{\rm loc}(I) = \Big\{ & k \in C(I) : \dot{k} \in L^1_{\rm loc}(I),\ k(t_2) = k(t_1) + \int_{t_1}^{t_2} \dot{k}(\tau) d\tau, \\ & \text{for all } t_1, t_2 \in I \Big\}, \end{aligned} \tag{2.3}$$

Equation (2.2) takes the form

$$x(t) = k(t) + \Big((x_0 - k_0) e^{-\int_{t_0}^{t} \beta(u)du} - e^{-\int_{t_0}^{t} \beta(u)du} \int_{t_0}^{t} \dot{k}(u) e^{\int_{t_0}^{u} \beta(\tau)d\tau} du\Big). \tag{2.4}$$

In the case where $\alpha(t)/\beta(t) = k$ does not depend on time, (2.4) reduces to

$$x(t) = k + (x_0 - k) e^{-\int_{t_0}^{t} \beta(u)du}. \tag{2.5}$$

Regarding x_0 and k we have the following situations:

- if $x_0 = k$, from (2.5) we have $x(t) = k$ for all $t \in I$;
- if $x_0 \neq k$, we have

$$\lim_{t \to +\infty} x(t) = k + (x_0 - k) e^{-\int_{t_0}^{+\infty} \beta(u)du} = x_\infty,$$

  and
  * if $x_0 < k$, then $x(t) < x_\infty$ for all $t \in I$, and the solution $x(t)$ grows up to x_∞,
  * if $x_0 > k$, then $x(t) > x_\infty$ for all $t \in I$, and the solution $x(t)$ decreases to x_∞.

When $\alpha(t)/\beta(t) = k(t)$ depends on time, (2.1) has no constant solution and the solution (2.2) may crosses $k(t)$. It happens when $\dot{x}(t) = 0$. For $t_* \in I$, let us consider the closed interval

$$J_{t_*} = \{t \in I : t \geq t_*, \text{ and } x(\tau) = x(t_*) \text{ for all } t_* \leq \tau \leq t\} = [t_*, t_{**}],$$

where $t_{**} = \sup_{\tau \in I} J_{t_*}$. Moreover, if $t_* < t_{**}$, then $\dot{x}(t) = 0$ for all $t \in J_{t_*}$ and consequently, from (2.1) and (2.4) we have

$$x(t) = k(t), \quad \text{and} \quad \dot{k}(t) = 0 \text{ for all } t \in J_{t_*}. \tag{2.6}$$

2.1. **Increasing case.** In this section, we suppose that $k(t)$ is a non-decreasing function

$$k(t_1) \le k(t_2) \quad \text{for all } t_1, t_2 \in I \text{ such that } t_1 < t_2. \tag{2.7}$$

We have the following result.

Lemma 2.1. *Let $k(t) = \alpha(t)/\beta(t) : I \to \mathbb{R}$ be defined such that $\beta(t) > 0$ for all t. Suppose that k satisfies the assumptions (2.3) and (2.7), and let $x(t)$ be the solution of (2.1) passing through the point (t_0, x_0). If for some $s \in I$ we have $x(s) = k(s)$, then $x(t) \le k(t)$ for all $t \ge s$. More precisely,*

- *$x(t) = k(t)$, and $\dot{x}(t) = 0$, for all $t \in J_s$, and*
- *$x(t) < k(t)$, and $\dot{x}(t) > 0$, for all $t > t_s = \sup J_s$.*

Proof. Let $x(t)$ be the solution of (2.1) passing through the point (t_0, x_0), and let $s \in I$ such that $x(s) = k(s)$. From (2.6), we have $x(t) = k(t)$ for all $t \in J_s$. Let $t_s = \sup J_s \in J_s$, it follows that $x(t_s) = k(t_s) = k(s)$ and $k(t) > k(t_s)$ for all $t > t_s$. Thus, by considering the solution passing through the point $(t_s, x(t_s))$, we have

$$\begin{aligned} x(t) &= k(t) + \Big((x(t_s) - k(t_s)) e^{-\int_{t_s}^t \beta(u) du} - e^{-\int_{t_s}^t \beta(u) du} \int_{t_s}^t \dot{k}(u) e^{\int_{t_s}^u \beta(\tau) d\tau} du \Big) \\ &= k(t) - \int_{t_s}^t \dot{k}(u) e^{\int_t^u \beta(\tau) d\tau} du. \end{aligned}$$

As $k(t) > k(t_s)$ for all $t > t_s$, it follows that $k(t_s) < x(t) < k(t)$ for all $t > t_s$. $\square$

The following result characterizes the behaviour of any solution of (2.1) in the case where $k(t)$ is non decreasing.

Theorem 2.2. *Let $k(t)$ and $x(t)$ be defined as in Lemma 2.1. Also let $k_\infty = \lim_{t \to +\infty} k(t)$. Then*

(a) *if $x_0 < k_0$, then $x(t) < k(t)$, and $\dot{x}(t) > 0$, for all $t \in I$;*
(b) *if $x_0 = k_0$, then $x(t) \le k(t)$, and $\dot{x}(t) \ge 0$, for all $t \in I$;*
(c) *if $x_0 > k_0$, we have two cases to consider*
   (i) *if $x_0 \le k_\infty$, then it will exists some $s > t_0$ where $s = \operatorname{argmin} x(t)$ such that $x(s) = k(s)$, and in this case $x(t)$ decreases if $t < s$ and increases if $t > s$;*
   (ii) *if $x_0 > k_\infty$, then either $x(t)$ has the same behaviour as in (i), or $x(t) > k(t)$, and $\dot{x}(t) < 0$, for all $t \in I$.*

Proof. The proofs of assertions (a) and (b) follow immediately from (2.4) and Lemma 2.1. Let us prove (c).

(i) If $k_0 < x_0 < k_\infty$, from (2.1), it follows that $\dot{x}(t_0) < 0$. By continuity, we also have $\dot{x}(t) < 0$ for all t provided that $x(t) > k(t)$ which is satisfied at least locally near t_0. We will prove that, there exists some $s > t_0$ such that $x(s) = k(s)$ and $x(t)$ increases for $t > s$ with $x(t) < k(t)$. Indeed, suppose that $x(t) > k(t)$ for all $t > t_0$. Thus, from (2.1), we have that $\dot{x}(t) < 0$ for all $t \ge t_0$ and hence, $k(t) < x(t) < x(t_0) = x_0$. By taking the limit at infinity we obtain $k_\infty \le x_0$ which contradicts our assumption on x_0. If $x_0 = k_\infty < \infty$, by arguing as in the proof above and by taking the limit at infinity we obtain $x_\infty = x_0$ which contradict the fact that $x(t) > k(t)$, i.e. $\dot{x}(t) < 0$, for all $t > t_0$.

