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$C^{2,\alpha}$ ESTIMATES AND EXISTENCE RESULTS FOR A NONCONCAVE PDE

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Abstract. We establish $C^{2,\alpha}$ estimates for PDE of the form convex + a sum of weakly concave functions of the Hessian, thus generalising a recent result of Collins which is in turn inspired by a theorem of Caffarelli and Yuan. We apply this result to prove a "unique continuation" result for a generalised Monge-Ampère PDE. Independently, we also prove an existence result for a special case.

1. Introduction

In the classic paper [9] Krylov studied the PDE on a convex domain

$$S_m(D^2u) = \sum_{k=0}^{m-1} (l_k^+)^{m-k+1}(x) S_k(D^2u) \tag{1.1}$$

where $S_m(A)$ is the mth elementary symmetric polynomial of the symmetric matrix A. He proved that the corresponding Dirichlet problem has a smooth solution in the ellipticity cone of the equation. This was accomplished by reducing the equation to a Bellman equation and then using the standard theory of Bellman equations. Motivated by complex-geometric considerations (Chern-Weil theory) a very special case of equation 1.1 was studied in [10] and an existence result was proven using the method of continuity. To this end, *a priori* estimates on the solution were necessary. The $C^{2,\alpha}$ estimate for such nonlinear PDE is usually given by the Evans-Krylov-Safonov theorem which applies to PDE of the form $F(D^2u) = 0$ where F is a concave function of symmetric matrices. However, it is not immediately obvious that equation 1.1 is concave. Yet, upon dividing by $\det(D^2u)$ and rearranging the equation one can see that it is actually concave and thus amenable to Evans-Krylov theory.

Unfortunately, not all PDE can be rewritten to be concave functions of the Hessian. Indeed, not all level sets have a positive second fundamental form. To remedy this partially, Caffarelli and Yuan [4] proved a result that roughly speaking, allows one of the eigenvalues of the second fundamental form of the level set of $F(D^2u)$ to be negative. Using similar ideas, Cabre and Caffarelli [2] proved $C^{2,\alpha}$ estimates for functions that are the minimum of convex and concave functions. Even these theorems cannot handle the following PDE that arises in the study of

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the J-flow on toric manifolds [5] (Actually, the Legendre transform of the solution occurs in the J-flow.).

$$\det(D^2u) + \Delta u = 1. \tag{1.2}$$

Moreover, equation 1.2 is also a real example of a "generalised Monge-Ampère" PDE introduced in [10]. Another example of a non-concave PDE is

$$\ln\det(u_{x_ix_j}) - \ln\det(-u_{y_iy_j}) = 0.$$

This equation was studied by Streets and Warren in [11] and they proved a $C^{2,\alpha}$ estimate using the Legendre transformation in the y-coordinates.

Collins and Székelyhidi [5] proved interior $C^{2,\alpha}$ estimates for equation 1.2 using ideas from [4]. In [6] Collins generalised that result to obtain the following theorem. (The precise definition of "twisted" type equations is recalled in section 2.).

Theorem 1.1 (Collins)**.** *Consider the equation*

$$F(D^2u, x) = F_{\cup}(D^2u, x) + F_{\cap}(D^2u, x) = 0$$

on the unit ball B_1 in $\mathbb{R}^n$. For each x, assume that F is of the twisted type. Let $0 < \lambda < \Lambda < \infty$ be ellipticity constants for both $F, F_{\cup}$. For every $0 < \alpha < 1$ we have the estimate

$$\begin{aligned} &\|D^2u\|_{C^\alpha(B_{1/2})} \\ &\le C(n, \lambda, \Lambda, \alpha, \gamma, \Gamma, \|F_{\cup}\|_{C^2(D^2u(\bar{B}_1))}, \|F_{\cap}\|_{C^2(D^2u(\bar{B}_1))}, \|D^2u\|_{L^\infty(B_1)}), \end{aligned} \tag{1.3}$$

where $0 < \gamma = \inf_{x\in F_{\cup}(D^2u)(B_1)} G'(-x)$ and $\Gamma = osc_{B_1} G(-F_{\cup}(D^2u))$. (G is defined in section 2.)

Motivated by these developments, in this paper we prove the following improvement of Collins' result.

