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EXISTENCE OF SOLITONS FOR DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. By using the Mountain Pass Lemma, we establish sufficient conditions for the existence of solitons for the discrete nonlinear Schrödinger equations.

1. Introduction

The discrete nonlinear Schrödinger (DNLS) equation is one of the most important inherently discrete models. DNLS equations play a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology [7, 8, 9]. For example, they have been successfully applied to the modeling of localized pulse propagation optical fibers and wave guides, to the study of energy relaxation in solids, to the behavior of amorphous material, to the modeling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand [9].

Below $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a and b in $\mathbb{Z}$, define $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$. This article concerns the DNLS equation

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - f_n(\psi_n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is discrete Laplacian operator, ε_n is real valued for each $n \in \mathbb{Z}$, $f_n \in C(\mathbb{R},\mathbb{R})$, $f_n(0) = 0$ and the nonlinearity $f_n(u)$ is gauge invariant, that is,

$$f_n(e^{i\theta}u) = e^{i\theta} f_n(u), \quad \theta \in \mathbb{R}. \tag{1.2}$$

Since solitons are spatially localized time-periodic solutions and decay to zero at infinity. Thus, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t},$$

and

$$\lim_{|n|\to\infty} \psi_n = 0,$$

where ψ_n is real valued for each $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.1) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.3}$$

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and

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.4}$$

holds.

Actually, our methods allow us to consider the following more general equation

$$-\Delta(p_n(\Delta u_{n-1})^\delta) + q_n u_n^\delta = f_n(u_{n+T}, u_n, u_{n-T}),\ n \in \digamma, \tag{1.5}$$

with the same boundary condition (1.4). Here, Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, p_n and q_n are real valued for each $n \in \digamma$, $\delta > 0$ is the ratio of odd positive integers, $f_n \in C(\mathbb{R}^4, \mathbb{R})$, T is a given nonnegative integer. When $\delta = 1, p_n \equiv 1, q_n \equiv \varepsilon_n - \omega$ and $T = 0$, we obtain (1.3). Naturally, if we look for solitons of (1.1), we just need to get the solutions of (1.5) satisfying (1.4).

When $f_n(u_{n+T}, u_n, u_{n-T}) = 0,\ n \in \digamma(0)$, (1.5) reduces to the equation

$$\Delta(p_n(\Delta u_{n-1})^\delta) + q_n u_n^\delta = 0, \tag{1.6}$$

which has been studied in [16] for results on oscillation, asymptotic behavior and the existence of positive solutions.

In 2008, Cai and Yu [1] obtained some sufficient conditions for the existence of periodic solutions of the nonlinear difference equation

$$\Delta(p_n(\Delta u_{n-1})^\delta) + q_n u_n^\delta = f_n(u_n), \quad n \in \digamma. \tag{1.7}$$

It is well known that critical point theory is an effective approach to study the behavior of differential equations [10, 11, 12, 13, 24, 27]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions for second order difference equations [14, 15]. Along this direction, Ma and Guo [20] (without periodicity assumption) and [21] (with periodicity assumption) applied variational methods to prove the existence of homoclinic orbits for the special form of (1.5) (with $\delta = 1$ and $T = 0$). Chen and Wang [6] studied the existence infinitely many homoclinic orbits of the following nonlinear difference equation

$$\Delta(p_n(\Delta u_{n-1})^\delta) - q_n u_n^\delta + f_n(u_n) = 0,\ n \in \digamma, \tag{1.8}$$

by using the Symmetric Mountain Pass Lemma.

In the past decade, the existence of solitons of the DNLS equations has drawn a great deal of interest [17, 18, 22, 23, 25, 26, 31, 32, 33, 34, 35]. The existence for the periodic DNLS equations with superlinear nonlinearity [22, 23, 25, 26], and with saturable nonlinearity [34, 35] has been studied. And the existence results of solitons of the DNLS equations without periodicity assumptions were established in [17, 18, 31, 32, 33]. As for the existence of the homoclinic orbits of nonlinear Schrödinger equations, we refer to [5, 28, 29, 30].