(ii) if $x_0 > k_\infty > k_0$, i.e. $k_\infty < \infty$, we must have $x_\infty = \lim_{t\to+\infty} x(t) < \infty$. In addition, if we set $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = l$, from (2.4), we have

$$(x_0 - k_0)e^{-l} - (k_\infty - k_0) \le x_\infty - k_\infty \le (x_0 - k_\infty)e^{-l}. \tag{2.8}$$

If $l = +\infty$, from (2.8), it follows that $x_\infty \le k_\infty$ and hence, $x(t)$ has the same asymptotic behaviour as in (i). If $l < +\infty$, from (2.8), it follows that $x_\infty \ge k_\infty$ if $x_0 \ge k_0 + (k_\infty - k_0)e^l$, and the solution $x(t)$ is always decreasing. If $x_0 \le k_0 + (k_\infty - k_0)e^l$, the solution decreases or has the same asymptotic behaviour as in (i). □

The following result gives us some information about the asymptotic behaviour of the solution to the linear problem (2.1) when the time dependent coefficients $\alpha(t)$ and $\beta(t)$ are not necessary bounded.

Theorem 2.3. *Let $k(t)$ and $x(t)$ be defined as in Lemma 2.1 and set*

$$k_\infty = \lim_{t\to+\infty} k(t), \quad x_\infty = \lim_{t\to+\infty} x(t).$$

Then

- (a) *If $\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)du < +\infty$, then $x_\infty < +\infty$.*
- (b) *If $\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)du = +\infty$, then*
  - (I) *if $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = l < +\infty$, then $x_\infty = +\infty$;*
  - (II) *if $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = +\infty$, then $x_\infty = k_\infty$, and*
    - (i) *if $k_\infty < +\infty$, then $\lim_{t\to+\infty}(x(t) - k(t)) = 0$;*
    - (ii) *if $k_\infty = +\infty$, then $\lim_{t\to+\infty}(x(t) - k(t)) = -\lim_{t\to+\infty} \frac{\dot{k}(t)}{\beta(t)}$ if this limit exists.*

Proof. (a) If $\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)du < +\infty$, then from the fact that $\alpha(t) = k(t)\beta(t)$, and $k(t) > k_0 > 0$ for all $t \in I$, it follows that $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du < +\infty$. Hence,

$$\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du \le \lim_{t\to+\infty} e^{\int_{t_0}^t \beta(u)du} \int_{t_0}^t \alpha(u)du < +\infty.$$

Thus, from (2.2) it follows that $x_\infty < +\infty$.

(b) Let us suppose that $\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)du = +\infty$. As

$$\int_{t_0}^t \alpha(u)du \le \int_{t_0}^t \alpha(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du,$$

it follows that

$$\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du = +\infty. \tag{2.9}$$

(I) If $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = l < +\infty$, then from (2.2), and (2.9), it follows that $\lim_{t\to+\infty} x(t) = \infty$.

(II) If $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = +\infty$, then from (2.2) and (2.9), and by using the L'Hôpital's rule we obtain

$$x_\infty = \lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \frac{\int_{t_0}^t \alpha(\tau)e^{\int_{t_0}^\tau \beta(u)du}d\tau}{e^{\int_{t_0}^t \beta(u)du}}$$

$$
\begin{aligned}
&= \lim_{t\to+\infty} \frac{\alpha(t)e^{\int_{t_0}^t \beta(u)du}}{\beta(t)e^{\int_{t_0}^t \beta(u)du}}\\
&= \lim_{t\to+\infty} \frac{\alpha(t)}{\beta(t)}\\
&= \lim_{t\to+\infty} k(t) = k_\infty.
\end{aligned}
$$

(i) If $k_\infty < \infty$, we have that $\lim_{t\to+\infty}(x(t)-k(t)) = x_\infty - k_\infty = 0$.
(ii) If $\lim_{t\to+\infty} k(t) = \infty$, as $\lim_{t\to+\infty}\int_{t_0}^t \beta(u)du = +\infty$ it follows that

$$
\begin{aligned}
&\lim_{t\to+\infty}(x(t)-k(t))\\
&= \lim_{t\to+\infty}\Big((x_0 - \frac{\alpha_0}{\beta_0})e^{-\int_{t_0}^t \beta(u)du} - e^{-\int_{t_0}^t \beta(u)du}\int_{t_0}^t \dot{k}(u)e^{\int_{t_0}^u \beta(\tau)d\tau}\Big)\\
&= -\lim_{t\to+\infty}\frac{\int_{t_0}^t \dot{k}(u)e^{\int_{t_0}^u \beta(\tau)d\tau}}{e^{\int_{t_0}^t \beta(u)du}}.
\end{aligned}
$$

In addition, as $k(t)$ is an non decreasing function, we have for all $t > t_0$,

$$
\dot{k}(u) \le \dot{k}(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du \le \dot{k}(u)e^{\int_{t_0}^t \beta(\tau)d\tau},
$$

thus

$$
k(t)-k(t_0) \le \int_{t_0}^t \dot{k}(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du \le (k(t)-k(t_0))e^{\int_{t_0}^t \beta(\tau)d\tau},
$$

and $\lim_{t\to+\infty}\int_{t_0}^t \dot{k}(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du = +\infty$. Thus

$$
\lim_{t\to+\infty} x(t)-k(t) = -\lim_{t\to+\infty}\frac{\int_{t_0}^t \dot{k}(u)e^{\int_{t_0}^u \beta(\tau)d\tau}}{e^{\int_{t_0}^t \beta(u)du}} = -\lim_{t\to+\infty}\frac{\dot{k}(t)}{\beta(t)},
$$

if this limit exists. □

Example 2.4 ([1], Linear asymptote)**.** Let $\beta(t) = a$, and $k(t) = pt + q$, where $a > 0$, $p > 0$, and $q \ge 0$. From (2.4) we have

$$
\begin{aligned}
x(t) &= e^{-a(t-t_0)}\Big(x_0 + \int_{t_0}^t ae^{a(x-t_0)}(px+q)dx\Big),\\
&= k(t) - p/a + (x_0 - (k_0 - p/a))e^{-a(t-t_0)},
\end{aligned}
$$

where $x_0 = x(t_0)$ and $k_0 = k(t_0)$. From Lemma 2.2 we have

(1) If $x_0 < k_0$, then $x(t)$ is increasing and $x(t) < k(t)$ for all t. In this case $x(t)$ is convex if $x_0 > k_0 - p/a$ and concave if not.
(2) If $x_0 \ge k_0$, then $x(t)$ is convex and intersects $k(t)$ at $t_* = t_0 + \ln(1 + \frac{a}{p}(x_0 - k_0))/a$, with $t_* = \operatorname{argmin} x(t)$. In this case, $x(t)$ increases if $t > t_*$ and decreases if not.