Theorem 1.2. *Consider the equation*

$$F(D^2u, x) = F_{\cup}(D^2u, x) + \sum_{\alpha=1}^{m} F_{\cap,\alpha}(D^2u, x) = 0$$

on the unit ball B_1 in $\mathbb{R}^n$. For each x, assume that F is of the "generalised" twisted type. Let $0 < \lambda < \Lambda < \infty$ be ellipticity constants for both $F, F_{\cup}$. For every $0 < \alpha < 1$ we have the estimate

$$\begin{aligned} &\|D^2u\|_{C^\alpha(B_{1/2})} \\ &\le C(n, \lambda, \Lambda, \alpha, \gamma, \|F_{\cup}\|_{C^2(D^2u(\bar{B}_1))}, \|F_{\cap}\|_{C^2(D^2u(\bar{B}_1))}, \|D^2u\|_{L^\infty(B_1)}, \|G\|_{L^\infty(W)}), \end{aligned} \tag{1.4}$$

where $0 < \gamma = \inf_{\{x\in W\}} G'(x)$ and

$$W = \cup_{\alpha=1}^{m} F_{\cap,\alpha}(D^2u(\bar{B}_1)) \cup_{1\le j\le m} \cup_{\{x\,\in\,\bar{B}(1)\}} \sum_{\alpha=1}^{j} F_{\cap,\alpha}(D^2u(x)).$$

The proof of theorem 1.2 follows the arguments (with some modifications) in [6, 4]. Applying this result we arrive at the following "unique continuation" result for equations of like 1.2.

Corollary 1.3. *Let D be a strictly convex domain in $\mathbb{R}^n$, i.e., there exists a smooth proper function $\rho : \bar{D} \to \mathbb{R}$ such that $\rho_{ij} > K\delta_{ij}$ for a constant $K > 0$, $\nabla\rho|_{\partial D} \neq 0$, $\rho^{-1}(0) = \partial D$ and $\rho^{-1}(-\infty, 0) = D$. Consider the family of equations depending on $t \in [0, 1]$.*

$$\begin{gathered} \begin{aligned} H(D^2u_t, x, t) &= \det(D^2u_t) + t\Big(\operatorname{tr}(AD^2u_t) + \sum_{k=2}^{n-1} f_k\sigma_{k,B_k}(D^2u_t)\Big) \\ &= g \quad \text{in } D \end{aligned} \\ u_t = 0 \quad \text{on } \partial D. \end{gathered} \tag{1.5}$$

where $g : \bar{\Omega} \to \mathbb{R}_{>0}$, $f_k : \bar{\Omega} \to \mathbb{R}_{\geq 0}$ are smooth functions. Also assume that A, B_k are smooth, positive-definite $n \times n$ real matrix-valued functions on $\bar{\Omega}$, and let $\sigma_{k,B}(A)$ be the coefficient of t^k in $\det(B + tA)$. There exists a number $T \in (0, 1]$ such that the equation has unique, smooth, strictly convex (i.e. $D^2u > 0$ on $\bar{\Omega}$) solutions for $t \in [0, T)$. For any number $t_* in (0, 1]$ such that the equation has a unique smooth strictly convex solutions in $[0, t_*)$, there exists unique smooth strictly convex solutions in $[0, t_* + \delta)$ for some $\delta > 0$.

Independently, we also prove the following existence result.

Proposition 1.4. *Consider the PDE*

$$\begin{gathered} \det(D^2u) + \sum_{k=2}^{n} S_k(D^2u) = f \quad \text{in } D \\ u|_{\partial D} = \phi, \end{gathered} \tag{1.6}$$

where S_k is the kth symmetric polynomial (for instance σ_n is the determinant), $f : \bar{D} \to (n-1, \infty)$ and ϕ are smooth functions (with ϕ being the restriction to ∂D of a smooth function on $\bar{D}$), and D is a strictly convex domain with a proper smooth defining function ρ, i.e., $\rho^{-1}(0) = \partial D$, $\rho^{-1}(-\infty, 0) = D$, $\nabla\rho \neq 0$ on ∂D, and $D^2\rho \geq CI$ ($C > 0$ is a constant). It has a unique smooth solution u such that $D^2u > -I$ and

$$\frac{\partial}{\partial\lambda_i}(\lambda_1\lambda_2 \dots \lambda_n + \sum_{k=2}^{n} \sigma_k(\vec{\lambda})) > 0 \quad \forall i,$$

where λ_i are the eigenvalues of D^2u.