Our main results are the following theorems.

Theorem 1.1. *Suppose that the following hypotheses are satisfied:*

- (A1) *for any* $n \in \mathbb{Z}$, $p_n > 0$;
- (A2) *for any* $n \in \mathbb{Z}$, $\underline{q} = \inf_{n\in\mathbb{Z}} q_n > 0$ *and* $\lim_{|n|\to+\infty} q_n = +\infty$;
- (A3) *there exists a function* $F_n(v_1, v_2) \in C^1(\mathbb{R}^3, \mathbb{R})$ *satisfies*

$$\frac{\partial F_{n-T}(v_2, v_3)}{\partial v_2} + \frac{\partial F_n(v_1, v_2)}{\partial v_2} = f_n(v_1, v_2, v_3),$$

$$\lim_{\beta_1\to 0}\frac{F_n(v_1,v_2)}{\beta_1^{\delta+1}}=0 \quad \textit{uniformly for } n\in\mathbb{Z}\setminus M,\ \beta_1=(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}},$$

$$\lim_{\beta_2\to 0}\frac{f_n(v_1,v_2,v_3)}{\beta_2^{\delta}}=0 \quad \textit{uniformly for } n\in\mathbb{Z}\setminus M,\ \beta_2=(v_1^{\delta+1}+v_2^{\delta+1}+v_3^{\delta+1})^{\frac{1}{\delta+1}};$$

(A4) *for each* $n\in\mathbb{Z}$, $F_n(v_1,v_2)=W_n(v_2)-H_n(v_1,v_2)$, W,H *are continuously differentiable in* v_2 *and* v_1,v_2 *respectively. Moreover, there is a bounded set* $M\subset\mathbb{Z}$ *such that* $H_n(v_1,v_2)\ge 0$*;*

(A5) *there is a constant* $\mu>\delta+1$ *such that*

$$0<\mu W_n(v_2)\le\frac{\partial W_n(v_2)}{\partial v_2}v_2,\quad \forall(n,v_2)\in\mathbb{Z}\times(\mathbb{R}\setminus\{0\});$$

(A6) $H_n(0,0)=0$ *and there is a constant* $\varrho\in(\delta+1,\mu)$ *such that*

$$\frac{\partial H_n(v_1,v_2)}{\partial v_1}v_1+\frac{\partial H_n(v_1,v_2)}{\partial v_2}v_2\le\varrho H_n(v_1,v_2);$$

(A7) *there exists a constant* c *such that*

$$H_n(v_1,v_2)\le c(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{\varrho}{\delta+1}}\quad \textit{for } n\in\mathbb{Z},\ v_1^{\delta+1}+v_2^{\delta+1}>1.$$

Then (1.5) *has a nontrivial solution satisfying* (1.4).

Theorem 1.2. *Suppose that* (A1)–(A3), (A5)–(A8)*, and the following hypothesis are satisfied:*

(A4') *for each* $n\in\mathbb{Z}$, $F_n(v_1,v_2)=W_n(v_2)-H_n(v_1,v_2)$, W,H *are continuously differentiable in* v_2 *and* v_1,v_2 *respectively;*

or

$$\lim_{\beta_1\to 0}\frac{F_n(v_1,v_2)}{\beta_1^{\delta+1}}=0 \quad \textit{uniformly for } n\in\mathbb{Z},\ \beta_1=(v_1^{\delta+1}+v_2^{\delta+1})^{\frac{1}{\delta+1}}.$$

Then (1.5) *has a nontrivial solution satisfying* (1.4).