Moreover we have

$$
\lim_{t\to+\infty}(x(t)-k(t)) = -\lim_{t\to+\infty}\frac{\dot{k}(t)}{\beta(t)} = -\frac{b}{a}.
$$

In the next example, we give some hypothesis on $\beta(t)$ which ensure that the limit at infinity between the solution to the differential equation (2.1) and a curvilinear carrying capacity $k(t)$ vanishes.

Example 2.5 (Curvilinear asymptote)**.** Let $\alpha(t)$, and $\beta(t)$ be two continuous functions such that

$$k(t) = \frac{\alpha(t)}{\beta(t)} = pt^{\lambda} + q,$$

with $t > 0$, $p > 0$, $q \geq 0$, and suppose that

$$\frac{\beta(t)}{t^{\gamma}} \geq c > 0, \quad \text{for all } t > 0,$$

with $0 < \lambda < \gamma + 1$. The unique solution to the initial value problem

$$\dot{x}(t) = \alpha(t) - \beta(t)x(t), \quad t \geq t_0,$$
$$x(t_0) = x_0,$$

has the following asymptotic property,

$$\lim_{t\to+\infty}(x(t) - k(t)) = 0.$$

Indeed, as $\beta(t) \geq ct^{\gamma}$, and $k(t)\beta(t) = \alpha(t)$ it follows that

$$\int_{t_0}^{+\infty} \beta(\tau)d\tau = \int_{t_0}^{+\infty} \alpha(\tau)d\tau = +\infty.$$

From Theorem 2.3, and $\lim_{t\to+\infty} k(t) = +\infty$, it follows that $\lim_{t\to+\infty}(x(t)-k(t)) = -\lim_{t\to+\infty} \frac{\dot{k}(t)}{\beta(t)}$. In addition, we have

$$0 \leq \frac{\dot{k}(t)}{\beta(t)} \leq \frac{\dot{k}(t)}{ct^{\gamma}} = \frac{\lambda p}{c} t^{\lambda-(\gamma+1)}.$$

As $0 < \lambda < \gamma + 1$, it follows that $\lim_{t\to+\infty} \frac{\dot{k}(t)}{\beta(t)} = 0$. Thus,

$$\lim_{t\to+\infty}(x(t) - k(t)) = -\lim_{t\to+\infty} \frac{\dot{k}(t)}{\beta(t)} = 0.$$

2.2. **Decreasing case.** In this section, we suppose that $k(t)$ is a non increasing function

$$k(t_1) \geq k(t_2) \quad \text{for all } t_1, t_2 \in I \text{ such that } t_1 < t_2. \tag{2.10}$$

We have the following result.

Lemma 2.6. *Let $k(t) = \alpha(t)/\beta(t) : I \to \mathbb{R}^+$ be defined such that $\alpha(t) > 0$, and $\beta(t) > 0$ for all t. Suppose that $k(t)$ satisfies the assumptions* (2.3) *and* (2.10)*, and let $x(t)$ be the solution of* (2.1) *passing through the point (t_0, x_0). If for some $s \in I$ we have $x(s) = k(s)$, then $x(t) \geq k(t)$ for all $t \geq s$. More precisely,*

- *$x(t) = k(t)$, and $\dot{x}(t) = 0$, for all $t \in J_s$, and*
- *$x(t) > k(t)$, and $\dot{x}(t) < 0$, for all $t > t_s = \sup J_s$.*

Proof. Let $x(t)$ be the solution of (2.1) passing through the point (t_0, x_0), and let $s \in I$ such that $x(s) = k(s)$. From (2.6), we have $x(t) = k(t)$ for all $t \in J_s$. Let $t_s = \sup J_s \in J_s$, it follows that $x(t_s) = k(t_s) = k(s)$ and $k(t) < k(t_s)$ for all $t > t_s$. Thus, by considering the solution passing through the point $(t_s, x(t_s))$, we have

$$\begin{aligned} x(t) &= k(t) + \Big((x(t_s) - k(t_s))e^{-\int_{t_s}^t \beta(u)du} - e^{-\int_{t_s}^t \beta(u)du} \int_{t_s}^t \dot{k}(u)e^{\int_{t_s}^u \beta(\tau)d\tau}du\Big), \\ &= k(t) - \int_{t_s}^t \dot{k}(u)e^{\int_t^u \beta(\tau)d\tau}du > k(t). \end{aligned}$$

□

The following result characterizes the behaviour of any solution of (2.1) in the case where $k(t)$ is decreasing.

Theorem 2.7. *Let $k(t)$ and $x(t)$ be defined as in Lemma 2.6. Also let $k_\infty = \lim_{t\to+\infty} k(t)$ and $x_\infty = \lim_{t\to+\infty} x(t)$. Then*

- (a) *if $x_0 > k_0$, then $x(t) > k(t)$, and $\dot{x}(t) < 0$, for all $t \in I$;*
- (b) *if $x_0 = k_0$, then $x(t) \geq k(t)$, and $\dot{x}(t) \leq 0$, for all $t \in I$;*
- (c) *if $0 \leq x_0 < k_0$, we have two cases to consider*
  - (i) *if $x_0 \geq k_\infty$, then it will exists some $s > t_0$ where $s = \operatorname{argmax} x(t)$ such that $x(s) = k(s)$, and in this case $x(t)$ increases if $t < s$ and decreases if $t > s$;*
  - (ii) *if $x_0 < k_\infty$, then either $x(t)$ has the same behaviour as in (i), or $x(t) < k(t)$, and $\dot{x}(t) > 0$, for all $t \in I$.*

Proof. The proofs of the assertions (a) and (b) follow immediately from (2.4) and Lemma 2.6. Let us prove (c).

(i) If $k_0 > x_0 > k_\infty$, from (2.1), it follows that $\dot{x}(t_0) > 0$. By continuity, we also have $\dot{x}(t) > 0$ for all t provided that $x(t) < k(t)$. We will prove that, there exists some $s > t_0$ such that $x(s) = k(s)$ and $x(t)$ decreases for $t > s$ with $x(t) > k(t)$. Indeed, suppose that $x(t) < k(t)$ for all $t > t_0$. Thus, from (2.1), we have that $\dot{x}(t) > 0$ for all $t \geq t_0$ and hence, $k(t) > x(t) > x(t_0) = x_0$. By taking the limit at infinity we obtain $k_\infty \geq x_0$ which contradicts our assumption on x_0.