The requirement $f > n - 1$ is not optimal. But we give a counterexample for finding solutions in the ellipticity cone in the case $f < 0$. Notice that this seemingly harder equation has an existence result but it is still not clear whether equation 1.2 does.

The layout of the paper is as follows. In section 2 we give the definitions of twisted type equations and give an example of its applicability. In section 4 we prove proposition 1.4 and discuss its hypotheses.

2. Preliminaries

In this section we present the definitions and prove some basic results. Firstly, we define what it means for a PDE to be of the generalised twisted type. The following definition generalises Collins' [6].

Definition 2.1. Let $F(D^2u) = 0$ be a uniformly elliptic equation on the unit ball B_1. It is said to be of the generalised twisted type if $F = F_\cup + \sum_{\alpha=1}^{m} F_{\cap,\alpha}$ where

(1) $F_{\cup}$ and $\forall\ 1 \leq \alpha \leq m$ $F_{\cap,\alpha}$ are (possibly degenerate) elliptic C^2 functions on an open set $\mathcal{O}$ containing $D^2u(\bar{B}_1)$.
(2) $F_{\cup}$ is convex and uniformly elliptic on the space of all symmetric matrices, and $\sum_{\alpha=1}^{m} F_{\cap,\alpha}$ is weakly concave on $\mathcal{O}$ in the sense of definition 2.2.

The definition of weak concavity in our case is as follows.

Definition 2.2. We say that $\sum_{\alpha=1}^{m} F_{\cap,\alpha}$ is weakly concave if there exists a function $G : U \to \mathbb{R}$ such that

(1) The domain U contains a connected open set V with compact closure containing
$$W = \cup_{\alpha=1}^{m} F_{\cap,\alpha}(D^2u(\bar{B}_1)) \cup_{1\leq j\leq m} \cup_{\{x\ \in\ \bar{B}(1)\}} \sum_{\alpha=1}^{j} F_{\cap,\alpha}(D^2u(x)).$$
(2) $G' > 0$, $G'' \leq 0$, and $G(F_{\cap,\alpha}(.))$ is concave for all $1 \leq \alpha \leq m$.
(3) For all $x \in \bar{B}(1)$ and $1 \leq \alpha \leq m$ consider $y_\alpha(x) = F_{\cap,\alpha}(D^2u(x))$. There exists a constant $1 \geq c > 0$ independent of x such that
$$\sum_{i=1}^{m} G(y_i(x)) \geq G\Big(\sum_{i=1}^{m} y_i(x)\Big) \geq c\sum_{i=1}^{m} G(y_i(x)).$$

Definition 2.2 might seem somewhat convoluted and unnatural compared to the analogous one in [6]. Firstly, we remark that condition (3) is actually redundant in many cases of interest (but we choose to impose it since it appears naturally in our proofs). Indeed,

Proposition 2.3. *Given a function $\tilde{G}$ that satisfies requirements (1), (2) of definition 2.2 such that $W \subseteq \mathbb{R}_{\geq 0}$ and $\tilde{G}(0) = 0$, automatically satisfies requirement* (3)*, i.e.,*
$$\sum_{\alpha=1}^{m} \tilde{G}(y_\alpha(x)) \geq \tilde{G}\Big(\sum_{al=1}^{m} y_\alpha(x)\Big) \geq \frac{1}{2^m}\sum_{\alpha=1}^{m} \tilde{G}(y_\alpha(x)).$$

Proof. Consider the function $T(y) = \tilde{G}(y+z) - \tilde{G}(y) - \tilde{G}(z)$ for a fixed $z \geq 0$. By the concavity of G we see that $T'(y) \leq 0$. Hence $\tilde{G}(y+z) - \tilde{G}(y) - \tilde{G}(z) \leq -\tilde{G}(0) = 0$. Using induction we see that
$$\sum_{\alpha=1}^{m} \tilde{G}(y_\alpha(x)) \geq \tilde{G}\Big(\sum_{\alpha=1}^{m} y_\alpha(x)\Big).$$
The concavity of G implies that
$$\tilde{G}\big(\frac{y+z}{2}\big) \geq \frac{\tilde{G}(y) + \tilde{G}(z)}{2}.$$
Since $\tilde{G}$ is increasing this implies that $\tilde{G}(y+z) \geq \frac{\tilde{G}(y)+\tilde{G}(z)}{2}$. Induction gives the desired result. ☐