Remark 1.3. Equations similar in structure to (1.5) are discussed by Zhang et al [31, 32] under the assumption that f satisfies:

$$0<(q-1)f(u)u\le f'(u)u^2,\quad \forall u\ne 0$$

holds for some constant $q\in(2,+\infty)$. This is a stronger condition than the classical Ambrosetti- Rabinowitz superlinear condition, i.e., there exist constants $q>2$ and $r>0$ such that

$$0<q\int_0^u f(s)ds\le uf(u),\ \forall|u|\ge r.$$

Thus, our results improves the corresponding results in [31, 32].

As it is well known, critical point theory is a powerful tool to deal with the homoclinic solutions of differential equations [10, 11, 12, 13] and is used to study homoclinic solutions of discrete systems in recent years [2, 3, 4, 6, 20, 21, 34]. Our aim in this article is to obtain the existence results of solitons for the discrete nonlinear Schrödinger equations by using the Mountain Pass Lemma. The main idea is to transfer the problem of solutions in E (defined in Section 2) of (1.5) into that of critical points of the corresponding functional. The motivation for the present work stems from the recent papers [3, 6, 11].

2. Preliminaries

In order to apply the critical point theory, we establish the variational framework corresponding to (1.5) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notation.

Let S be the vector space of all real sequences of the form

$$u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty},$$

namely

$$S = \{\{u_n\} : u_n \in \mathbb{R},\ n \in \digamma\}.$$

Define

$$E = \big\{u \in S : \sum_{n=-\infty}^{+\infty} \big[p_n(\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}\big] < +\infty\big\}.$$

The space is a Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{n=-\infty}^{+\infty} \big[p_n(\Delta u_{n-1})^{\delta} \Delta v_{n-1} + q_n u_n^{\delta} v_n\big], \quad \forall u, v \in E, \tag{2.1}$$

and the corresponding norm

$$\|u\| = \Big\{\sum_{n=-\infty}^{+\infty} \big[p_n(\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}\big]\Big\}^{\frac{1}{\delta+1}}, \quad \forall u \in E. \tag{2.2}$$

On the other hand, we define the space of real sequences,

$$l^s = \big\{u \in S : \|u\|_s = (\sum_{n=-\infty}^{+\infty} |u_n|^s)^{1/s} < +\infty\big\}, \quad 1 \le s < +\infty,$$

with $\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u_n|$ when $s = +\infty$.

For all $u \in E$, define the functional J on E as follows:

$$\begin{aligned} J(u) :=& \frac{1}{\delta+1} \sum_{n=-\infty}^{+\infty} \big[p_n(\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}\big] - \sum_{n=-\infty}^{+\infty} F_n(u_{n+T}, u_n) \\ =& \frac{1}{\delta+1} \|u\|^{\delta+1} - \sum_{n=-\infty}^{+\infty} F_n(u_{n+T}, u_n). \end{aligned} \tag{2.3}$$

Standard arguments show that the functional J is a well-defined C^1 functional on E and (1.5) is easily recognized as the corresponding Euler-Lagrange equation for J. Thus, to find nontrivial solutions to (1.5) satisfying (1.4), we need only to look for nonzero critical points of J in E.

For the derivative of J we have the following formula,

$$\langle J'(u), v\rangle = \sum_{n=-\infty}^{+\infty} \Big[p_n(\Delta u_{n-1})^{\delta} \Delta v_{n-1} + q_n u_n^{\delta} v_n - f_n(u_{n+T}, u_n, u_{n-T}) v_n\Big], \tag{2.4}$$

for all $u, v \in E$.