If $x_0 = k_\infty$, by arguing as in the proof above and by taking the limit at infinity we obtain $x_\infty = x_0$ which contradict the fact that $x(t) < k(t)$, i.e. $\dot{x}(t) > 0$, for all $t > t_0$.

(ii) Suppose that $0 \leq x_0 < k_\infty < k_0$. If we set $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = l$, from (2.4), we have

$$(x_0 - k_\infty)e^{-l} \leq x_\infty - k_\infty \leq (x_0 - k_0)e^{-l} - (k_\infty - k_0). \tag{2.11}$$

If $l = +\infty$, from (2.11), it follows that $x_\infty \geq k_\infty$ and hence, $x(t)$ has the same asymptotic behaviour as in (i). If $l < +\infty$, from (2.11), it follows that $x_\infty \leq k_\infty$ if $x_0 \leq k_0 + (k_\infty - k_0)e^l$, and the solution $x(t)$ is always increasing. If $x_0 \geq k_0 + (k_\infty - k_0)e^l$, the solution increases or has the same asymptotic behaviour as in (i). □

The following result gives us some information about the asymptotic behaviour of the solution to the linear problem (2.1) when the time dependent coefficients $\alpha(t)$ and $\beta(t)$ are not necessary bounded.

Theorem 2.8. *Let $k(t)$ and $x(t)$ be defined as in Lemma 2.6. Also let $k_\infty = \lim_{t\to+\infty} k(t)$, and $x_\infty = \lim_{t\to+\infty} x(t)$. Then*

- (a) *If $\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)du = +\infty$, then $x_\infty = k_\infty \geq 0$.*
- (b) *If $\lim_{t\to+\infty} \int_{t_0}^t \alpha(u)du < +\infty$, then*
  - (I) *if $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = +\infty$, then $x_\infty = k_\infty = 0$;*
  - (II) *if $\lim_{t\to+\infty} \int_{t_0}^t \beta(u)du = l < +\infty$, then*

$$k_\infty + (x_0 - k_\infty)e^{-l} \leq x_\infty \leq k_0 + (x_0 - k_0)e^{-l}.$$

Proof. (a) If $\lim_{t\to+\infty}\int_{t_0}^t \alpha(u)du = +\infty$, as $\alpha(t) = k(t)\beta(t)$, and $k(t) \le k_0$ for all $t \in I$, it follows that $\lim_{t\to+\infty}\int_{t_0}^t \beta(u)du = +\infty$. In addition, $\int_{t_0}^t \alpha(u)du \le \int_{t_0}^t \alpha(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du$, hence

$$\lim_{t\to+\infty}\int_{t_0}^t \alpha(u)e^{\int_{t_0}^u \beta(\tau)d\tau}du = +\infty.$$

Thus, by using L'Hôpital's rule we have

$$\begin{aligned}
x_\infty = \lim_{t\to+\infty} x(t) &= \lim_{t\to+\infty}\frac{\int_{t_0}^t \alpha(\tau)e^{\int_{t_0}^\tau \beta(u)du}d\tau}{e^{\int_{t_0}^t \beta(u)du}}\\
&= \lim_{t\to+\infty}\frac{\alpha(t)e^{\int_{t_0}^t \beta(u)du}}{\beta(t)e^{\int_{t_0}^t \beta(u)du}}\\
&= \lim_{t\to+\infty}\frac{\alpha(t)}{\beta(t)}\\
&= \lim_{t\to+\infty} k(t) = k_\infty \ge 0.
\end{aligned}$$

(b) Let us suppose that $\lim_{t\to+\infty}\int_{t_0}^t \alpha(u)du < +\infty$.

(I) If $\lim_{t\to+\infty}\int_{t_0}^t \beta(u)du = +\infty$, then from (2.2) and the dominated convergence Theorem, it follows that

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty}\int_{t_0}^t \alpha(\tau)e^{-\int_\tau^t \beta(u)du}d\tau = 0.$$

As $x_\infty \ge k_\infty \ge 0$ it follows that $k_\infty = 0$.

(II) If $\lim_{t\to+\infty}\int_{t_0}^t \beta(u)du = l < +\infty$, then from (2.4) the result follows. □

Example 2.9 (Hyperbolic asymptote, first case)**.** Let $k(t) = 1/(pt+q)$, and $\alpha(t) = 1/(t+1)^\gamma$ where $p > 0$, $q > 0$, and $x(t)$ be the solution of (2.1) passing through the point (t_0, x_0) where $t_0 = 0$. From Theorem 2.8 we have

If $\gamma \le 1$ then $\lim_{t\to+\infty}\int_{t_0}^t \alpha(u)du = +\infty$ and hence $x_\infty = k_\infty = 0$.

If $\gamma > 1$, then $\alpha(t)$ is decreasing with $\alpha(0) = 1$ and $\lim_{t\to+\infty}\int_{t_0}^t \alpha(u)du = \frac{1}{\gamma-1}$. Moreover, $\beta(t) = (pt+q)/(t+1)^\gamma$, $\beta(0) = q$, and we have

$$\lim_{t\to+\infty}\int_{t_0}^t \beta(u)du = \begin{cases} l = \frac{p}{\gamma-2} + \frac{q-p}{\gamma-1} & \text{if } \gamma > 2,\\ +\infty & \text{if } 1 < \gamma \le 2.\end{cases}$$

Hence,

- If $1 < \gamma \le 2$, then $x_\infty = k_\infty = 0$.
- If $\gamma > 2$, then
$$x_0e^{-l} \le x_\infty \le x_0e^{-l} + \frac{1}{q}(1 - e^{-l}).$$

Example 2.10 (Hyperbolic asymptote, second case)**.** Let $k(t) = 1/(pt^\lambda + q)$, and $\alpha(t) = 1/t^\gamma$ where $\lambda > 0$, $p > 0$, $q > 0$, and $x(t)$ be the solution of (2.1) passing through the point (t_0, x_0) where $t_0 = 1$. From Theorem 2.8 we have

- If $\gamma \le 1$ then $\lim_{t\to+\infty}\int_{t_0}^t \alpha(u)du = +\infty$ and hence $x_\infty = k_\infty = 0$.