Remark 2.4. Furthermore, it is more natural to have a different G_α that works for $F_{\cap,\alpha}$. However, under mild conditions on such G_α one may produce a G that works for all $1 \leq \alpha \leq m$. Indeed, assume that $\bar{V} \subset \mathbb{R}_{\geq 0}$, and G_α are such that on the appropriate compact sets $G_\alpha \geq 0$, $G'_\alpha \geq 1$ and $G_1(\bar{V}) \subseteq \mathrm{dom}(G_2)$, $G_2(G_1(\bar{V})) \subseteq \mathrm{dom}(G_3) \ldots$.

Consider the function $H_k = G_k \circ G_{k-1} \dots \circ G_1$. Note that

$$\begin{aligned} D^2 H_k(F_{\cap,k}) &= H_k'' DF_{\cap,k} DF_{\cap,k} + H_k' D^2 F_{\cap,k} \\ &= (G_k''(H_{k-1}')^2 + G_k' H_{k-1}'') DF_{\cap,k} DF_{\cap,k} + G_k' H_{k-1}' D^2 F_{\cap,k} \end{aligned}$$

Inductively we may assume that $H_{k-1}' \geq 1$. Thus we obtain

$$D^2 H_k(F_{\cap,k}) \leq H_{k-1}'(G_k'' DF_{\cap,k} DF_{\cap,k} + G_k' D^2 F_{\cap,k}) + G_k' H_{k-1}'' DF_{\cap,k} DF_{\cap,k} \leq 0$$

where we used the facts that $G_k \circ F_{\cap,k}$ is concave, $H_{k-1}' > 0$, $G_k' > 0$, and H_{k-1} is concave. Now notice that if H is any concave increasing function and $Y(A)$ is any concave function of symmetric matrices, then $D^2(H \circ Y) = H'' DY DY + H' D^2 Y \leq 0$. This means that $H_m \circ F_{\cap,\alpha}$ is concave for all $1 \leq \alpha \leq m$. Using proposition 2.3 we are done.

Now we give an example of an equation that satisfies the conditions imposed by theorem 1.2.

Proposition 2.5. *Consider the following equation on a domain Ω.*

$$H(D^2 u, x) = \operatorname{tr}(AD^2 u) + \sum_{k=2}^{n} f_k \sigma_{k,B_k}(D^2 u) = g \tag{2.1}$$

where $g : \bar{\Omega} \to \mathbb{R}_{>0}$, $f_k : \bar{\Omega} \to \mathbb{R}_{\geq 0}$ are smooth functions. Also assume that A, B_k are smooth, positive-definite $n \times n$ real matrix-valued functions on $\bar{\Omega}$. $\sigma_{k,B}(A)$ be the coefficient of t^k in $\det(B + tA)$. Equation 2.1 is of the generalised twisted type on every ball $B_r(x_0) \subseteq \Omega$ if $D^2 u > 0$ on $\bar{\Omega}$.

Proof. Fix an x. In equation 2.1 $F_{\cup}(D^2 u) = \operatorname{tr}(AD^2 u)$ which is obviously smooth and uniformly elliptic. As for $F_{\cap,\alpha}(D^2 u) = \sigma_{\alpha,B_\alpha}(D^2 u)$, firstly by means of diagonalising the quadratic form B_α we may assume that it is the identity matrix. Thus, at the point x we see that $F_{\cap,\alpha}(D^2 u)$ is a positive multiple of the αth symmetric polynomial. Hence it is elliptic if $CI > D^2 u > 0$ (It may not be uniformly elliptic because we do not have a given lower bound on $D^2 u$, but that is not a requirement anyway.). Therefore $F(D^2 u)$ is uniformly elliptic. Moreover, the function $G(x) = x^{1/n}$ defined on $\mathbb{R}_{>0}$ satisfies the conditions required by definition 2.2. Indeed, since $(\sigma_{k,B_k})^{1/k}$ is concave it is clear that $(\sigma_{k,B_k})^{1/n}$ is too. $\square$

Proposition 2.5 may be used to prove corollary 1.3.