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition ((PS) condition for short) if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [27]). *Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfy the (PS) condition. If $J(0)=0$ and*

(1) *there exist constants $\rho,\ \alpha>0$ such that $J|_{\partial B_\rho} \geq \alpha$, and*
(2) *there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.*

Then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{2.5}$$

where

$$\Gamma = \{g \in C([0,1],E) | g(0)=0,\ g(1)=e\}. \tag{2.6}$$

Lemma 2.2. *For $u \in E$,*

$$\underline{q}\|u\|_\infty^{\delta+1} \leq \underline{q}\|u\|_{\delta+1}^{\delta+1} \leq \|u\|^{\delta+1}. \tag{2.7}$$

Proof. Since $u \in E$, it follows that $\lim_{|n|\to\infty} |u_n| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u_{n^*}| = \max_{n\in\mathbb{Z}} |u_n|.$$

By (A2) and (2.2), we have

$$\|u\|^{\delta+1} = \sum_{n\in\mathbb{Z}} \big[p_n(\Delta u_{n-1})^{\delta+1} + q_n u_n^{\delta+1}\big] \geq \underline{q} \sum_{n\in\mathbb{Z}} u_n^{\delta+1} \geq \underline{q}\|u\|_\infty^{\delta+1}.$$

The proof is complete. ☐

Lemma 2.3. *Suppose that* (A5) *holds. Then for each $(n,u) \in \mathbb{Z}\times\mathbb{R}$, $s^{-\mu}W_n(su)$ is nondecreasing on $(0,+\infty)$.*

The proof of the above lemma is routine and so we omit it.

Lemma 2.4. *Suppose that* (A1)–(A8) *are satisfied. Then J satisfies the (PS) condition.*

Proof. Let $\{u^{(k)}\}_{k\in\mathbb{N}} \subset E$ be such that $\{J(u^{(k)})\}_{k\in\mathbb{N}}$ is bounded and $J'(u^{(k)}) \to 0$ as $k \to \infty$. Then there is a positive constant K such that

$$|J(u^{(k)})| \leq K, \quad \|J'(u^{(k)})\|_{E^*} \leq \rho K \quad \text{for } k \in \mathbb{N}.$$

Thus, by (2.3), (A5) and (A6), we have

$$\begin{aligned}
&(\delta+1)K + (\delta+1)K\|u^{(k)}\| \\
&\geq (\delta+1)J(u^{(k)}) - \frac{(\delta+1)}{\varrho}\langle J'(u^{(k)}), u^{(k)}\rangle \\
&= \frac{\varrho-(\delta+1)}{\varrho}\|u^{(k)}\|^{\delta+1} - (\delta+1)\sum_{n=-\infty}^{+\infty}\Big[W_n(u_n^{(k)}) - \frac{1}{\varrho}\frac{\partial W_n(u_n^{(k)})}{\partial v_2}u_n^{(k)}\Big] \\
&\quad + (\delta+1)\sum_{n=-\infty}^{+\infty} H_n(u_{n+T}^{(k)}, u_n^{(k)}) \\
&\quad - \frac{(\delta+1)}{\varrho}\sum_{n=-\infty}^{+\infty}\Big[\frac{\partial H_n(u_{n+T}^{(k)}, u_n^{(k)})}{\partial v_1}u_{n+T}^{(k)} + \frac{\partial H_n(u_{n+T}^{(k)}, u_n^{(k)})}{\partial v_2}u_n^{(k)}\Big]
\end{aligned}$$

$$\geq \frac{\varrho-(\delta+1)}{\varrho}\|u^{(k)}\|^{\delta+1}.$$

Since $\varrho > \delta+1$, it is not difficult to know that $\{u^{(k)}\}_{k\in\mathbb{N}}$ is a bounded sequence in E, i.e., there exists a constant $K_1 > 0$ such that

$$\|u^{(k)}\| \leq K_1, \quad k \in \mathbb{N}. \tag{2.8}$$

So passing to a subsequence if necessary, it can be assumed that $u^{(k)} \rightharpoonup u^{(0)}$ in E. For any given number $\varepsilon > 0$, by (A3), we can choose $\zeta > 0$ such that

$$|f_n(u_{n+T}, u_n, u_{n-T})| \leq \varepsilon(u_{n+T}^{\delta+1} + u_n^{\delta+1} + u_{n-T}^{\delta+1})^{\frac{\delta}{\delta+1}}, \quad \forall n \in \mathbb{Z}\setminus M,\ u \in \mathbb{R}, \tag{2.9}$$

where $(u_{n+T}^{\delta+1} + u_n^{\delta+1} + u_{n-T}^{\delta+1})^{\frac{1}{\delta+1}} \leq \zeta$.