- If $\gamma > 1$, then $\alpha(t)$ is decreasing with $\alpha(1) = 1$ and $\lim_{t\to+\infty} \int_{t_0}^{t} \alpha(u)du = \frac{1}{\gamma-1} < +\infty$. Moreover, $\beta(t) = (pt^{\lambda} + q)/t^{\gamma}$, and we have

$$\lim_{t\to+\infty} \int_{t_0}^{t} \beta(u)du = \begin{cases} l = \frac{p}{\gamma-\lambda-1} + \frac{q}{\gamma-1} & \text{if } \gamma > \lambda + 1, \\ +\infty & \text{if } 1 < \gamma \le \lambda + 1. \end{cases}$$

Hence,

- If $1 < \gamma \le \lambda + 1$, then $x_\infty = k_\infty = 0$.
- If $\gamma > \lambda + 1$, then

$$x_0 e^{-l} \le x_\infty \le x_0 e^{-l} + \frac{1}{q}(1 - e^{-l}).$$

3. Subsolution and supersolution

The following theorem gives us some information on the boundedness of the solution $x(t)$ of (2.1) passing through a point (t_0, x_0) when the carrying capacity $k(t) = \alpha(t)/\beta(t)$ is bounded above and below by given positive functions.

Theorem 3.1. *Let $k : I \to (0, +\infty)$ be defined such that $\alpha(t) > 0$, and $\beta(t) > 0$ for all t, and let $x(t)$ be the solution of (2.1) passing through the point (t_0, x_0). Suppose that there exist two functions $\underline{k}, \overline{k} : I \to (0, +\infty)$ such that*

$$\underline{k}(t) \le k(t) \le \overline{k}(t), \quad \text{for all } t \ge t_0. \tag{3.1}$$

In addition, let $\underline{x}(t)$ and $\overline{x}(t)$ be respectively the solutions of the auxiliary problems

$$\begin{aligned} \dot{\underline{x}}(t) &= \beta(t)(\underline{k}(t) - \underline{x}(t)), \\ \dot{\overline{x}}(t) &= \beta(t)(\overline{k}(t) - \overline{x}(t)), \end{aligned} \tag{3.2}$$

with $\underline{x}(t_0) = \overline{x}(t_0) = x(t_0)$. Then, we have

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \quad \text{for all } t \ge t_0. \tag{3.3}$$

Proof. The solutions $\underline{\epsilon}(t) = x(t) - \underline{x}(t)$, and $\overline{\epsilon}(t) = \overline{x}(t) - x(t)$ of the auxiliary problems

$$\begin{aligned} \dot{\underline{\epsilon}}(t) &= \beta(t)((k(t) - \underline{k}(t)) - (x(t) - \underline{x}(t)))), \\ \dot{\overline{\epsilon}}(t) &= \beta(t)((\overline{k}(t) - k(t)) - (\overline{x}(t) - x(t))), \end{aligned}$$

with $\underline{\epsilon}(t_0) = \overline{\epsilon}(t_0) = 0$ are respectively

$$\begin{aligned} \underline{\epsilon}(t) &= e^{-\int_{t_0}^{t} \beta(u)du} \int_{t_0}^{t} \beta(\tau)(k(\tau) - \underline{k}(\tau)) e^{\int_{t_0}^{\tau} \beta(u)du d\tau}, \\ \overline{\epsilon}(t) &= e^{-\int_{t_0}^{t} \beta(u)du} \int_{t_0}^{t} \beta(\tau)(\overline{k}(\tau) - k(\tau)) e^{\int_{t_0}^{\tau} \beta(u)du d\tau}. \end{aligned}$$

From (3.1) it follows that $\underline{\epsilon}(t) \ge 0$, $\overline{\epsilon}(t) \ge 0$, and hence the inequalities (3.3) follow. $\square$

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2 (Increasing case)**.** *Let $\overline{k}, \underline{k} : I \to (0, +\infty)$ be defined by*

$$\overline{k}(t) = \max\{k(\tau) : t_0 \le \tau \le t\}, \quad \text{and} \quad \underline{k}(t) = \min\{k(\tau) : \tau \ge t\},$$

and let $\underline{x}(t)$ *and* $\overline{x}(t)$ *be respectively the solutions of the auxiliary problems* (3.2), *with* $\underline{x}(t_0) = \overline{x}(t_0) = x(t_0)$. *Then,*

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \quad \text{for all } t \ge t_0. \tag{3.4}$$

Proof. It will be noted that $\overline{k}(t)$ and $\underline{k}(t)$ are respectively the least upper and the greatest lower bounds of the sets of all increasing function $f(t)$, such that $k(t) \le f(t)$, respectively $k(t) \ge f(t)$. Hence, $\overline{k}(t)$ and $\underline{k}(t)$ are positive and increasing functions with $\underline{k}(t) \le k(t) \le \overline{k}(t)$ for all $t \ge t_0$. From Theorem 3.1 the inequalities (3.4) follow immediately. □

Example 3.3. *Let* $k(t) = pt + q(1 + r\sin(\omega t))$ *and* $\beta(t) = a$, *where* $p \ge 0$, $q \ge 0$, $0 < r \le 1$, $\omega > 0$, *and* $a > 0$. *From* (2.4), *and Example* 2.4 *we have*

$$x(t) = k(t) - p/a + (x_0 - (k_0 - p/a))e^{-a(t-t_0)} - \frac{rq\omega}{a^2+\omega^2}[\phi(t) - \phi(t_0)e^{-a(t-t_0)}],$$

where $x_0 = x(t_0)$, $k_0 = k(t_0)$, *and* $\phi(t) = (\omega\sin(\omega t) + a\cos(\omega t))$. *It will be noted that, for a large values of* t, *we have*

$$x(t) - k(t) \approx -p/a - \frac{rq\omega}{a^2+\omega^2}\phi(t).$$

On the other hand, let $\underset{\sim}{k}(t)$ *and* $\widetilde{k}(t)$ *be defined by*

$$\underset{\sim}{k}(t) = pt + q(1-r), \quad \text{and} \quad \widetilde{k}(t) = pt + q(1+r).$$

Obviously, $\underset{\sim}{k}$ *and* $\widetilde{k}$ *are non decreasing functions and satisfy the inequalities*

$$\underset{\sim}{k}(t) \le \underline{k}(t) \le k(t) \le \overline{k}(t) \le \widetilde{k}(t), \quad \text{for all } t \ge t_0.$$

It follows from Corollary 3.2, *that*

$$\underset{\sim}{x}(t) \le x(t) \le \widetilde{x}(t), \quad \text{for all} t \ge t_0,$$

where

$$\underset{\sim}{x}(t) = \underset{\sim}{k}(t) - p/a + (\underset{\sim}{x}_0 - (\underset{\sim}{k}_0 - p/a))e^{-a(t-t_0)},$$
$$\widetilde{x}(t) = \widetilde{k}(t) - p/a + (\widetilde{x}_0 - (\widetilde{k}_0 - p/a))e^{-a(t-t_0)},$$

are respectively the solutions of (2.1) *with carrying capacities* $\underset{\sim}{k}(t)$ *and* $\widetilde{k}(t)$ *respectively. Figures* 1 *and* 2 *illustrates respectively the cases* $p > 0$ *and* $p = 0$.