Proof of corollary 1.3. Uniqueness of solutions satisfying $D^2 u_t > 0$ on $\bar{\Omega}$ is standard. At $t = 0$ the equation boils down to the usual Monge-Ampère equation and hence has a smooth solution. A standard implicit function theorem argument shows that the set of $t \in [0, 1]$ for which the solution exists is open. Hence solutions exist for $t \in [0, T)$ for some $T > 0$. To prove "continuation" at t_*, we need *a priori* estimates as usual. At least some of these are obtained by following the arguments of [3]. $\square$

Lemma 2.6. *If u_t is a smooth convex solution of equation 1.5 then $\|u_t\|_{C^2(\bar{D})} \leq C$ where C depends only on the C^1 norm of the coefficients of the equation and $\|\rho\|_{C^2(\bar{D})}$.*

Proof. We omit the subscript t in what follows.
C^0 estimate: Since $D^2u > 0$, by the maximum principle $u \leq 0$. Choose a constant $R \gg 1$ so that $R\rho$ satisfies $F(D^2(R\rho), x) \geq g$. Upon subtraction we obtain

$$H(D^2u, x) - H(D^2\rho, x) = \int_0^1 H^{ij}(tD^2u + (1-t)D^2\rho, x)(u-\rho)_{x_i x_j} dt \leq 0.$$

This means (by the minimum principle) that $u \geq R\rho$ on $\bar{D}$.
C^2 estimate : Since $D^2u > 0$ and $\operatorname{tr}(AD^2u) \leq C$, we see that $\|D^2u\|_{L^\infty(\bar{D})} \leq C$. Since $0 < \Delta u \leq C$ and $\|u\|_{C^0} \leq C$, by the L^p regularity of elliptic equations we see that $\|u\|_{C^1} \leq C$ as well.

Notice that this does not guarantee uniform (independent of t) lower boundedness of D^2u away from zero. $\square$

Using proposition 2.5 we see that for every compact subset K of D, $\|u\|_{C^{2,\alpha}(K)} \leq C_K$. The interior estimates together with the uniform ellipticity of equation 2.1 actually imply boundary $C^{2,\alpha}$ estimates thanks to a theorem of Krylov whose simplified proof may be found in [8] for instance. This completes the proof of corollary 1.3.

3. Proof of theorem 1.2

As mentioned in the introduction we prove a stronger version of Theorem 1.1, i.e. instead of $F_\cup + F_\cap = 0$ we have $F_\cup + \sum_{\alpha=1}^m F_{\cap,\alpha} = 0$ where there exists a G so that $G(F_{\cap,\alpha})$ is concave for every α. The strategy to prove theorem 1.2 is exactly the one used in [4, 5, 6]. Here is a high-level overview:

(1) One first reduces the content of theorem 1.2 to the case where $F(D^2u, x)$ does not depend on x. Indeed, one can use a blowup argument à la [6] to conclude this. This reduction step requires F to be uniformly elliptic which it is by assumption.
(2) In the case of $F(D^2u) = 0$, one proves that the level set of u is very "close" to a quadratic polynomial satisfying $F(D^2P) = 0$ (after "zooming" in so to say). This is done by proving that $F_\cup(D^2u)$ concentrates in measure near its level set using the Krylov-Safonov weak Harnack inequality, and using the Alexandrov-Bakelmann-Pucci estimate in conjunction with the usual Evans-Krylov theory to conclude the existence of a polynomial close to u. Then one perturbs the polynomial to make it satisfy $F(D^2P) = 0$.
(3) Then it may be proven that one can find a family of such quadratic polynomials with the "closeness" improving in a quantitative way on the size (the smaller the better) of the neighbourhood of the point in consideration.
(4) This can be used to prove that the second derivative does not change too much, i.e., the desired estimate on $\|D^2u\|_{C^\alpha(B_{1/2})}$.

Out of these, only step 2 needs modification in our case. To this end, we need the following lemma.