By (A2), we can also choose a positive integer $D > \max\{\max\{|n| : n \in M\}, T\}$ such that

$$q_n \geq \frac{K_1^{\delta+1}}{\zeta^{\delta+1}},\ |n| \geq D. \tag{2.10}$$

By (2.8) and (2.10), we obtain

$$(u_n^{(k)})^{\delta+1} = \frac{1}{q_n} q_n (u_n^{(k)})^{\delta+1} \leq \frac{\zeta^{\delta+1}}{K_1^{\delta+1}}\|u^{(k)}\|^{\delta+1} \leq \zeta^{\delta+1}, \quad |n| \geq D. \tag{2.11}$$

Since $u^{(k)} \rightharpoonup u^{(0)}$ in E, it is easy to verify that $u_n^{(k)}$ converges to $u_n^{(0)}$ pointwise for all $n \in \mathbb{Z}$; that is,

$$\lim_{k\to\infty} u_n^{(k)} = u_n^{(0)}, \quad \forall n \in \mathbb{Z}. \tag{2.12}$$

Combining with (2.11), we have

$$(u_n^{(0)})^{\delta+1} \leq \zeta^{\delta+1}, \quad |n| \geq D. \tag{2.13}$$

It follows from (2.12) and the continuity of $f_n(v_1, v_2, v_3)$ on v_1, v_2, v_3 that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-D}^{D} \left|f_n\big(u_{n+T}^{(k)}, u_n^{(k)}, u_{n-T}^{(k)}\big) - f_n\big(u_{n+T}^{(0)}, u_n^{(0)}, u_{n-T}^{(0)}\big)\right| < \varepsilon, \quad k \geq k_0. \tag{2.14}$$

On the other hand, it follows from (A3), (2.7), (2.8), (2.9), (2.11) and (2.13) that

$$
\begin{aligned}
&\sum_{|n|\geq D} \big|f_n\big(u_{n+T}^{(k)}, u_n^{(k)}, u_{n-T}^{(k)}\big) - f_n\big(u_{n+T}^{(0)}, u_n^{(0)}, u_{n-T}^{(0)}\big)\big||u_n^{(k)} - u_n^{(0)}| \\
&\leq \sum_{|n|\geq D} \big[\big|f_n\big(u_{n+T}^{(k)}, u_n^{(k)}, u_{n-T}^{(k)}\big)\big| + \big|f_n\big(u_{n+T}^{(0)}, u_n^{(0)}, u_{n-T}^{(0)}\big)\big|\big](|u_n^{(k)}| + |u_n^{(0)}|) \\
&\leq \varepsilon \sum_{|n|\geq D} \Big\{\big[(u_{n+T}^{(k)})^{\delta+1} + (u_n^{(k)})^{\delta+1} + (u_{n-T}^{(k)})^{\delta+1}\big]^{\frac{\delta}{\delta+1}} \\
&\quad + \big[(u_{n+T}^{(0)})^{\delta+1} + (u_n^{(0)})^{\delta+1} + (u_{n-T}^{(0)})^{\delta+1}\big]^{\frac{\delta}{\delta+1}}\Big\}(|u_n^{(k)}| + |u_n^{(0)}|) \\
&\leq 3\varepsilon \sum_{n=-\infty}^{+\infty} \big[|u_n^{(k)}|^{\delta} + |u_n^{(0)}|^{\delta}\big](|u_n^{(k)}| + |u_n^{(0)}|) \\
&\leq 6\varepsilon \sum_{n=-\infty}^{+\infty} \big[(u_n^{(k)})^{\delta+1} + (u_n^{(0)})^{\delta+1}\big] \\
&\leq \frac{6\varepsilon}{\underline{q}}(K_1^{\delta+1} + \|u^{(0)}\|^{\delta+1}).
\end{aligned} \tag{2.15}
$$