The following corollary is an immediate consequence of Theorem 3.1.

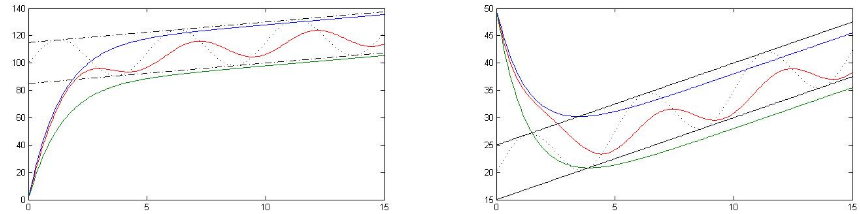
Corollary 3.4 (Decreasing case)**.** *Let* $\overline{k}, \underline{k} : I \to (0, +\infty)$ *be defined by*

$$\overline{k}(t) = \max\{k(\tau) : \tau \ge t\} \quad \text{and} \quad \underline{k}(t) = \min\{k(\tau) : t_0 \le \tau \le t\},$$

and let $\underline{x}(t)$ *and* $\overline{x}(t)$ *be respectively the solutions of the auxiliary problems* (3.2), *with* $\underline{x}(t_0) = \overline{x}(t_0) = x(t_0)$. *Then*

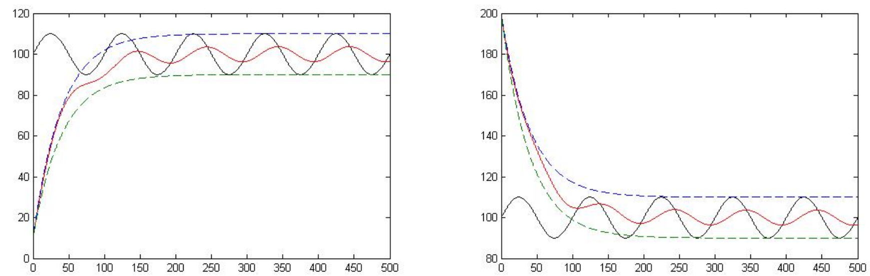
$$\underline{x}(t) \le x(t) \le \overline{x}(t), \quad \text{for all } t \ge t_0.$$

The proof can be done in a similar way as done in the proof of Corollary 3.2. Hence it is omitted.



(a) $x_0 < k_0$ (b) $x_0 > k_0$

Figure 1. Representation of the solution $x(t)$ (solid line) with its lower and upper bound solutions $\underset{\sim}{x}(t)$ and $\widetilde{x}(t)$ (long dashed line), and the carrying capacity $k(t)$ (black line) when $p > 0$.



(a) $x_0 < k_0$ (b) $x_0 > k_0$

Figure 2. Representation of the solution $x(t)$ (solid line) with its lower and upper bound solutions $\underset{\sim}{x}(t)$ and $\widetilde{x}(t)$ (long dashed line), and the carrying capacity $k(t)$ (black line) when $p = 0$.

Example 3.5. Let $k(t) = pe^{-\sigma(t-t_0)}(1 + r\sin(\omega t))$ and $\beta(t) = a$, where $p > 0$, $\sigma \geq 0$, $0 < r \leq 1$, $\omega > 0$, and $a > 0$. Two cases would be considered. If $a \neq \sigma$, the solution of the problem (2.1) is

$$x(t) = e^{-a(t-t_0)}\big[\frac{ap}{a-\sigma}(e^{(a-\sigma)(t-t_0)} - 1) + \frac{rap}{(a-\sigma)^2+\omega^2}(e^{(a-\sigma)(t-t_0)}\varphi(t) - \varphi(t_0))\big] \\ + x_0 e^{-a(t-t_0)},$$

where $\varphi(t) = (a-\sigma)\sin(\omega t) - \omega\cos(\omega t)$. Otherwise, the solution is

$$x(t) = e^{-a(t-t_0)}\big[ap(t - \frac{r}{\omega}\cos(\omega t)) - ap(t_0 - \frac{r}{\omega}\cos(\omega t_0)) + x_0\big].$$

In both cases, let $\underset{\sim}{k}(t)$ and $\widetilde{k}(t)$ be defined by

$$\underset{\sim}{k}(t) = pe^{-\sigma(t-t_0)}(1-r), \quad \widetilde{k}(t) = pe^{-\sigma(t-t_0)}(1+r).$$

Obviously, $\underset{\sim}{k}$ and $\widetilde{k}$ are non increasing functions and satisfy the inequalities

$$\underset{\sim}{k}(t) \le \underline{k}(t) \le k(t) \le \overline{k}(t) \le \widetilde{k}(t), \quad \text{for all } t \ge t_0.$$

It follows from Corollary 3.4, that

$$\underset{\sim}{x}(t) \le x(t) \le \widetilde{x}(t), \quad \text{for all } t \ge t_0,$$

where

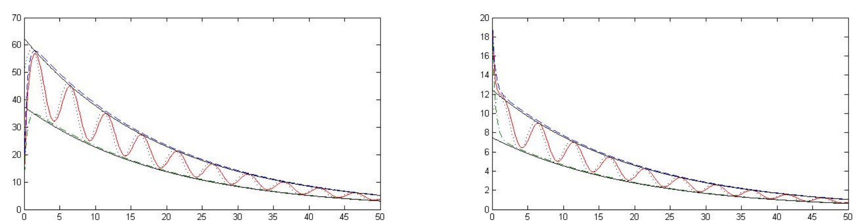
$$\underset{\sim}{x}(t) = \frac{(1-r)ap}{a-\sigma}\big(e^{-\sigma(t-t_0)} - e^{-a(t-t_0)}\big) + x_0 e^{-a(t-t_0)},$$

$$\widetilde{x}(t) = \frac{(1+r)ap}{a-\sigma}\big(e^{-\sigma(t-t_0)} - e^{-a(t-t_0)}\big) + x_0 e^{-a(t-t_0)},$$

are respectively the solutions of (2.1) with carrying capacities $\underset{\sim}{k}(t)$ and $\widetilde{k}(t)$ respectively for the case $a \ne \sigma$. In the case $a = \sigma$, $\underset{\sim}{x}(t)$ and $\widetilde{x}(t)$ are given by

$$\underset{\sim}{x}(t) = e^{-a(t-t_0)}(ap(1-r)(t-t_0) + x_0), \quad \widetilde{x}(t) = e^{-a(t-t_0)}(ap(1+r)(t-t_0) + x_0).$$

An illustration is given in Figure 3.