Lemma 3.1. *Let L be the linearisation of $F = F_\cup + \sum_\alpha F_{\cap,\alpha}$, i.e. $L^{ab} = F_\cup^{ab} + \sum_\alpha F_{\cap,\alpha}^{ab}$. Then*

$$L\Big(\sum_\alpha G(F_{\cap,\alpha}(D^2u))\Big) \leq 0.$$

Proof. We compute

$$\partial_a G(F_{\cap,\alpha}(D^2u)) = G' F^{ij}_{\cap,\alpha} u_{x_a x_i x_j}$$

$$\begin{aligned}\partial_{ab} G(F_{\cap,\alpha}(D^2u)) &= G'' F^{ij}_{\cap,\alpha} u_{x_a x_i x_j} F^{rs}_{\cap,\alpha} u_{x_b x_r x_s} + G' F^{ijrs}_{\cap,\alpha} u_{x_a x_i x_j} u_{x_b x_r x_s} \\ &\quad + G' F_{\cap,\alpha} u_{x_a x_b x_i x_j}.\end{aligned}$$

Moreover, using the equation itself we obtain

$$L^{ab} u_{x_a x_b x_i} = (F^{ab}_{\cup} + \sum_{\alpha} F^{ab}_{\cap,\alpha}) u_{x_a x_b x_i} = 0 \tag{3.1}$$

$$L^{ab} u_{x_a x_b x_i x_j} + (F^{abrs}_{\cup} + \sum_{\alpha} F^{abrs}_{\cap,\alpha}) u_{x_a x_b x_i} u_{x_r x_s x_j} = 0. \tag{3.2}$$

Then we obtain

$$\begin{aligned}
&L\Big(\sum_{\alpha=1}^{m} G(F_{\cap,\alpha}(D^2u))\Big) \\
&= \sum_{\alpha=1}^{m} L^{ab}(G'' F^{ij}_{\cap,\alpha} u_{x_a x_i x_j} F^{rs}_{\cap,\alpha} u_{x_b x_r x_s} + G' F^{ijrs}_{\cap,\alpha} u_{x_a x_i x_j} u_{x_b x_r x_s} \\
&\quad + G' F^{ij}_{\cap,\alpha} u_{x_a x_b x_i x_j}) \\
&= \sum_{\alpha=1}^{m} L^{ab}(G'' F^{ij}_{\cap,\alpha} F^{rs}_{\cap,\alpha} + G' F^{ijrs}_{\cap,\alpha}) u_{x_a x_i x_j} u_{x_b x_r x_s} + G' L^{ab} F^{ij}_{\cap,\alpha} u_{x_a x_b x_i x_j} \\
&= \sum_{\alpha=1}^{m} \Big((F^{ab}_{\cup} + \sum_{\beta} F^{ab}_{\cap,\beta})(G'' F^{ij}_{\cap,\alpha} F^{rs}_{\cap,\alpha} + G' F^{ijrs}_{\cap,\alpha}) u_{x_a x_i x_j} u_{x_b x_r x_s} \\
&\quad - G' F^{ab}_{\cap,\alpha}(F^{ijrs}_{\cup} + \sum_{\beta} F^{ijrs}_{\cap,\beta}) u_{x_i x_j x_a} u_{x_r x_s x_b} \Big) \qquad (3.3) \\
&= \sum_{\alpha=1}^{m} \Big(F^{ab}_{\cup}(G'' F^{ij}_{\cap,\alpha} F^{rs}_{\cap,\alpha} + G' F^{ijrs}_{\cap,\alpha}) u_{x_a x_i x_j} u_{x_b x_r x_s} \\
&\quad + \sum_{\beta} F^{ab}_{\cap,\beta} G'' F^{ij}_{\cap,\alpha} F^{rs}_{\cap,\alpha} u_{x_i x_j x_a} u_{x_r x_s x_b} - G' F^{ab}_{\cap,\alpha} F^{ijrs}_{\cup} u_{x_i x_j x_a} u_{x_r x_s x_b} \Big) \qquad (3.4)
\end{aligned}$$

At this point we note that since $G \circ F_{\cap,\alpha}$ is concave and $F_{\cup}$ is elliptic the first term in 3.4 is negative. Likewise, so is the second term because $G'' \le 0$ and $F_{\cap}$ is also elliptic. Since $F_{\cup}$ is convex, so is the third term. Hence we see that

$$L\Big(\sum_{\alpha} G(F_{\cap,\alpha}(D^2u))\Big) \le 0.$$

Note that in equation 3.3 the terms of the form $F^{ab}_{\cap,\alpha} F^{ijrs}_{\cap,\beta}$ cancelled out. This is perhaps the main point of this calculation. If we had different G_α for each α this would not have happened. □

Secondly, we need the following proposition that actually addresses step 2 in the strategy described above.