Since ε is arbitrary, we obtain

$$
\sum_{n=-\infty}^{+\infty} \big|f_n\big(u_{n+T}^{(k)}, u_n^{(k)}, u_{n-T}^{(k)}\big) - f_n\big(u_{n+T}^{(0)}, u_n^{(0)}, u_{n-T}^{(0)}\big)\big| \to 0, \quad k \to \infty. \tag{2.16}
$$

It follows from (2.2), (2.4) and (2.7) that

$$
\begin{aligned}
&\langle J'(u^{(k)}) - J'(u^{(0)}), u^{(k)} - u^{(0)}\rangle \\
&= \|u^{(k)} - u^{(0)}\|^{\delta+1} \\
&\quad - \sum_{n=-\infty}^{+\infty} \big[f_n\big(u_{n+T}^{(k)}, u_n^{(k)}, u_{n-T}^{(k)}\big) - f_n\big(u_{n+T}^{(0)}, u_n^{(0)}, u_{n-T}^{(0)}\big)\big](u^{(k)} - u^{(0)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|u^{(k)} - u^{(0)}\|^{\delta+1} \\
&\leq \langle J'(u^{(k)}) - J'(u^{(0)}), u^{(k)} - u^{(0)}\rangle \\
&\quad + \sum_{n=-\infty}^{+\infty} \big[f_n\big(u_{n+T}^{(k)}, u_n^{(k)}, u_{n-T}^{(k)}\big) - f_n\big(u_{n+T}^{(0)}, u_n^{(0)}, u_{n-T}^{(0)}\big)\big](u^{(k)} - u^{(0)}).
\end{aligned}
$$

Since $\langle J'(u^{(k)}) - J'(u^{(0)}), u^{(k)} - u^{(0)}\rangle \to 0$ as $k \to \infty$, we have $u^{(k)} \to u^{(0)}$ in E. The proof is complete. $\square$

3. Proofs of theorems

In this section, we shall obtain the existence of a nontrivial solution of (1.5) satisfying (1.4) by using the critical point method.

Proof of Theorem 1.1. We shall prove the existence of a nontrivial solution to (1.5) satisfying (1.4). It is clear that $J(0) = 0$. We have already known that $J \in C^1(E, \mathbb{R})$

and J satisfies the (PS) condition. Hence, it suffices to prove that J satisfies the conditions for the (PS) condition. By (A3), there exists $\eta \in (0,1)$ such that

$$|F_n(u_{n+T},u_n)| \leq \frac{1}{4(\delta+1)}(u_{n+T}^{\delta+1}+u_n^{\delta+1}), \quad \forall n \in \mathbb{Z}\setminus M,\ (u_{n+T}^{\delta+1}+u_n^{\delta+1})^{\frac{1}{\delta+1}} \leq \eta. \tag{3.1}$$

Set

$$G = \sup\{W_n(v_2) | v_2 \in \mathbb{R},\ v_2^{\delta+1} = 1\}, \tag{3.2}$$

and

$$\theta = \min\big\{[\frac{\underline{q}}{8(\delta+1)(G+1)}]^{\mu-(\delta+1)}, \eta\big\}.$$