(a) $x_0 < k_0$ (b) $x_0 > k_0$

FIGURE 3. Representation of the solution $x(t)$ (solid line) with its lower and upper bound solutions $\underset{\sim}{x}(t)$ and $\widetilde{x}(t)$ (long dashed line), and the carrying capacity $k(t)$ (black line) when $a \ne \sigma$.

4. A product decomposition of the solution

In this section, we present a decomposition for the solution of (1.1) and (1.2) as the product of the carrying capacity and the solution to a corresponding differential equation with a constant carrying capacity. The next two theorems present these results.

Theorem 4.1 (Linear equation). *Let $\alpha, \beta \in C^1(I; (0,+\infty))$, $k(t) = \alpha(t)/\beta(t) > 0$ for all $t \ge t_0$, and $k(t) \in AC^1_{\rm loc}(I)$. We have the following product decomposition of the solution.*

(i) The unique solution to

$$\begin{gathered}\dot{\tilde{x}}(t)=\alpha(t)-\big(\beta(t)-\frac{\dot{k}(t)}{k(t)}\big)\tilde{x}(t),\quad t\geq t_0,\\ \tilde{x}(t_0)=\tilde{x}_0,\end{gathered}\tag{4.1}$$

is

$$\tilde{x}(t)=k(t)\omega(t),\tag{4.2}$$

where $\omega(t)$ is the solution of

$$\begin{gathered}\dot{\omega}(t)=\beta(t)(1-\omega(t)),\quad t\geq t_0,\\ \omega(t_0)=\tilde{x}_0/k_0.\end{gathered}\tag{4.3}$$

(ii) Let $\tilde{\beta}(t)$ be the solution of

$$\begin{gathered}\dot{\tilde{\beta}}(t)=1-\big(\frac{\dot{\alpha}(t)}{\alpha(t)}+\beta(t)\big)\tilde{\beta}(t),\quad t\geq t_0,\\ \tilde{\beta}(t_0)>0.\end{gathered}\tag{4.4}$$

Let us set

$$\tilde{k}(t)=\alpha(t)\tilde{\beta}(t),\tag{4.5}$$

and let $\tilde{\omega}(t)$ be the solution of

$$\begin{gathered}\dot{\tilde{\omega}}(t)=\frac{1}{\tilde{\beta}(t)}(1-\tilde{\omega}(t)),\quad t\geq t_0,\\ \tilde{\omega}(t_0)=\tilde{\omega}_0=x_0/\tilde{k}_0.\end{gathered}\tag{4.6}$$

Then the solution of (2.1) passing through the point (t_0,x_0) is

$$x(t)=\tilde{k}(t)\tilde{\omega}(t),\tag{4.7}$$

where $\tilde{k}(t)$ is a solution of (2.1) passing through the point $(t_0,\tilde{k}_0)$. Moreover,

$$\lim_{t\to+\infty}(x(t)-\tilde{k}(t))=\begin{cases}0 & if\lim_{t\to+\infty}\int_{t_0}^t\beta(u)du=+\infty,\\ (x_0-\tilde{k}_0)e^{-l} & if\lim_{t\to+\infty}\int_{t_0}^t\beta(u)du=l.\end{cases}\tag{4.8}$$

Proof. (i) From (2.2), the unique solution of (4.1) is

$$\begin{aligned}\tilde{x}(t)&=e^{-\int_{t_0}^t(\beta(u)-\frac{\dot{k}(u)}{k(u)})du}\Big(\tilde{x}_0+\int_{t_0}^t\alpha(\tau)e^{\int_{t_0}^\tau(\beta(u)-\frac{\dot{k}(u)}{k(u)})du}d\tau\Big)\\ &=e^{-\int_{t_0}^t\beta(u)du}e^{\ln\frac{k(t)}{k_0}}\Big(\tilde{x}_0+\int_{t_0}^t\alpha(\tau)e^{\int_{t_0}^\tau\beta(u)du}e^{-\ln\frac{k(\tau)}{k_0}}d\tau\Big)\\ &=\frac{k(t)}{k_0}e^{-\int_{t_0}^t\beta(u)du}\Big(\tilde{x}_0+k_0\int_{t_0}^t\frac{\alpha(\tau)}{k(\tau)}e^{\int_{t_0}^\tau\beta(u)du}d\tau\Big)\\ &=\frac{k(t)}{k_0}e^{-\int_{t_0}^t\beta(u)du}\Big(\tilde{x}_0+k_0(e^{\int_{t_0}^t\beta(u)du}-1)\Big)\\ &=k(t)\Big(1+\big(\frac{\tilde{x}_0}{k_0}-1\big)e^{-\int_{t_0}^t\beta(u)du}\Big)\\ &=k(t)\omega(t),\end{aligned}\tag{4.9}$$

where $\omega(t)$ is the solution of the initial value problem (4.3).

(ii) From (4.5) and (4.4), we show that $\tilde{k}(t)$ is the solution of (2.1) because

$$\dot{\tilde{k}}(t) = \dot{\alpha}(t)\tilde{\beta}(t) + \alpha(t)\dot{\tilde{\beta}}(t) = \alpha(t) - \beta(t)\tilde{k}(t). \tag{4.10}$$

Then using (4.7), (4.10) and (4.6), we have

$$\begin{aligned}
\dot{x}(t) &= \dot{\tilde{k}}(t)\tilde{\omega}(t) + \tilde{k}(t)\dot{\tilde{\omega}}(t) \\
&= \big(\alpha(t) - \beta(t)\tilde{k}(t)\big)\tilde{\omega}(t) + \frac{\tilde{k}(t)}{\tilde{\beta}(t)}(1 - \tilde{\omega}(t)) \\
&= \alpha(t)\tilde{\omega}(t) - \beta(t)\tilde{k}(t)\tilde{\omega}(t) + \alpha(t) - \alpha(t)\tilde{\omega}(t) \\
&= \alpha(t) - \beta(t)\tilde{k}(t)\tilde{\omega}(t) \\
&= \alpha(t) - \beta(t)x(t).
\end{aligned}$$

Thus, $x(t)$ is the solution of (2.1) passing through the point (t_0, x_0).

In addition, since both $x(t)$ and $\tilde{k}(t)$ are solution of (2.1), from (2.2) we have

$$x(t) - \tilde{k}(t) = (x_0 - \tilde{k}_0)e^{-\int_{t_0}^{t} \beta(u)du}$$

and (4.8) follows directly. □

The results presented in Theorem 4.1 can then be applied to the logistic equation (1.2) and we have the following theorem.