Proposition 3.2. *Under the assumptions of the main theorem, for any given $\epsilon > 0$ there exists a positive constant*

$$\eta = \eta(c, m, \|G\|_{L^\infty}, \quad \|F_{\cap,\alpha}\|_{L^\infty}, n, \lambda, \Lambda, \epsilon, \gamma, \Gamma, \|D^2u\|_{L^\infty})$$

quadratic polynomial P so that for all x in B_1,

$$|\frac{1}{\eta^2}u(\eta x) - P(x)| \leq \epsilon$$
$$F(D^2P) = 0$$

Proof. We shall determine k_0, ρ, ξ, δ in the course of the proof. Let $1 \leq k \leq k_0$ and $t_k = \max_{\bar{B}(1/2^k)} F_\cup(D^2u)$ and

$$s_k = \min_{\bar{B}(1/2^k)} \sum_{\alpha=1}^{m} G(F_{\cap,\alpha}(D^2u)).$$

Also define $w_k(x) = 2^{2k}u(\frac{x}{2^k})$. Hence $D^2w_k(x) = D^2u(\frac{x}{2^k})$.
Note that since G is increasing,

$$G(-t_k) = G\Big(\min_{\bar{B}(1/2^k)} \sum_{\alpha=1}^{m} F_{\cap,\alpha}(D^2u)\Big) = \min_{\bar{B}(1/2^k)} G\Big(\sum_{\alpha=1}^{m} F_{\cap,\alpha}(D^2u)\Big) \geq cs_k.$$

Moreover, $s_k \geq G(-t_k)$.

If there exists an l such that $1 \leq l \leq k_0$ such that

$$|E_k| \leq \delta |B_{1/2^l}| \tag{3.5}$$

where E_k is the set of x in $B_{1/2^{k+1}}$ such that $F_\cup$ is "close" to t_k, i.e. $F_\cup(D^2u) \leq t_k - \xi$, then we are done by the arguments of [6]. If not, we shall arrive at a contradiction by actually proving the existence of such a δ, k and l. Indeed, assume the contrary. By lemma 3.1 we see that $L\big(\sum_\alpha G(F_{\cap,\alpha}(D^2w_k)) - s_k\big) \leq 0$. By applying the weak Harnack inequality we see that for all x in $B_{1/2}$

$$\sum_\alpha G(F_{\cap,\alpha}(D^2w_k))(x) - s_k \geq C(n,\lambda)\|\sum_\alpha G(F_{\cap,\alpha}(D^2w_k))(x) - s_k\|_{L^{p_0}(B_{1/2})},$$

where p_0 depends on n, λ, Λ. On E_k we recall that $\sum_\alpha F_{\cap,\alpha}(D^2w_k) \geq -t_k + \xi$, and hence

$$\sum_\alpha G(F_{\cap,\alpha}(D^2w_k)) \geq G\Big(\sum_\alpha F_{\cap,\alpha}(D^2w_k)\Big) \geq G(-t_k+\xi) \geq G(-t_k)+\gamma\xi \geq cs_k+\gamma\xi.$$

Choose ξ to be large enough so that $(c-1)s_k + \gamma\xi \geq \theta_0 > 0$ where θ_0 does not depend on k. Of course such a θ_0 would depend on $\|D^2u\|_{L^\infty(B_1)}$, $\|F_{\cap,\alpha}\|_{L^\infty}$, and $\|G\|_{L^\infty}$. This means that

$$\sum_\alpha G(F_{\cap,\alpha}(D^2w_k))(x) \geq s_k + C(n,\lambda)\theta_0\delta^{1/p_0} = s_k + \theta$$

In particular this means that $s_{k+1} \leq s_k + \theta$. At this point it follows that after

$$k_0 = \frac{\mathrm{Osc}_{B_1}\left(\sum_\alpha F_{\cap,\alpha}(D^2u)\right)}{\theta}$$

iterations condition 3.5 ought to hold. $\square$

The rest of the proof of theorem 1.2 is exactly the same as in [4].