If $\|u\| = \underline{q}^{\frac{1}{\delta+1}}\theta := \rho$, then by Lemma 2.3, $|u_n| \leq \theta \leq \eta < 1$ for $n \in \mathbb{Z}$. By (A3), (3.1), (3.2) and Lemma 2.3, we have

$$\begin{aligned} \sum_{n\in M} W_n(u_n) &\leq \sum_{n\in M,\ u_n\neq 0} W_n(\frac{u_n}{|u_n|})|u_n|^\mu \\ &\leq G\sum_{n\in M}|u_n|^\mu \\ &\leq G\theta^{\mu-(\delta+1)}\sum_{n\in M}u_n^{\delta+1} \\ &\leq \frac{G\theta^{\mu-(\delta+1)}}{\underline{q}}\sum_{n\in M}q_n u_n^{\delta+1} \\ &\leq \frac{1}{8(\delta+1)}\sum_{n\in M}q_n u_n^{\delta+1}. \end{aligned} \tag{3.3}$$

Set $\alpha = \frac{1}{2(\delta+1)}\theta^{\delta+1}$. Hence, from (2.3), (3.1), (3.2), (A2)–(A4), we have

$$\begin{aligned} J(u) &\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}\setminus M}F_n(u_{n+T},u_n) - \sum_{n\in M}F_n(u_{n+T},u_n) \\ &\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{8(\delta+1)}\sum_{n\in\mathbb{Z}\setminus M}(u_{n+T}^{\delta+1}+u_n^{\delta+1}) - \sum_{n\in M}W_n(u_n) \\ &\quad + \sum_{n\in M}H_n(u_{n+T},u_n) \\ &\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{4(\delta+1)}\sum_{n\in\mathbb{Z}\setminus M}q_n u_n^{\delta+1} - \frac{1}{4(\delta+1)}\sum_{n\in M}q_n u_n^{\delta+1} \\ &\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{4(\delta+1)}\|u\|^{\delta+1} - \frac{1}{4(\delta+1)}\|u\|^{\delta+1} \\ &= \frac{1}{2(\delta+1)}\|u\|^{\delta+1} = \alpha. \end{aligned} \tag{3.4}$$

This inequality shows that $\|u\| = \rho$ implies that $J(u) \geq \alpha$, i.e., J satisfies assumption (1) in Lemma 2.1.

Next we shall verify the condition (2). Take $\tau \in E$ such that

$$|\tau_n| = \begin{cases} 1, & \text{for } |n| \leq 1, \\ 0, & \text{for } |n| \geq 2, \end{cases} \tag{3.5}$$

and $|\tau_n| \le 1$ for $|n| \in (1,2)$. For any $u \in E$, it follows from (2.7) and (A7) that

$$\begin{aligned}
&\sum_{n=-2}^{2} H_n(u_{n+T}, u_n)\\
&= \sum_{n\in\mathbb{Z}(-2,2),\ u_{n+T}^{\delta+1}+u_n^{\delta+1}>1} H_n(u_{n+T}, u_n) + \sum_{n\in\mathbb{Z}(-2,2),\ u_{n+T}^{\delta+1}+u_n^{\delta+1}\le 1} H_n(u_{n+T}, u_n)\\
&\le c \sum_{n\in\mathbb{Z}(-2,2),\ u_{n+T}^{\delta+1}+u_n^{\delta+1}>1} (u_{n+T}^{\delta+1} + u_n^{\delta+1})^{\frac{\varrho}{\delta+1}}\\
&\quad + \sum_{n\in\mathbb{Z}(-2,2),\ u_{n+T}^{\delta+1}+u_n^{\delta+1}\le 1} H_n(u_{n+T}, u_n)\\
&\le 2c\underline{q}^{-\frac{\varrho}{\delta+1}} \|u\|^{\varrho} + K_2,
\end{aligned} \tag{3.6}$$

where

$$K_2 = \sum_{n\in\mathbb{Z}(-2,2),\ u_{n+T}^{\delta+1}+u_n^{\delta+1}\le 1} H_n(u_{n+T}, u_n).$$