Theorem 4.2 (Logistic equation). *Let $\alpha, \beta \in C^1(I; (0, +\infty))$, $k(t) = \alpha(t)/\beta(t) > 0$ for all $t \geq t_0$, and suppose $k(t) \in AC^1_{\rm loc}(I)$. We have the following product decomposition of the solution.*

(i) The unique solution to

$$\begin{gathered}
\dot{\tilde{x}}(t) = \big(\beta(t) - \frac{\dot{k}(t)}{k(t)}\big)\tilde{x}(t) - \alpha(t)\tilde{x}^2(t), \quad t \geq t_0, \\
\tilde{x}(t_0) = \tilde{x}_0,
\end{gathered} \tag{4.11}$$

is

$$\tilde{x}(t) = \frac{w(t)}{k(t)}, \tag{4.12}$$

where $w(t)$ is the solution of

$$\begin{gathered}
\dot{w}(t) = \beta(t)w(t)(1 - w(t)), \quad t \geq t_0, \\
w(t_0) = k_0\tilde{x}_0.
\end{gathered} \tag{4.13}$$

(ii) Let $\tilde{\beta}(t)$ be the solution of

$$\begin{gathered}
\dot{\tilde{\beta}}(t) = (\frac{\dot{\alpha}(t)}{\alpha(t)} + \beta(t))\tilde{\beta}(t) - \tilde{\beta}^2(t), \quad t \geq t_0, \\
\tilde{\beta}(t_0) > 0.
\end{gathered} \tag{4.14}$$

Let us set

$$\tilde{k}(t) = \frac{\tilde{\beta}(t)}{\alpha(t)},$$

and let $\tilde{w}(t)$ be the solution of the initial value problem

$$\begin{gathered}
\dot{\tilde{w}}(t) = \tilde{\beta}(t)\tilde{w}(t)(1 - \tilde{w}(t)), \quad t \geq t_0, \\
\tilde{w}(t_0) = \tilde{k}_0\tilde{x}_0.
\end{gathered} \tag{4.15}$$

Then the solution of (1.2) *passing through the point* (t_0, x_0) *is*

$$x(t) = \tilde{k}(t)\tilde{w}(t), \tag{4.16}$$

where $\tilde{k}(t)$ *is the solution of* (1.2) *passing through the point* $(t_0, \tilde{k}_0)$*.*

Proof. (i) and (ii) are directly obtained from Theorem 4.1 using the application $x \mapsto y = x^{-1}$. □

Conclusion. The goal of this study has been to make a qualitative study of the solutions of the linear and logistic differential equations. We have obtained new results on the behaviour and the asymptotic behaviour of any solution to these differential equations in the case where the coefficients are time dependent. We have studied the monotone case and also the non monotone case when it is possible to construct subsolution and supersolution. Finally we obtain a product decomposition of the solution for some special form of these models.

References

[1] R. B. Banks; *Growth and diffusion phenomena: Mathematical frameworks and applications*, Springer-Verlag, New York, 1994.

[2] B. D. Coleman; *Nonautonomous logistic equations as models of the adjustment of populations to environmental change*, Mathematical Biosciences **45** (1979), 159–173.

[3] B. D. Coleman; *On optimal intrinsic growth rates for populations in periodically changing environments*, Journal Mathematical Biology **12** (1981), 343–354.

[4] B. D. Coleman, Y. H. Hsieh, G. P. Knowles; *On the optimal choice of r for a population in a periodic environment*, Mathematical Biosciences **46** (1979), 71–85.

[5] F. Dubeau, Y. Mir; *Exponential growth model : from horizontal to linear asymptote*, Communications in Statistics - Simulation and Computation (2013), To appear.

[6] F. Dubeau, Y. Mir; *Growth models with oblique asymptote*, Mathematical Modelling and Analysis **8** (2013), 204–218.

[7] F. Dubeau, Y. Mir, A. A. Assani, A. Chalifour; *Least squares fitting with single inflection point growth curve II - an application*, Mathematical Modelling and Applied Computing **2** (2011), 283–301.

[8] F. Dubeau, Y. Mir, A. A. Assani, A. Chalifour; *Modelling stage-discharge relationship with single inflection point nonlinear functions*, International Journal of Hydrology Science and Technology **2** (2012), 153–167.

[9] G. N. Galanis, P. K. Palamides; *Global positive solutions of a generalized logistic equation with bounded and unbounded coefficients*, Electronic Journal of Differential Equations **119** (2003), 1–13.

[10] S. Huet, E. Jolivet, A. Messéan; *La régression non linéaire: méthodes et applications en biologie*, INRA, Paris, 1992.

[11] P. S. Meyer, J. H. Ausubel; *Carrying capacity: a model with logistically varying limits*, Technological Forecasting and Social Change **61** (1999), 209–214.

[12] P. S. Meyer, J. W. Yung, J. H. Ausubel; *A primer on logistic growth and substitution: the mathematics of the Loglet Lab Software*, Technological Forecasting and Social Change **61** (1999), 247–271.

[13] Y. Mir and F. Dubeau; *Least squares fitting of the stage-discharge relationship using smooth models with curvilinear asymptotes*, Hydrological Sciences Journal **DOI: 10.1080/02626667.2014.935779** (2014).

[14] J. D. Murray, *Mathematical biology*, Springel, Berlin, 1989.

[15] M. N. Nkashama; *Dynamics of logistic equations with non-autonomous bounded coefficients*, Electronic Journal of Differential Equations **02** (2000), 1–8.

[16] D. A. Ratkowsky; *Nonlinear regression modeling*, Marcel Dekker, New York, 1983.

[17] D. A. Ratkowsky; *Handbook of nonlinear regression models*, Marcel Dekker, New York, 1989.

[18] S. P. Rogovchenko, Y. V. Rogovchenko; *Effect of periodic environmental fluctuations on the pearl-verhulst model*, Chaos, Solitons and Fractals **39** (2009), 1169–1181.

[19] H. M. Safuan, Z. Jovanoski, I. N. Towers, H. S. Sidhu; *Exact solution of a non-autonomous logistic population model*, Ecological Modelling **251** (2013), 99–102.

[20] H. M. Safuan, I. N. Towers, Z. Jovanoski, H. S. Sidhu; *Coupled logistic carrying capacity model*, ANZIAM Journal **53** (2012), 172–184.

[21] J. J. Shepherd, L. Stojkov; *The logistic population model with slowly varying carrying capacity*, ANZIAM J. **47** (2007), 492–506.

[22] H. R. Thieme; *Mathematics in population biology*, Princeton and Oxford: Princeton University Press, 2003.

[23] J. H. M. Thornleya, J. France; *An open-ended logistic-based growth function*, Ecological Modelling **184** (2005), 257–261.

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