4. Proof of proposition 1.4

We reduce theorem 1.4 to Krylov's equation 1.1 and invoke the existence result in [9]. Indeed, define $v = u + \frac{1}{2}\sum_{i=1}^{n} x_i^2$. Then $D^2v = D^2u + I$. The eigenvalues of D^2v are $\mu_i = \lambda_i + 1$. Consider the equation

$$\begin{aligned} \mu_1\mu_2\dots\mu_n - \sum_{i=1}^{n}\mu_i &= f - n + 1 \quad \text{in } D \\ v|_{\partial D} &= \phi + \frac{1}{2}\sum_{i=1}^{n} x_i^2. \end{aligned} \tag{4.1}$$

Writing equation 4.1 in terms of λ_i we see quite easily that equation 1.6 is recovered. Thus, Krylov's theorem [9] states that there is a unique smooth solution to 4.1 in the ellipticity cone as long as the right hand side is positive. This proves proposition 1.4.

As mentioned in the introduction, the restriction $f > n-1$ may not be optimal (as is easily seen by considering a radial solution in the case of the ball with a constant f). However, the following counterexample shows that the case $f < 0$ does not admit solutions in the ellipticity cone.

Proposition 4.1. *There is no smooth solution u of the following equation satisfying $\mu_1\dots\mu_{i-1}\mu_{i+1}\dots\mu_n > 1$ and $\mu_i > 0$ where μ_i are the eigenvalues of D^2v.*

$$\begin{aligned} \det(D^2v) - \Delta v &= -c \quad \text{in } B(1) \\ v|_{\partial B(1)} &= 0 \end{aligned} \tag{4.2}$$

where $c > n-1$ is a constant.

Proof. We first show that such a solution has to be radially symmetric. To this end, we use the standard method of moving planes [7]. For $0 \le t \le 1$ consider the plane $P_t : x_n = t$. Let the reflection of the point x across the plane P_t be $x_t = (x_1, \dots, x_{n-1}, 2t - x_n)$ and let $E_t = \{x \in B(1) | t < x_n \le 1\}$. We prove that

$$u(x) > u(x_t) \quad \forall x \in E_t \quad \text{(property (L))}.$$

Near any boundary point the function is strictly increasing as a function of x_n because $\frac{\partial u}{\partial n} \ge 0$ and $D^2u > 0$. Hence (L) holds for $t < 1$ sufficiently close to 1. Let the infimum of all such t be t_0. If $t_0 > 0$, then consider $w(x) = u(x) - u(x_{t_0})$ where $x \in E_{t_0}$. Upon subtracting the equations for $u(x)$ and $u(x_{t_0}$ we see that

$$\begin{aligned} &\det(D^2u(x)) - \Delta(u(x)) - (\det(D^2u(x_{t_0})) - \Delta u(x_{t_0})) = 0 \\ &\Rightarrow \int_0^1 \frac{d}{ds}(\det(D^2(su(x) + (1-s)u(x_{t_0}))) - \Delta(su(x) + (1-s)u(x_{t_0}))) = 0 \\ &\Rightarrow L^{ij}w_{ij}(x) = 0, \end{aligned} \tag{4.3}$$

where L^{ij} is a positive definite matrix depending on u. Note that we have used the assumption that D^2u is in the ellipticity cone and the fact that the cone is convex for this equation. Since $w \ge 0$ in E_{t_0} and $w = 0$ on the plane P_{t_0}, by applying the strong minimum principle we see that $w > 0$ in E_{t_0}. Applying the Hopf lemma to points on the plane P_{t_0} we see that $w_{x_n} > 0$ on $P_{t_0} \cap B(1)$. Since $w_{x_n} = 2u_{x_n}$ on the plane, we see that for t slightly less than t_0 property (L) holds. This is a contradiction. Thus $t_0 = 0$. Since the problem is rotationally symmetric, u is radial. The unique radial solution to the problem (if it exists) is easily seen to be of the form $\frac{A(r^2-1)}{2}$ for some constant $A > 0$. This means that $A^n - nA + c = 0$. It is easy to see that this equation admits no positive solutions. □

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