For $\sigma > 1$, by Lemma 2.4 and (3.5), we have

$$\sum_{n=-1}^{1} W_n(\sigma u_n) \ge \sigma^{\mu} \sum_{n=-1}^{1} W_n(u_n) = K_3\sigma^{\mu}, \tag{3.7}$$

where $K_3 = \sum_{n=-1}^{1} W_n(u_n) > 0$. By (2.3), (3.5), (3.6) and (3.7), for $\sigma > 1$, we have

$$\begin{aligned}
J(\sigma\tau) &= \frac{1}{\delta+1}\|\sigma\tau\|^{\delta+1} + \sum_{n=-\infty}^{+\infty} [H_n(\sigma\tau_{n+T}, \sigma\tau_n) - W_n(\sigma\tau_n)]\\
&\le \frac{\sigma^{\delta+1}}{\delta+1}\|\tau\|^{\delta+1} + \sum_{n=-2}^{2} H_n(\sigma\tau_{n+T}, \sigma\tau_n) - \sum_{n=-1}^{1} W_n(\sigma\tau_n)\\
&\le \frac{\sigma^{\delta+1}}{\delta+1}\|\tau\|^{\delta+1} + 2c\underline{q}^{-\frac{\varrho}{\delta+1}}\|u\|^{\varrho} + K_2 - K_3\sigma^{\mu}.
\end{aligned} \tag{3.8}$$

Since $\mu > \varrho > \delta + 1$ and $K_3 > 0$, (3.8) implies that there exists $\sigma_0 > 1$ such that $\sigma_0\tau > \rho$ and $J(\sigma_0\tau) < 0$. Set $e = \sigma_0\tau$. Then $e \in E$, $\|e\| = \|\sigma_0\tau\| > \rho$ and $J(e) = J(\sigma_0\tau) < 0$. By Lemma 2.1, J possesses a critical value $d \ge \alpha$ given by

$$d = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}.$$

Hence, there exists $u^* \in E$ such that

$$J(u^*) = d, \quad J'(u^*) = 0.$$

Then function u^* is a desired solution of (1.5) satisfying (1.4). Since $d > 0$, u^* is a nontrivial solution. The desired results follow. $\square$

Proof of Theorem 1.2. In the proof of Theorem 1.1, the condition that $H_n(v_1, v_2) \ge 0$ for $(n, v_1, v_2) \in M \times \mathbb{R}^2$, $\beta_1 = (v_1^{\delta+1} + v_2^{\delta+1})^{\frac{1}{\delta+1}}$ in (A4) is only used in the proof of hypothesis (1) of Lemma 2.1. Thus, we only prove hypothesis (1) of Lemma 2.1 still hold replacing (A4) by (A4'). By (A4'), we have

$$|F_n(u_{n+T}, u_n)| \leq \frac{1}{4(\delta+1)}(u_{n+T}^{\delta+1} + u_n^{\delta+1}), \quad \forall n \in \mathbb{Z}, \ (u_{n+T}^{\delta+1} + u_n^{\delta+1})^{\frac{1}{\delta+1}} \leq \eta. \tag{3.9}$$

If $\|u\| = \underline{q}^{\frac{1}{\delta+1}}\eta := \rho$, then by Lemma 2.3, $|u_n| \leq \eta$ for $n \in \mathbb{Z}$. Set $\alpha = \frac{1}{2(\delta+1)}\eta^{\delta+1}$. Hence, from (2.3) and (3.9), we have

$$\begin{aligned}
J(u) &\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n=-\infty}^{+\infty} F_n(u_{n+T}, u_n) \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{4(\delta+1)}\sum_{n=-\infty}^{+\infty}(u_{n+T}^{\delta+1} + u_n^{\delta+1}) \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\sum_{n=-\infty}^{+\infty} q_n u_n^{\delta+1} \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\|u\|^{\delta+1} \\
&= \frac{1}{2(\delta+1)}\|u\|^{\delta+1} = \alpha.
\end{aligned} \tag{3.10}$$

This inequality shows that $\|u\| = \rho$ implies that $J(u) \geq \alpha$, i.e., J satisfies assumption (1) of Lemma 2.1. The proof is complete. $\square$

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